# Calculus

## An Applied Approach

**EIGHTH EDITION**

**LARSON**

# Applications

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# Calculus

An Applied Approach

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# Calculus

## An Applied Approach

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**Eighth Edition**

# Contents

*Available at the text-specific website at
*college.cengage.com/pic/larsonCAA8e*

# A Word from the Author

Welcome to *Calculus: An Applied Approach*, Eighth Edition. In this revision, I focused not only on providing a meaningful revision to the text, but also a completely integrated learning program. Applied calculus students are a diverse group with varied interests and backgrounds. The revision strives to address the diversity and the different learning styles of students. I also aimed to alleviate and remove obstacles that prevent students from mastering the material.

## An Enhanced Text

The table of contents was streamlined to enable instructors to spend more time on each topic. This added time will give students a better understanding of the concepts and help them to master the material.

Real data and applications were updated, rewritten, and added to address more modern topics, and data was gathered from news sources, current events, industry, world events, and government. Exercises derived from other disciplines' textbooks are included to show the relevance of the calculus to students' majors. I hope these changes will give students a clear picture that the math they are learning exists beyond the classroom.

Two new chapter tests were added: a *Mid-Chapter Quiz* and a *Chapter Test*. The Mid-Chapter quiz gives students the opportunity to discover any topics they might need to study further before they progress too far into the chapter. The Chapter Test allows students to identify and strengthen any weaknesses in advance of an exam.

Several new section-level features were added to promote further mastery of the concepts.

- *Concept Checks* appear at the end of each section, immediately before the exercise sets. They ask non-computational questions designed to test students' basic understanding of that sections' concepts.

- *Make a Decision* exercises and examples ask open-ended questions that force students to apply concepts to real-world situations.

- *Extended Applications* are more in-depth, applied exercises requiring students to work with large data sets and often involve work in creating or analyzing models.

I hope the combination of these new features with the existing features will promote a deeper understanding of the mathematics.

## Enhanced Resources

Although the textbook often forms the basis of the course, today's students often find greater value in an integrated text and technology program. With that in mind, I worked with the publisher to enhance the online and media resources available to students, to provide them with a complete learning program.

An online course has been developed with dynamic, algorithmic exercises tied to exercises within the text. These exercises provide students with unlimited practice for complete mastery of the topics.

An additional resource for the 8th edition is a Multimedia Online eBook. This eBook breaks the physical constraints of a traditional text and binds a number of multimedia assets and features to the text itself. Based in Flash, students can read the text, watch the videos when they need extra explanation, view enlarged math graphs, and more. The eBook promotes multiple learning styles and provides students with an engaging learning experience.

For students who work best in groups or whose schedules don't allow them to come to office hours, Calc Chat is now available with this edition. Calc Chat (located at *www.CalcChat.com*) provides solutions to exercises. Calc Chat also has a moderated online forum for students to discuss any issues they may be having with their calculus work.

I hope you enjoy the enhancements made to the eighth edition. I believe the whole suite of learning options available to students will enable any student to master applied calculus.

Ron Larson

# A Plan for You as a Student

## Study Strategies

Your success in mathematics depends on your active participation both in class and outside of class. Because the material you learn each day builds on the material you have learned previously, it is important that you keep up with your course work every day and develop a clear plan of study. This set of guidelines highlights key study strategies to help you learn how to study mathematics.

*Preparing for Class*    The syllabus your instructor provides is an invaluable resource that outlines the major topics to be covered in the course. Use it to help you prepare. As a general rule, you should set aside two to four hours of study time for each hour spent in class. Being prepared is the first step toward success. Before class:

- Review your notes from the previous class.
- Read the portion of the text that will be covered in class.

*Keeping Up*    Another important step toward success in mathematics involves your ability to keep up with the work. It is very easy to fall behind, especially if you miss a class. To keep up with the course work, be sure to:

- Attend every class. Bring your text, a notebook, a pen or pencil, and a calculator (scientific or graphing). If you miss a class, get the notes from a classmate as soon as possible and review them carefully.
- Participate in class. As mentioned above, if there is a topic you do not understand, ask about it before the instructor moves on to a new topic.
- Take notes in class. After class, read through your notes and add explanations so that your notes make sense to *you*. Fill in any gaps and note any questions you might have.

*Getting Extra Help*    It can be very frustrating when you do not understand concepts and are unable to complete homework assignments. However, there are many resources available to help you with your studies.

- Your instructor may have office hours. If you are feeling overwhelmed and need help, make an appointment to discuss your difficulties with your instructor.
- Find a study partner or a study group. Sometimes it helps to work through problems with another person.
- Special assistance with algebra appears in the *Algebra Reviews*, which appear throughout each chapter. These short reviews are tied together in the larger *Algebra Review* section at the end of each chapter.

*Preparing for an Exam*    The last step toward success in mathematics lies in how you prepare for and complete exams. If you have followed the suggestions given above, then you are almost ready for exams. Do not assume that you can cram for the exam the night before—this seldom works. As a final preparation for the exam:

- When you study for an exam, first look at all definitions, properties, and formulas until you know them. Review your notes and the portion of the text that will be covered on the exam. Then work as many exercises as you can, especially any kinds of exercises that have given you trouble in the past, reworking homework problems as necessary.
- Start studying for your exam well in advance (at least a week). The first day or two, study only about two hours. Gradually increase your study time each day. Be completely prepared for the exam two days in advance. Spend the final day just building confidence so you can be relaxed during the exam.

For a more comprehensive list of study strategies, please visit *college.cengage.com/pic/larsonCAA8e*.

## Get more value from your textbook!

## Supplements for the Instructor

### Digital Instructor's Solution Manual

Found on the instructor website, this manual contains the complete, worked-out solutions for all the exercises in the text.

## Supplements for the Student

### Student Solutions Guide

This guide contains complete solutions to all odd-numbered exercises in the text.

### Excel Made Easy CD

This CD uses easy-to-follow videos to help students master mathematical concepts introduced in class. Electronic spreadsheets and detailed tutorials are included.

### Instructor and Student Websites

The Instructor and Student websites at *college.cengage.com/pic/larsonCAA8e* contain an abundance of resources for teaching and learning, such as Note Taking Guides, a Graphing Calculator Guide, Digital Lessons, ACE Practice Tests, and a graphing calculator simulator.

### Instruction DVDs

Hosted by Dana Mosely and captioned for the hearing-impaired, these DVDs cover all sections in the text. Ideal for promoting individual study and review, these comprehensive DVDs also support students in online courses or those who have missed a lecture.

### The Online Study Center

The Online Study Center encompasses the interactive online products and services integrated with Cengage Learning mathematics programs. Students and instructors can access Online Study Center content through text-specific Student and Instructor websites and via online learning platforms including WebAssign as well as Blackboard®, WebCT®, and other course management systems.

### Diploma Testing

Diploma Testing provides instructors with a wide array of algorithmic items along with improved functionality and ease of use. Diploma Testing offers all the tools needed to create, deliver, and customize multiple types of tests—including authoring and editing algorithmic questions. In addition to producing an unlimited number of tests for each chapter, including cumulative tests and final exams, Diploma Testing also offers instructors the ability to deliver tests online, or by paper and pencil.

### Online Course Content for Blackboard®, WebCT®, and eCollege®

Deliver program or text-specific Cengage Learning content online using your institution's local course management system. Cengage Learning offers homework, tutorials, videos, and other resources formatted for Blackboard®, WebCT®, eCollege®, and other course management systems. Add to an existing online course or create a new one by selecting from a wide range of powerful learning and instructional materials.

# Acknowledgments

# How to get the most out of your textbook . . .

## CHAPTER OPENERS

Each opener has an applied example of a core topic from the chapter. The section outline provides a comprehensive overview of the material being presented.

**2**  **Differentiation**

Higher-order derivatives are used to determine the acceleration function of a sports car. The acceleration function shows the changes in the car's velocity. As the car reaches its "cruising" speed, is the acceleration increasing or decreasing? (See Section 2.6, Exercise 45.)

2.1   The Derivative and the Slope of a Graph
2.2   Some Rules for Differentiation
2.3   Rates of Change: Velocity and Marginals
2.4   The Product and Quotient Rules
2.5   The Chain Rule
2.6   Higher-Order Derivatives
2.7   Implicit Differentiation
2.8   Related Rates

### Applications

Differentiation has many real-life applications. The applications listed below represent a sample of the applications in this chapter.

■ Sales, Exercise 61, page 137
■ Political Fundraiser, Exercise 63, page 137
■ Make a Decision: Inventory Replenishment, Exercise 65, page 163
■ Modeling Data, Exercise 51, page 180
■ Health: U.S. HIV/AIDS Epidemic, Exercise 47, page 187

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## Section 2.1

### The Derivative and the Slope of a Graph

■ Identify tangent lines to a graph at a point.
■ Approximate the slopes of tangent lines to graphs at points.
■ Use the limit definition to find the slopes of graphs at points.
■ Use the limit definition to find the derivatives of functions.
■ Describe the relationship between differentiability and continuity.

## SECTION OBJECTIVES

A bulleted list of learning objectives allows you the opportunity to preview what will be presented in the upcoming section.

### Tangent Line to a Graph

**FIGURE 2.1** The slope of a nonlinear graph changes from one point to another.

Calculus is a branch of mathematics that studies rates of change of functions. In this course, you will learn that rates of change have many applications in real life. In Section 1.3, you learned how the slope of a line indicates the rate at which the line rises or falls. For a line, this rate (or slope) is the same at every point on the line. For graphs other than lines, the rate at which the graph rises or falls changes from point to point. For instance, in Figure 2.1, the parabola is rising more quickly at the point $(x_1, y_1)$ than it is at the point $(x_2, y_2)$. At the vertex $(x_3, y_3)$, the graph levels off, and at the point $(x_4, y_4)$, the graph is falling.

To determine the rate at which a graph rises or falls at a *single point*, you can find the slope of the **tangent line** at the point. In simple terms, the tangent line to the graph of a function $f$ at a point $P(x_1, y_1)$ is the line that best approximates the graph at that point, as shown in Figure 2.1. Figure 2.2 shows other examples of tangent lines.

## NEW!  MAKE A DECISION

Multi-step exercises reinforce your problem-solving skills and mastery of concepts, as well as taking a real-life application further by testing what you know about a given problem to make a decision within the context of the problem.

**45. *MAKE A DECISION: FUEL COST***   A car is driven 15,000 miles a year and gets $x$ miles per gallon. Assume that the average fuel cost is $2.95 per gallon. Find the annual cost of fuel $C$ as a function of $x$ and use this function to complete the table.

| $x$ | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|
| $C$ | | | | | | | |
| $dC/dx$ | | | | | | | |

1 mile per gallon increase
o gets 15 miles per gallon
per gallon? Explain.

**61. *MAKE A DECISION: NEGOTIATING A PRICE***   You decide to form a partnership with another business. Your business determines that the demand $x$ for your product is inversely proportional to the square of the price for $x \geq 5$.

(a) The price is $1000 and the demand is 16 units. Find the demand function.

(b) Your partner determines that the product costs $250 per unit and the fixed cost is $10,000. Find the cost function.

(c) Find the profit function and use a graphing utility to graph it. From the graph, what price would you negotiate with your partner for this product? Explain your reasoning.

### CONCEPT CHECK

1. What is the name of the line that best approximates the slope of a graph at a point?

2. What is the name of a line through the point of tangency and a second point on the graph?

3. Sketch a graph of a function whose derivative is always negative.

4. Sketch a graph of a function whose derivative is always positive.

## NEW!  CONCEPT CHECK

These non-computational questions appear at the end of each section and are designed to check your understanding of the concepts covered in that section.

## DEFINITIONS AND THEOREMS

All definitions and theorems are highlighted for emphasis and easy recognition.

**The Sum and Difference Rules**

The derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$
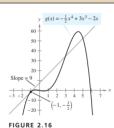
$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

**Definition of Average Rate of Change**

If $y = f(x)$, then the **average rate of change** of $y$ with respect to $x$ on the interval $[a, b]$ is

$$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\Delta y}{\Delta x}.$$

Note that $f(a)$ is the value of the function at the *left* endpoint of the interval, $f(b)$ is the value of the function at the *right* endpoint of the interval, and $b - a$ is the width of the interval, as shown in Figure 2.18.



**FIGURE 2.16**

*Example 9*   **Using the Sum and Difference Rules**

Find an equation of the tangent line to the graph of

$$g(x) = -\frac{1}{2}x^4 + 3x^3 - 2x$$

at the point $\left(-1, -\frac{3}{2}\right)$.

**SOLUTION**   The derivative of $g(x)$ is $g'(x) = -2x^3 + 9x^2 - 2$, which implies that the slope of the graph at the point $\left(-1, -\frac{3}{2}\right)$ is

$$\text{Slope} = g'(-1) = -2(-1)^3 + 9(-1)^2 - 2$$
$$= 2 + 9 - 2$$
$$= 9$$

as shown in Figure 2.16. Using the point-slope form, you can write the equation of the tangent line at $\left(-1, -\frac{3}{2}\right)$ as shown.

## EXAMPLES

There are a wide variety of relevant examples in the text, each titled for easy reference. Many of the solutions are presented graphically, analytically, and/or numerically to provide further insight into mathematical concepts. Examples using a real-life situation are identified with the ⓡ symbol.

## CHECKPOINT

After each example, a similar problem is presented to allow for immediate practice, and to further reinforce your understanding of the concepts just learned.

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### Application

*Example 10*   **Modeling Revenue**   ⓡ

From 2000 through 2005, the revenue $R$ (in millions of dollars per year) for Microsoft Corporation can be modeled by

$$R = -110.194t^3 + 993.98t^2 + 1155.6t + 23,036, \quad 0 \le t \le 5$$

where $t$ represents the year, with $t = 0$ corresponding to 2000. At what rate was Microsoft's revenue changing in 2001?   *(Source: Microsoft Corporation)*

**SOLUTION**   One way to answer this question is to find the derivative of the revenue model with respect to time.

$$\frac{dR}{dt} = -330.582t^2 + 1987.96t + 1155.6, \quad 0 \le t \le 5$$

In 2001 (when $t = 1$), the rate of change of the revenue with respect to time is given by

$$-330.582(1)^2 + 1987.96(1) + 1155.6 \approx 2813.$$

Because $R$ is measured in millions of dollars and $t$ is measured in years, it follows that the derivative $dR/dt$ is measured in millions of dollars per year. So, at the end of 2001, Microsoft's revenue was increasing at a rate of about $2813 million per year, as shown in Figure 2.17.



**FIGURE 2.17**

**✓ CHECKPOINT 10**

From 1998 through 2005, the revenue per share $R$ (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44, \quad 8 \le t \le 15$$

where $t$ represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?   *(Source: McDonald's Corporation)* ∎

## DISCOVERY

These projects appear before selected topics and allow you to explore concepts on your own. These boxed features are optional, so they can be omitted with no loss of continuity in the coverage of material.

### DISCOVERY

Use a graphing utility to graph $f(x) = 2x^3 - 4x^2 + 3x - 5$. On the same screen, sketch the graphs of $y = x - 5$, $y = 2x - 5$, and $y = 3x - 5$. Which of these lines, if any, appears to be tangent to the graph of $f$ at the point $(0, -5)$? Explain your reasoning.

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### TECHNOLOGY

**Modeling a Demand Function**

To model a demand function, you need data that indicate how many units of a product will sell at a given price. As you might imagine, such data are not easy to obtain for a new product. After a product has been on the market awhile, however, its sales history can provide the necessary data.

As an example, consider the two bar graphs shown below. From these graphs, you can see that from 2001 through 2005, the number of prerecorded DVDs sold increased from about 300 million to about 1100 million. During that time, the price per unit dropped from an average price of about $18 to an average price of about $15. *(Source: Kagan Research, LLC)*

The information in the two bar graphs is combined in the table, where $x$ represents the units sold (in millions) and $p$ represents the price (in dollars).

| $t$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $x$ | 215 | 505 | 130 | 66 | 1024 |
| $p$ | 1840 | 111 | 1583 | 1551 | 144 |

By entering the ordered pairs $(x, p)$ into a graphing utility, you can find that the power model for the demand for prerecorded DVDs is:
$p = 44.55x^{-0.155}$, $291.5 \le x \le 1072.4$. A graph of this demand function and its data points is shown below

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### TECHNOLOGY BOXES

These boxes appear throughout the text and provide guidance on using technology to ease lengthy calculations, present a graphical solution, or discuss where using technology can lead to misleading or wrong solutions.

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**SECTION 2.1**    The Derivative and the Slope of a Graph    121

In many applications, it is convenient to use a variable other than $x$ as the independent variable. Example 7 shows a function that uses $t$ as the independent variable.

### TECHNOLOGY

You can use a graphing utility to confirm the result given in Example 7. One way to do this is to choose a point on the graph of $y = 2/t$, such as $(1, 2)$, and find the equation of the tangent line at that point. Using the derivative found in the example, you know that the slope of the tangent line when $t = 1$ is $m = -2$. This means that the tangent line at the point $(1, 2)$ is

$y - y_1 = m(t - t_1)$

$y - 2 = -2(t - 1)$ or

$y = -2t + 4.$

By graphing $y = 2/t$ and $y = -2t + 4$ in the same viewing window, as shown below, you can confirm that the line is tangent to the graph at the point $(1, 2)$.*

**Example 7**    Finding a Derivative

Find the derivative of $y$ with respect to $t$ for the function

$y = \dfrac{2}{t}.$

**SOLUTION**    Consider $y = f(t)$, and use the limit process as shown.

$\dfrac{dy}{dt} = \lim_{\Delta t \to 0} \dfrac{f(t + \Delta t) - f(t)}{\Delta t}$    Set up difference quotient.

$= \lim_{\Delta t \to 0} \dfrac{\dfrac{2}{t + \Delta t} - \dfrac{2}{t}}{\Delta t}$    Use $f(t) = 2/t$.

$= \lim_{\Delta t \to 0} \dfrac{\dfrac{2t - 2t - 2\Delta t}{t(t + \Delta t)}}{\Delta t}$    Expand terms.

$= \lim_{\Delta t \to 0} \dfrac{-2\Delta t}{t(\Delta t)(t + \Delta t)}$    Factor and divide out.

$= \lim_{\Delta t \to 0} \dfrac{-2}{t(t + \Delta t)}$    Simplify.

$= -\dfrac{2}{t^2}$    Evaluate the limit.

So, the derivative of $y$ with respect to $t$ is

$\dfrac{dy}{dt} = -\dfrac{2}{t^2}.$

Remember that the derivative of a function gives you a formula for finding the slope of the tangent line at any point on the graph of the function. For example, the slope of the tangent line to the graph of $f$ at the point $(1, 2)$ is given by

$f'(1) = -\dfrac{2}{1^2} = -2.$

To find the slopes of the graph at other points, substitute the $t$-coordinate of the point into the derivative, as shown below.

| Point | $t$-Coordinate | Slope |
|---|---|---|
| $(2, 1)$ | $t = 2$ | $m = f'(2) = -\dfrac{2}{2^2} = -\dfrac{1}{2}$ |
| $(-2, -1)$ | $t = -2$ | $m = f'(-2) = -\dfrac{2}{(-2)^2} = -\dfrac{1}{2}$ |

✓**CHECKPOINT 7**

Find the derivative of $y$ with respect to $t$ for the function $y = 4/t$. ∎

*Specific calculator keystroke instructions for operations in this and other technology boxes can be found at college.cengage.com/info/larsonapplied.

## TECHNOLOGY EXERCISES

Many exercises in the text can be solved with or without technology. The ⓉR symbol identifies exercises for which students are specifically instructed to use a graphing calculator or a computer algebra system to solve the problem. Additionally, the Ⓢ symbol denotes exercises best solved by using a spreadsheet.

Ⓣ **78. Credit Card Rate**    The average annual rate $r$ (in percent form) for commercial bank credit cards from 2000 through 2005 can be modeled by

$$r = \sqrt{-1.7409t^4 + 18.070t^3 - 52.68t^2 + 10.9t + 249}$$

where $t$ represents the year, with $t = 0$ corresponding to 2000.    *(Source: Federal Reserve Bulletin)*

(a) Find the derivative of this model. Which differentiation

(b) Ⓣ **Graphical, Numerical, and Analytic Analysis**    In Exercises 63–66, use a graphing utility to graph $f$ on the interval $[-2, 2]$. Complete the table by graphically estimating the slopes of the graph at the given points. Then evaluate the slopes analytically and compare your results with those obtained graphically.

(c)

(d)

| $x$ | $-2$ | $-\frac{3}{2}$ | $-1$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ | $\frac{3}{2}$ | $2$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | | | | |
| $f'(x)$ | | | | | | | | | |

**63.** $f(x) = \frac{1}{4}x^3$          **64.** $f(x) = \frac{1}{2}x^2$

Ⓢ **57. Income Distribution**    Using the Lorenz curve in Exercise 56 and a spreadsheet, complete the table, which lists the percent of total income earned by each quintile in the United States in 2005.

| Quintile | Lowest | 2nd | 3rd | 4th | Highest |
|---|---|---|---|---|---|
| Percent | | | | | |

### BUSINESS CAPSULES

Business Capsules appear at the ends of numerous sections. These capsules and their accompanying exercises deal with business situations that are related to the mathematical concepts covered in the chapter.

**Business Capsule**



AP/Wide World Photos

In 1978 Ben Cohen and Jerry Greenfield used their combined life savings of $8000 to convert an abandoned gas station in Burlington, Vermont into their first ice cream shop. Today, Ben & Jerry's Homemade Holdings, Inc. has over 600 scoop shops in 16 countries. The company's three-part mission statement emphasizes product quality, economic reward, and a commitment to the community. Ben & Jerry's contributes a minimum of $1.1 million annually through corporate philanthropy that is primarily employee led.

**73. Research Project**    Use your school's library, the Internet, or some other reference source to find information on a company that is noted for its philanthropy and community commitment. (One such business is described above.) Write a short paper about the company.

## ALGEBRA REVIEWS

These appear throughout each chapter and offer algebraic support at point of use. Many of the reviews are then revisited in the Algebra Review at the end of the chapter, where additional details of examples with solutions and explanations are provided.

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### Algebra Review

**Simplifying Algebraic Expressions**

To be successful in using derivatives, you must be good at simplifying algebraic expressions. Here are some helpful simplification techniques.

1. Combine *like terms*. This may involve expanding an expression by multiplying factors.
2. Divide out *like factors* in the numerator and denominator of an expression.
3. Factor an expression.
4. Rationalize a denominator.
5. Add, subtract, multiply, or divide fractions.

**TECHNOLOGY**

Ⓣ Symbolic algebra systems can simplify algebraic expressions. If you have access to such a system, try using it to simplify the expressions in this Algebra Review.

*Example 1*  **Simplifying a Fractional Expression**

**a.** $\dfrac{(x + \Delta x)^2 - x^2}{\Delta x} = \dfrac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x}$    Expand expression.

$= \dfrac{2x(\Delta x) + (\Delta x)^2}{\Delta x}$    Combine like terms.

$= \dfrac{\Delta x(2x + \Delta x)}{\Delta x}$    Factor.

$= 2x + \Delta x, \qquad \Delta x \neq 0$    Divide out like factors.

**b.** $\dfrac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2)}{(x^2 - 1)^2}$

$= \dfrac{(-2x^2 - 2x^3 + 2 + 2x) - (6 - 4x - 2x^2)}{(x^2 - 1)^2}$    Expand expression.

$= \dfrac{-2x^2 - 2x^3 + 2 + 2x - 6 + 4x + 2x^2}{(x^2 - 1)^2}$    Remove parentheses.

$= \dfrac{-2x^3 + 6x - 4}{(x^2 - 1)^2}$    Combine like terms.

**c.** $2\left(\dfrac{2x + 1}{3x}\right)\left[\dfrac{3x(2) - (2x + 1)(3)}{(3x)^2}\right]$

$= 2\left(\dfrac{2x + 1}{3x}\right)\left[\dfrac{6x - (6x + 3)}{(3x)^2}\right]$    Multiply factors.

$= \dfrac{2(2x + 1)(6x - 6x - 3)}{(3x)^3}$    Multiply fractions and remove parentheses.

$= \dfrac{2(2x + 1)(-3)}{3(9)x^3}$    Combine like terms and factor.

$= \dfrac{-2(2x + 1)}{9x^3}$    Divide out like factors.

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### Algebra Review

For help in evaluating the expressions in Examples 3–6, see the review of simplifying fractional expressions on page 196.

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### STUDY TIP

When differentiating functions involving radicals, you should rewrite the function with rational exponents. For instance, you should rewrite $y = \sqrt[3]{x}$ as $y = x^{1/3}$, and you should rewrite

$y = \dfrac{1}{\sqrt[3]{x^4}}$  as  $y = x^{-4/3}$.

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## STUDY TIPS

Scattered throughout the text, study tips address special cases, expand on concepts, and help you to avoid common errors.

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### STUDY TIP

In real-life problems, it is important to list the units of measure for a rate of change. The units for $\Delta y/\Delta x$ are "$y$-units" per "$x$-units." For example, if $y$ is measured in miles and $x$ is measured in hours, then $\Delta y/\Delta x$ is measured in *miles per hour*.

## SKILLS REVIEW

These exercises at the beginning of each exercise set help students review skills covered in previous sections. The answers are provided at the back of the text to reinforce understanding of the skill sets learned.

## EXERCISE SETS

These exercises offer opportunities for practice and review. They progress in difficulty from skill-development problems to more challenging problems, to build confidence and understanding.

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**SECTION 2.3** Rates of Change: Velocity and Marginals    149

**Skills Review 2.3**  The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.1 and 2.2.

In Exercises 1 and 2, evaluate the expression.

1. $\dfrac{-63 - (-105)}{21 - 7}$

2. $\dfrac{-37 - 54}{16 - 3}$

In Exercises 3–10, find the derivative of the function.

3. $y = 4x^2 - 2x + 7$

4. $y = -3t^3 + 2t^2 - 8$

5. $s = -16t^2 + 24t + 30$

6. $y = -16x^2 + 54x + 70$

7. $A = \frac{1}{10}(-2t^3 + 3t^2 + 5t)$

8. $y = \frac{1}{9}(6x^3 - 18x^2 + 63x - 15)$

9. $y = 12x - \dfrac{x^2}{5000}$

10. $y = 138 + 74x - \dfrac{x^3}{10{,}000}$

**Exercises 2.3**    See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. **Research and Development**  The table shows the amounts $A$ (in billions of dollars per year) spent on R&D in the United States from 1980 through 2004, where $t$ is the year, with $t = 0$ corresponding to 1980. Approximate the average rate of change of $A$ during each period. *(Source: U.S. National Science Foundation)*

(a) 1980–1985   (b) 1985–1990   (c) 1990–1995
(d) 1995–2000   (e) 1980–2004   (f) 1990–2004

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $A$ | 63 | 72 | 81 | 90 | 102 | 115 | 120 |

(c) Imports: 1990–2000   (d) Exports: 1990–2000
(e) Imports: 1980–2005   (f) Exports: 1980–2005

**Trade Deficit**

Figure for 2

Ⓣ In Exercises 3–12, use a graphing utility to graph the function and find its average rate of change on the interval. Compare this rate with the instantaneous rates of change at the endpoints of the interval.

3. $f(t) = 3t + 5; [1, 2]$    4. $h(x) = 2 - x; [0, 2]$
5. $h(x) = x^2 - 4x + 2; [-2, 2]$

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13. **Consumer Trends**  The graph shows the number of visitors $V$ to a national park in hundreds of thousands during a one-year period, where $t = 1$ represents January.

**Visitors to a National Park**

(a) Estimate the rate of change of $V$ over the interval [9, 12] and explain your results.

(b) Over what interval is the average rate of change approximately equal to the rate of change at $t = 8$? Explain your reasoning.

14. **Medicine**  The graph shows the estimated number of milligrams of a pain medication $M$ in the bloodstream $t$ hours after a 1000-milligram dose of the drug has been given.

**Pain Medication in Bloodstream**

(a) Estimate the one-hour interval over which the average rate of change is the greatest.

(b) Over what interval is the average rate of change approximately equal to the rate of change at $t = 4$? Explain your reasoning.

15. **Medicine**  The effectiveness $E$ (on a scale from 0 to 1) of a pain-killing drug $t$ hours after entering the bloodstream is given by

$$E = \frac{1}{27}(9t + 3t^2 - t^3), \quad 0 \le t \le 4.5.$$

Find the average rate of change of $E$ on each indicated interval and compare this rate with the instantaneous rates of change at the endpoints of the interval.

(a) [0, 1]    (b) [1, 2]    (c) [2, 3]    (d) [3, 4]

16. **Chemistry: Wind Chill**  At 0° Celsius, the heat loss $H$ (in kilocalories per square meter per hour) from a person's body can be modeled by

$$H = 33(10\sqrt{v} - v + 10.45)$$

where $v$ is the wind speed (in meters per second).

(a) Find $\dfrac{dH}{dv}$ and interpret its meaning in this situation.

(b) Find the rates of change of $H$ when $v = 2$ and when $v = 5$.

17. **Velocity**  The height $s$ (in feet) at time $t$ (in seconds) silver dollar dropped from the top of the V Monument is given by

$$s = -16t^2 + 555.$$

(a) Find the average velocity on the interval [2, $t = 3$.

(b) Find the instantaneous velocities when $t = $

(c) How long will it take the dollar to hit the gr

(d) Find the velocity of the dollar when it hits t

Ⓑ 18. **Physics: Velocity**  A racecar travels north straight, level track at a constant speed, trave kilometer in 20.0 seconds. The return trip ove track is made in 25.0 seconds.

(a) What is the average velocity of the car in second for the first leg of the run?

(b) What is the average velocity for the total tr

*(Source: Shipman/Wilson/Todd, An Introductio cal Science, Eleventh Edition)*

**Marginal Cost**  In Exercises 19–22, find the cost for producing $x$ units. (The cost is me dollars.)

19. $C = 4500 + 1.47x$    20. $C = 205{,}000$
21. $C = 55{,}000 + 470x - 0.25x^2, \quad 0 \le x \le 940$
22. $C = 100(9 + 3\sqrt{x})$

**Marginal Revenue**  In Exercises 23–26, marginal revenue for producing $x$ units. (The is measured in dollars.)

23. $R = 50x - 0.5x^2$    24. $R = 30x - $
25. $R = -6x^3 + 8x^2 + 200x$    26. $R = 50(20x$

**Marginal Profit**  In Exercises 27–30, find the profit for producing $x$ units. (The profit is me dollars.)

27. $P = -2x^2 + 72x - 145$
28. $P = -0.25x^2 + 2000x - 1{,}250{,}000$
29. $P = -0.00025x^2 + 12.2x - 25{,}000$
30. $P = -0.5x^3 + 30x^2 - 164.25x - 1000$

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40. **Marginal Cost**  The cost $C$ of producing $x$ units is modeled by $C = v(x) + k$, where $v$ represents the variable cost and $k$ represents the fixed cost. Show that the marginal cost is independent of the fixed cost.

41. **Marginal Profit**  When the admission price for a baseball game was $6 per ticket, 36,000 tickets were sold. When the price was raised to $7, only 33,000 tickets were sold. Assume that the demand function is linear and that the variable and fixed costs for the ballpark owners are $0.20 and $85,000, respectively.

(a) Find the profit $P$ as a function of $x$, the number of tickets sold.

Ⓣ (b) Use a graphing utility to graph $P$, and comment about the slopes of $P$ when $x = 18{,}000$ and when $x = 36{,}000$.

(c) Find the marginal profits when 18,000 tickets are sold and when 36,000 tickets are sold.

42. **Marginal Profit**  In Exercise 41, suppose ticket sales decreased to 30,000 when the price increased to $7. How would this change the answers?

43. **Profit**  The demand function for a product is given by $p = 50/\sqrt{x}$ for $1 \le x \le 8000$, and the cost function is given by $C = 0.5x + 500$ for $0 \le x \le 8000$.

Find the marginal profits for (a) $x = 900$, (b) $x = 1600$, (c) $x = 2500$, and (d) $x = 3600$.

If you were in charge of setting the price for this product, what price would you set? Explain your reasoning.

44. **Inventory Management**  The annual inventory cost for a manufacturer is given by

$$C = 1{,}008{,}000/Q + 6.3Q$$

where $Q$ is the order size when the inventory is replenished. Find the change in annual cost when $Q$ is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.

45. **MAKE A DECISION: FUEL COST**  A car is driven 15,000 miles a year and gets $x$ miles per gallon. Assume that the average fuel cost is $2.95 per gallon. Find the annual cost of fuel $C$ as a function of $x$ and use this function to complete the table.

| $x$ | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|
| $C$ | | | | | | | |
| $dC/dx$ | | | | | | | |

Who would benefit more from a 1 mile per gallon increase in fuel efficiency—the driver who gets 15 miles per gallon or the driver who gets 35 miles per gallon? Explain.

46. **Gasoline Sales**  The number $N$ of gallons of regular unleaded gasoline sold by a gasoline station at a price of $p$ dollars per gallon is given by $N = f(p)$.

(a) Describe the meaning of $f'(2.959)$.

(b) Is $f'(2.959)$ usually positive or negative? Explain.

47. **Dow Jones Industrial Average**  The table shows the year-end closing prices $p$ of the Dow Jones Industrial Average (DJIA) from 1992 through 2006, where $t$ is the year, and $t = 2$ corresponds to 1992. *(Source: Dow Jones Industrial Average)*

| $t$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $p$ | 3301.11 | 3754.09 | 3834.44 | 5117.12 | 6448.26 |

| $t$ | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|
| $p$ | 7908.24 | 9181.43 | 11,497.12 | 10,786.85 | 10,021.50 |

| $t$ | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|
| $p$ | 8341.63 | 10,453.92 | 10,783.01 | 10,717.50 | 12,463.15 |

(a) Determine the average rate of change in the value of the DJIA from 1992 to 2006.

(b) Estimate the instantaneous rate of change in 1998 by finding the average rate of change from 1996 to 2000.

(c) Estimate the instantaneous rate of change in 1998 by finding the average rate of change from 1997 to 1999.

(d) Compare your answers for parts (b) and (c). Which interval do you think produced the best estimate for the instantaneous rate of change in 1998?

Ⓑ 48. **Biology**  Many populations in nature exhibit logistic growth, which consists of four phases, as shown in the figure. Describe the rate of growth of the population in each phase, and give possible reasons as to why the rates might be changing from phase to phase. *(Source: Adapted from Levine/Miller, Biology: Discovering Life, Second Edition)*

## MID-CHAPTER QUIZ

Appearing in the middle of each chapter, this one page test allows you to practice skills and concepts learned in the chapter. This opportunity for self-assessment will uncover any potential weak areas that might require further review of the material.

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### Mid-Chapter Quiz

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, use the limit definition to find the derivative of the function. Then find the slope of the tangent line to the graph of $f$ at the given point.

**1.** $f(x) = -x + 2;\ (2, 0)$     **2.** $f(x) = \sqrt{x + 3};\ (1, 2)$     **3.** $f(x) = \frac{4}{x};\ (1, 4)$

In Exercises 4–12, find the derivative of the function.

**4.** $f(x) = 12$     **5.** $f(x) = 19x + 9$     **6.** $f(x) = 5 - 3x^2$

**7.** $f(x) = 12x^{1/4}$     **8.** $f(x) = 4x^{-2}$     **9.** $f(x) = 2\sqrt{x}$

**10.** $f(x) = \frac{2x + 3}{3x + 2}$     **11.** $f(x) = (x^2 + 1)(-2x + 4)$     **12.** $f(x) = \frac{4 - x}{x + 5}$

**T** In Exercises 13–16, use a graphing utility to graph the function and find its average rate of change on the interval. Compare this rate with the instantaneous rates of change at the endpoints of the interval.

**13.** $f(x) = x^2 - 3x + 1;\ [0, 3]$

**14.** $f(x) = 2x^3 + x^2 - x + 4;\ [-1, 1]$

**15.** $f(x) = \frac{1}{2x};\ [2, 5]$

**16.** $f(x) = \sqrt[3]{x};\ [8, 27]$

**17.** The profit (in dollars) from selling $x$ units of a product is given by

$$P = -0.0125x^2 + 16x - 600$$

(a) Find the additional profit when the sales increase from 175 to 176 units.

(b) Find the marginal profit when $x = 175$.

(c) Compare the results of parts (a) and (b).

**T** In Exercises 18 and 19, find an equation of the tangent line to the graph of $f$ at the given point. Then use a graphing utility to graph the function and the equation of the tangent line in the same viewing window.

**18.** $f(x) = 5x^2 + 6x - 1;\ (-1, -2)$

**19.** $f(x) = (x - 1)(x + 1);\ (0, -1)$

**20.** From 2000 through 2005, the sales per share $S$ (in dollars) for CVS Corporation can be modeled by

$$S = 0.18390t^3 - 0.8242t^2 + 3.492t + 25.60,\ 0 \le t \le 5$$

where $t$ represents the year, with $t = 0$ corresponding to 2000. *(Source: CVS Corporation)*

(a) Find the rate of change of the sales per share with respect to the year.

(b) At what rate were the sales per share changing in 2001? in 2004? in 2005?

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### Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, use the limit definition to find the derivative of the function. Then find the slope of the tangent line to the graph of $f$ at the given point.

**1.** $f(x) = x^2 + 1;\ (2, 5)$     **2.** $f(x) = \sqrt{x} - 2;\ (4, 0)$

In Exercises 3–11, find the derivative of the function. Simplify your result.

**3.** $f(t) = t^3 + 2t$     **4.** $f(x) = 4x^2 - 8x + 1$     **5.** $f(x) = x^{3/2}$

**6.** $f(x) = (x + 3)(x - 3)$     **7.** $f(x) = -3x^{-3}$     **8.** $f(x) = \sqrt{x}(5 + x)$

**9.** $f(x) = (3x^2 + 4)^2$     **10.** $f(x) = \sqrt{1 - 2x}$     **11.** $f(x) = \frac{(5x - 1)^3}{x}$

**T 12.** Find an equation of the tangent line to the graph of $f(x) = x - \frac{1}{x}$ at the point $(1, 0)$. Then use a graphing utility to graph the function and the tangent line in the same viewing window.

**13.** The annual sales $S$ (in millions of dollars per year) of Bausch & Lomb for the years 1999 through 2005 can be modeled by

$$S = -2.9667t^3 + 135.008t^2 - 1824.42t + 9426.3,\ 9 \le t \le 15$$

where $t$ represents the year, with $t = 9$ corresponding to 1999. *(Source: Bausch & Lomb, Inc.)*

(a) Find the average rate of change for the interval from 2001 through 2005.

(b) Find the instantaneous rates of change of the model for 2001 and 2005.

(c) Interpret the results of parts (a) and (b) in the context of the problem.

**14.** The monthly demand and cost functions for a product are given by

$$p = 1700 - 0.016x \quad \text{and} \quad C = 715,000 + 240x.$$

Write the profit function for this product.

In Exercises 15–17, find the third derivative of the function. Simplify your result.

**15.** $f(x) = 2x^2 + 3x + 1$     **16.** $f(x) = \sqrt{3 - x}$     **17.** $f(x) = \frac{2x + 1}{2x - 1}$

In Exercises 18–20, use implicit differentiation to find $dy/dx$.

**18.** $x + xy = 6$     **19.** $y^2 + 2x - 2y + 1 = 0$     **20.** $x^2 - 2y^2 = 4$

**21.** The radius $r$ of a right circular cylinder is increasing at a rate of 0.25 centimeter per minute. The height $h$ of the cylinder is related to the radius by $h = 20r$. Find the rate of change of the volume when (a) $r = 0.5$ centimeter and (b) $r = 1$ centimeter.

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## CHAPTER TEST

Appearing at the end of the chapter, this test is designed to simulate an in-class exam. Taking these tests will help you to determine what concepts require further study and review.

## CHAPTER SUMMARY AND STUDY STRATEGIES

The *Chapter Summary* reviews the skills covered in the chapter and correlates each skill to the Review Exercises that test the skill. Following each *Chapter Summary* is a short list of Study Strategies for addressing topics or situations in the chapter.

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### Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills. The exercise numbers are keyed to the Review Exercises that begin on page 200. Answers to odd-numbered Review Exercises are given in the back of the text.*

| Section 2.1 | Review Exercises |
|---|---|
| ■ Approximate the slope of the tangent line to a graph at a point. | 1–4 |
| ■ Interpret the slope of a graph in a real-life setting. | 5–8 |
| ■ Use the limit definition to find the derivative of a function and the slope of a graph at a point. $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$ | 9–16 |
| ■ Use the derivative to find the slope of a graph at a point. | 17–24 |
| ■ Use the graph of a function to recognize points at which the function is not differentiable. | 25–28 |

| Section 2.2 | |
|---|---|
| ■ Use the Constant Multiple Rule for differentiation. $$\frac{d}{dx}[cf(x)] = cf'(x)$$ | 29, 30 |
| ■ Use the Sum and Difference Rules for differentiation. $$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$ | 31–38 |

| Section 2.3 | |
|---|---|
| ■ Find the average rate of change of a function over an interval and the instantaneous rate of change at a point. $$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$ $$\text{Instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$ | 39, 40 |
| ■ Find the average and instantaneous rates of change of a quantity in a real-life problem. | 41–44 |
| ■ Find the velocity of an object that is moving in a straight line. | 45, 46 |
| ■ Create mathematical models for the revenue, cost, and profit for a product. $$P = R - C, \quad R = xp$$ | 47, 48 |
| ■ Find the marginal revenue, marginal cost, and marginal profit for a product. | 49–58 |

valuable study aids to help you master the material in this chapter. The *Student* ...ludes step-by-step solutions to all odd-numbered exercises to help you review ...dent website at *college.cengage.com/info/larsonapplied* offers algebra help and a ...gy Guide. The *Graphing Technology Guide* contains step-by-step commands ...a wide variety of graphing calculators, including the most recent models.

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## APPLICATION INDEX

This list, found on the front and back end sheets, is an index of all the applications presented in the text Examples and Exercises.

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### Applications

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# Calculus

## An Applied Approach

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# A Precalculus Review

The annual operating costs of each van owned by a utility company can be determined by solving an inequality. (See Section 0.1, Exercise 36.)

### Applications

Topics in precalculus have many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Sales, Exercise 35, page 7
- Quality Control, Exercise 51, page 12
- Production Level, Exercise 75, page 24
- Make a Decision: Inventory, Exercise 48, page 32

# The Real Number Line and Order

- Represent, classify, and order real numbers.
- Use inequalities to represent sets of real numbers.
- Solve inequalities.
- Use inequalities to model and solve real-life problems.

## The Real Number Line



| Negative direction | Positive direction |
| (x decreases) | (x increases) |
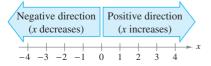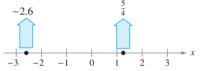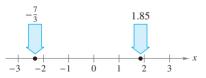
**FIGURE 0.1**    The Real Number Line

Real numbers can be represented with a coordinate system called the **real number line** (or *x*-axis), as shown in Figure 0.1. The **positive direction** (to the right) is denoted by an arrowhead and indicates the direction of increasing values of *x*. The real number corresponding to a particular point on the real number line is called the **coordinate** of the point. As shown in Figure 0.1, it is customary to label those points whose coordinates are integers.

The point on the real number line corresponding to zero is called the **origin.** Numbers to the right of the origin are **positive,** and numbers to the left of the origin are **negative.** The term **nonnegative** describes a number that is either positive or zero.

The importance of the real number line is that it provides you with a conceptually perfect picture of the real numbers. That is, each point on the real number line corresponds to one and only one real number, and each real number corresponds to one and only one point on the real number line. This type of relationship is called a **one-to-one correspondence** and is illustrated in Figure 0.2.



Every point on the real number line corresponds to one and only one real number.



Every real number corresponds to one and only one point on the real number line.

**FIGURE 0.2**

Each of the four points in Figure 0.2 corresponds to a real number that can be expressed as the ratio of two integers.

$$-2.6 = -\frac{13}{5} \qquad \frac{5}{4} \qquad -\frac{7}{3} \qquad 1.85 = \frac{37}{20}$$

Such numbers are called **rational.** Rational numbers have either terminating or infinitely repeating decimal representations.

*Terminating Decimals*

$$\frac{2}{5} = 0.4$$

$$\frac{7}{8} = 0.875$$

*Infinitely Repeating Decimals*

$$\frac{1}{3} = 0.333 \ldots = 0.\overline{3}*$$

$$\frac{12}{7} = 1.714285714285 \ldots = 1.\overline{714285}$$
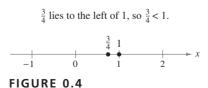
Real numbers that are not rational are called **irrational,** and they cannot be represented as the ratio of two integers (or as terminating or infinitely repeating decimals). So, a decimal approximation is used to represent an irrational number. Some irrational numbers occur so frequently in applications that mathematicians have invented special symbols to represent them. For example, the symbols $\sqrt{2}$, $\pi$, and $e$ represent irrational numbers whose decimal approximations are as shown. (See Figure 0.3.)



**FIGURE 0.3**

$$\sqrt{2} \approx 1.4142135623 \qquad \pi \approx 3.1415926535 \qquad e \approx 2.7182818284$$

---

*The bar indicates which digit or digits repeat infinitely.

## Order and Intervals on the Real Number Line

One important property of the real numbers is that they are **ordered:** 0 is less than 1, $-3$ is less than $-2.5$, $\pi$ is less than $\frac{22}{7}$, and so on. You can visualize this property on the real number line by observing that $a$ is less than $b$ if and only if $a$ lies to the left of $b$ on the real number line. Symbolically, "$a$ is less than $b$" is denoted by the inequality $a < b$. For example, the inequality $\frac{3}{4} < 1$ follows from the fact that $\frac{3}{4}$ lies to the left of 1 on the real number line, as shown in Figure 0.4.

$\frac{3}{4}$ lies to the left of 1, so $\frac{3}{4} < 1$.



**FIGURE 0.4**

When three real numbers $a$, $x$, and $b$ are ordered such that $a < x$ and $x < b$, we say that $x$ is **between** $a$ and $b$ and write

$a < x < b.$            *x is between a and b.*

The set of *all* real numbers between $a$ and $b$ is called the **open interval** between $a$ and $b$ and is denoted by $(a, b)$. An interval of the form $(a, b)$ does not contain the "endpoints" $a$ and $b$. Intervals that include their endpoints are called **closed** and are denoted by $[a, b]$. Intervals of the form $[a, b)$ and $(a, b]$ are neither open nor closed. Figure 0.5 shows the nine types of intervals on the real number line.
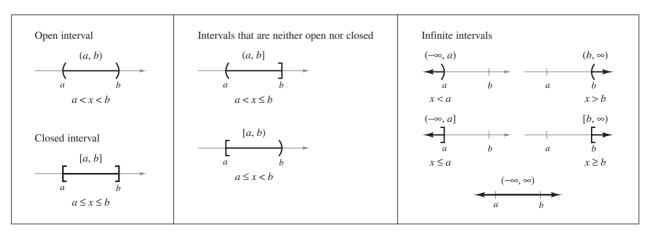


**FIGURE 0.5**    Intervals on the Real Number Line

**STUDY TIP**

Note that a square bracket is used to denote "less than or equal to" ($\leq$) or "greater than or equal to" ($\geq$). Furthermore, the symbols $\infty$ and $-\infty$ denote **positive** and **negative infinity.** These symbols do not denote real numbers; they merely let you describe unbounded conditions more concisely. For instance, the interval $[b, \infty)$ is unbounded to the right because it includes *all* real numbers that are greater than or equal to $b$.

## Solving Inequalities
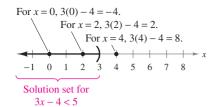
In calculus, you are frequently required to "solve inequalities" involving variable expressions such as $3x - 4 < 5$. The number $a$ is a **solution** of an inequality if the inequality is true when $a$ is substituted for $x$. The set of all values of $x$ that satisfy an equality is called the **solution set** of the inequality. The following properties are useful for solving inequalities. (Similar properties are obtained if < is replaced by ≤ and > is replaced by ≥.)

**Properties of Inequalities**

Let $a$, $b$, $c$, and $d$ be real numbers.

**1.** Transitive property: $a < b$ and $b < c$ $\Longrightarrow$ $a < c$

**2.** Adding inequalities: $a < b$ and $c < d$ $\Longrightarrow$ $a + c < b + d$

**3.** Multiplying by a (positive) constant: $a < b$ $\Longrightarrow$ $ac < bc$, $c > 0$

**4.** Multiplying by a (negative) constant: $a < b$ $\Longrightarrow$ $ac > bc$, $c < 0$

**5.** Adding a constant: $a < b$ $\Longrightarrow$ $a + c < b + c$

**6.** Subtracting a constant: $a < b$ $\Longrightarrow$ $a - c < b - c$

Note that you *reverse the inequality* when you multiply by a negative number. For example, if $x < 3$, then $-4x > -12$. This principle also applies to division by a negative number. So, if $-2x > 4$, then $x < -2$.

**Example 1**   **Solving an Inequality**

Find the solution set of the inequality $3x - 4 < 5$.

**SOLUTION**

| | |
|---|---|
| $3x - 4 < 5$ | Write original inequality. |
| $3x - 4 + 4 < 5 + 4$ | Add 4 to each side. |
| $3x < 9$ | Simplify. |
| $\dfrac{1}{3}(3x) < \dfrac{1}{3}(9)$ | Multiply each side by $\frac{1}{3}$. |
| $x < 3$ | Simplify. |

So, the solution set is the interval $(-\infty, 3)$, as shown in Figure 0.6.

For $x = 0$, $3(0) - 4 = -4$.
For $x = 2$, $3(2) - 4 = 2$.
For $x = 4$, $3(4) - 4 = 8$.

Solution set for
$3x - 4 < 5$

**FIGURE 0.6**

**✓ CHECKPOINT 1**

Find the solution set of the inequality $2x - 3 < 7$. ■

In Example 1, all five inequalities listed as steps in the solution have the same solution set, and they are called **equivalent inequalities.**

The inequality in Example 1 involves a first-degree polynomial. To solve inequalities involving polynomials of higher degree, you can use the fact that a polynomial can change signs *only* at its real zeros (the real numbers that make the polynomial zero). Between two consecutive real zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into **test intervals** in which the polynomial has no sign changes. That is, if a polynomial has the factored form

$$(x - r_1)(x - r_2), \ldots, (x - r_n), \qquad r_1 < r_2 < r_3 < \cdots < r_n$$

then the test intervals are

$$(-\infty, r_1), \quad (r_1, r_2), \quad \ldots, \quad (r_{n-1}, r_n), \quad \text{and} \quad (r_n, \infty).$$

For example, the polynomial

$$x^2 - x - 6 = (x - 3)(x + 2)$$

can change signs only at $x = -2$ and $x = 3$. To determine the sign of the polynomial in the intervals $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$, you need to test only *one value* from each interval.

Sign of $(x - 3)(x + 2)$

| $x$ | *Sign* | $< 0$? |
|-----|--------|--------|
| $-3$ | $(-)(-)$ | No |
| $-2$ | $(-)(0)$ | No |
| $-1$ | $(-)(+)$ | Yes |
| $0$ | $(-)(+)$ | Yes |
| $1$ | $(-)(+)$ | Yes |
| $2$ | $(-)(+)$ | Yes |
| $3$ | $(0)(+)$ | No |
| $4$ | $(+)(+)$ | No |



**FIGURE 0.7**   Is $(x - 3)(x + 2) < 0$?

## *Example 2*  Solving a Polynomial Inequality

Find the solution set of the inequality $x^2 < x + 6$.

**SOLUTION**

$$
\begin{aligned}
x^2 &< x + 6 & &\text{Write original inequality.}\\
x^2 - x - 6 &< 0 & &\text{Polynomial form}\\
(x - 3)(x + 2) &< 0 & &\text{Factor.}
\end{aligned}
$$

So, the polynomial $x^2 - x - 6$ has $x = -2$ and $x = 3$ as its zeros. You can solve the inequality by testing the sign of the polynomial in each of the following intervals.

$$x < -2, \quad -2 < x < 3, \quad x > 3$$

To test an interval, choose a representative number in the interval and compute the sign of each factor. For example, for any $x < -2$, both of the factors $(x - 3)$ and $(x + 2)$ are negative. Consequently, the product (of two negative numbers) is positive, and the inequality is *not* satisfied in the interval

$$x < -2.$$

A convenient testing format is shown in Figure 0.7. Because the inequality is satisfied only by the center test interval, you can conclude that the solution set is given by the interval

$$-2 < x < 3. \qquad\qquad \text{Solution set}$$

✓ **CHECKPOINT 2**

Find the solution set of the inequality $x^2 > 3x + 10$. ■

## Application

Inequalities are frequently used to describe conditions that occur in business and science. For instance, the inequality

$$144 \le W \le 180$$

describes the recommended weight $W$ for a man whose height is 5 feet 10 inches. Example 3 shows how an inequality can be used to describe the production levels in a manufacturing plant.

### *Example 3*    Production Levels    (R)
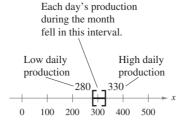
In addition to fixed overhead costs of $500 per day, the cost of producing $x$ units of an item is $2.50 per unit. During the month of August, the total cost of production varied from a high of $1325 to a low of $1200 per day. Find the high and low *production levels* during the month.

**SOLUTION**    Because it costs $2.50 to produce one unit, it costs $2.5x$ to produce $x$ units. Furthermore, because the fixed cost per day is $500, the total daily cost of producing $x$ units is $C = 2.5x + 500$. Now, because the cost ranged from $1200 to $1325, you can write the following.

| | | | |
|---|---|---|---|
| $1200 \le$ | $2.5x + 500$ | $\le 1325$ | Write original inequality. |
| $1200 - 500 \le$ | $2.5x + 500 - 500$ | $\le 1325 - 500$ | Subtract 500 from each part. |
| $700 \le$ | $2.5x$ | $\le 825$ | Simplify. |
| $\dfrac{700}{2.5} \le$ | $\dfrac{2.5x}{2.5}$ | $\le \dfrac{825}{2.5}$ | Divide each part by 2.5. |
| $280 \le$ | $x$ | $\le 330$ | Simplify. |

So, the daily production levels during the month of August varied from a low of 280 units to a high of 330 units, as shown in Figure 0.8.

Each day's production during the month fell in this interval.

Low daily production    High daily production

280    330

$$\xrightarrow{\quad\quad\quad} x$$

0    100    200    300    400    500

**FIGURE 0.8**

### ✓ CHECKPOINT 3

Use the information in Example 3 to find the high and low production levels if, during October, the total cost of production varied from a high of $1500 to a low of $1000 per day. ■

The symbol (R) indicates an example that uses or is derived from real-life data.

## Exercises 0.1

In Exercises 1–10, determine whether the real number is rational or irrational.

*1. $0.25$

2. $-3678$

3. $\dfrac{3\pi}{2}$

4. $3\sqrt{2} - 1$

5. $4.3\overline{451}$

6. $\dfrac{22}{7}$

7. $\sqrt[3]{64}$

8. $0.\overline{8177}$

9. $\sqrt[3]{60}$

10. $2e$

In Exercises 11–14, determine whether each given value of $x$ satisfies the inequality.

11. $5x - 12 > 0$

   (a) $x = 3$    (b) $x = -3$    (c) $x = \frac{5}{2}$

12. $x + 1 < \dfrac{x}{3}$

   (a) $x = 0$    (b) $x = 4$    (c) $x = -4$

13. $0 < \dfrac{x - 2}{4} < 2$

   (a) $x = 4$    (b) $x = 10$    (c) $x = 0$

14. $-1 < \dfrac{3 - x}{2} \le 1$

   (a) $x = 0$    (b) $x = 1$    (c) $x = 5$

In Exercises 15–28, solve the inequality and sketch the graph of the solution on the real number line.

15. $x - 5 \ge 7$

16. $2x > 3$

17. $4x + 1 < 2x$

18. $2x + 7 < 3$

19. $4 - 2x < 3x - 1$

20. $x - 4 \le 2x + 1$

21. $-4 < 2x - 3 < 4$

22. $0 \le x + 3 < 5$

23. $\dfrac{3}{4} > x + 1 > \dfrac{1}{4}$

24. $-1 < -\dfrac{x}{3} < 1$

25. $\dfrac{x}{2} + \dfrac{x}{3} > 5$

26. $\dfrac{x}{2} - \dfrac{x}{3} > 5$

27. $2x^2 - x < 6$

28. $2x^2 + 1 < 9x - 3$

In Exercises 29–32, use inequality notation to describe the subset of real numbers.

29. A company expects its earnings per share $E$ for the next quarter to be no less than $4.10 and no more than $4.25.

30. The estimated daily oil production $p$ at a refinery is greater than 2 million barrels but less than 2.4 million barrels.

31. According to a survey, the percent $p$ of Americans that now conduct most of their banking transactions online is no more than 40%.

32. The net income $I$ of a company is expected to be no less than $239 million.

33. **Physiology**  The maximum heart rate of a person in normal health is related to the person's age by the equation

$$r = 220 - A$$

where $r$ is the maximum heart rate in beats per minute and $A$ is the person's age in years. Some physiologists recommend that during physical activity a person should strive to increase his or her heart rate to at least 60% of the maximum heart rate for sedentary people and at most 90% of the maximum heart rate for highly fit people. Express as an interval the range of the target heart rate for a 20-year-old.

34. **Profit**  The revenue for selling $x$ units of a product is $R = 115.95x$, and the cost of producing $x$ units is $C = 95x + 750$. To obtain a profit, the revenue must be *greater than* the cost. For what values of $x$ will this product return a profit?

35. **Sales**  A doughnut shop at a shopping mall sells a dozen doughnuts for $4.50. Beyond the fixed cost (for rent, utilities, and insurance) of $220 per day, it costs $2.75 for enough materials (flour, sugar, etc.) and labor to produce each dozen doughnuts. If the daily profit *varies between* $60 and $270, between what levels (in dozens) do the daily sales vary?

36. **Annual Operating Costs**  A utility company has a fleet of vans. The annual operating cost $C$ (in dollars) of each van is estimated to be $C = 0.35m + 2500$, where $m$ is the number of miles driven. The company wants the annual operating cost of each van to be less than $13,000. To do this, $m$ must be less than what value?

In Exercises 37 and 38, determine whether each statement is true or false, given $a < b$.

37. (a) $-2a < -2b$

   (b) $a + 2 < b + 2$

   (c) $6a < 6b$

   (d) $\dfrac{1}{a} < \dfrac{1}{b}$

38. (a) $a - 4 < b - 4$

   (b) $4 - a < 4 - b$

   (c) $-3b < -3a$

   (d) $\dfrac{a}{4} < \dfrac{b}{4}$

# Absolute Value and Distance on the Real Number Line

- Find the absolute values of real numbers and understand the properties of absolute value.
- Find the distance between two numbers on the real number line.
- Define intervals on the real number line.
- Find the midpoint of an interval and use intervals to model and solve real-life problems.

## Absolute Value of a Real Number

**TECHNOLOGY**

(T) Absolute value expressions can be evaluated on a graphing utility. When an expression such as $|3 - 8|$ is evaluated, parentheses should surround the expression, as in abs$(3 - 8)$.

**Definition of Absolute Value**

The **absolute value** of a real number $a$ is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

At first glance, it may appear from this definition that the absolute value of a real number can be negative, but this is not possible. For example, let $a = -3$. Then, because $-3 < 0$, you have

$$|a| = |-3| = -(-3) = 3.$$

The following properties are useful for working with absolute values.

**Properties of Absolute Value**

1. Multiplication: $|ab| = |a||b|$

2. Division: $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

3. Power: $|a^n| = |a|^n$

4. Square root: $\sqrt{a^2} = |a|$

Be sure you understand the fourth property in this list. A common error in algebra is to imagine that by squaring a number and then taking the square root, you come back to the original number. But this is true only if the original number is nonnegative. For instance, if $a = 2$, then

$$\sqrt{2^2} = \sqrt{4} = 2$$

but if $a = -2$, then

$$\sqrt{(-2)^2} = \sqrt{4} = 2.$$
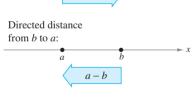
The reason for this is that (by definition) the square root symbol $\sqrt{\phantom{x}}$ denotes only the nonnegative root.

## Distance on the Real Number Line

Directed distance
from $a$ to $b$:



Directed distance
from $b$ to $a$:



Distance between
$a$ and $b$:



**FIGURE 0.9**

Consider two distinct points on the real number line, as shown in Figure 0.9.

**1.** The **directed distance from $a$ to $b$** is $b - a$.

**2.** The **directed distance from $b$ to $a$** is $a - b$.

**3.** The **distance between $a$ and $b$** is $|a - b|$ or $|b - a|$.

In Figure 0.9, note that because $b$ is to the right of $a$, the directed distance from $a$ to $b$ (moving to the right) is positive. Moreover, because $a$ is to the left of $b$, the directed distance from $b$ to $a$ (moving to the left) is negative. The distance *between* two points on the real number line can never be negative.
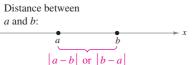
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**Distance Between Two Points on the Real Number Line**

The distance $d$ between points $x_1$ and $x_2$ on the real number line is given by

$$d = |x_2 - x_1| = \sqrt{(x_2 - x_1)^2}.$$

---

Note that the order of subtraction with $x_1$ and $x_2$ does not matter because

$$|x_2 - x_1| = |x_1 - x_2| \quad \text{and} \quad (x_2 - x_1)^2 = (x_1 - x_2)^2.$$
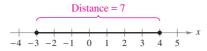
**Example 1**   Finding Distance on the Real Number Line

Determine the distance between $-3$ and $4$ on the real number line. What is the directed distance from $-3$ to $4$? What is the directed distance from $4$ to $-3$?

**SOLUTION**    The distance between $-3$ and $4$ is given by

$$|-3 - 4| = |-7| = 7 \quad \text{or} \quad |4 - (-3)| = |7| = 7 \qquad {\color{magenta}|a - b| \text{ or } |b - a|}$$

as shown in Figure 0.10.



**FIGURE 0.10**

The directed distance from $-3$ to $4$ is

$$4 - (-3) = 7. \qquad {\color{magenta}b - a}$$

The directed distance from $4$ to $-3$ is

$$-3 - 4 = -7. \qquad {\color{magenta}a - b}$$

**✓ CHECKPOINT 1**

Determine the distance between $-2$ and $6$ on the real number line. What is the directed distance from $-2$ to $6$? What is the directed distance from $6$ to $-2$? ■

## Intervals Defined by Absolute Value

**Example 2** Defining an Interval on the Real Number Line

Find the interval on the real number line that contains all numbers that lie no more than two units from 3.

**SOLUTION** Let $x$ be any point in this interval. You need to find all $x$ such that the distance between $x$ and 3 is less than or equal to 2. This implies that

$$|x - 3| \leq 2.$$

Requiring the absolute value of $x - 3$ to be less than or equal to 2 means that $x - 3$ must lie between $-2$ and 2. So, you can write

$$-2 \leq x - 3 \leq 2.$$

Solving this pair of inequalities, you have

$$-2 + 3 \leq x - 3 + 3 \leq 2 + 3$$

$$1 \leq \quad x \quad \leq 5. \qquad \text{Solution set}$$

So, the interval is $[1, 5]$, as shown in Figure 0.11.

$|x - 3| \leq 2$

2 units   2 units
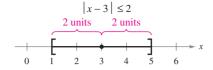
**FIGURE 0.11**

✓ **CHECKPOINT 2**
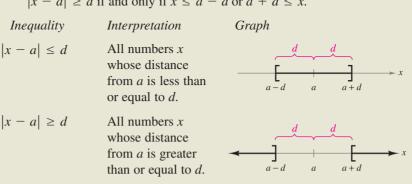
Find the interval on the real number line that contains all numbers that lie no more than four units from 6. ■

**STUDY TIP**

Be sure you see that inequalities of the form $|x - a| \geq d$ have solution sets consisting of two intervals. To describe the two intervals without using absolute values, you must use *two* separate inequalities, connected by an "or" to indicate union.

**Two Basic Types of Inequalities Involving Absolute Value**

Let $a$ and $d$ be real numbers, where $d > 0$.

$$|x - a| \leq d \text{ if and only if } a - d \leq x \leq a + d.$$

$$|x - a| \geq d \text{ if and only if } x \leq a - d \text{ or } a + d \leq x.$$

| *Inequality* | *Interpretation* | *Graph* |
|---|---|---|
| $|x - a| \leq d$ | All numbers $x$ whose distance from $a$ is less than or equal to $d$. | |
| $|x - a| \geq d$ | All numbers $x$ whose distance from $a$ is greater than or equal to $d$. | |

## Application

*Example 3*
**MAKE A DECISION** Quality Control Ⓡ

A large manufacturer hired a quality control firm to determine the reliability of a product. Using statistical methods, the firm determined that the manufacturer could expect 0.35% ± 0.17% of the units to be defective. If the manufacturer offers a money-back guarantee on this product, how much should be budgeted to cover the refunds on 100,000 units? (Assume that the retail price is $8.95.) Will the manufacturer have to establish a refund budget greater than $5000?

**SOLUTION** Let $r$ represent the percent of defective units (written in decimal form). You know that $r$ will differ from 0.0035 by at most 0.0017.

$$0.0035 - 0.0017 \le r \le 0.0035 + 0.0017$$
$$0.0018 \le r \le 0.0052 \qquad \text{Figure 0.12(a)}$$
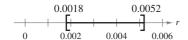
Now, letting $x$ be the number of defective units out of 100,000, it follows that $x = 100{,}000r$ and you have

$$0.0018(100{,}000) \le 100{,}000r \le 0.0052(100{,}000)$$
$$180 \le \quad x \quad \le 520. \qquad \text{Figure 0.12(b)}$$
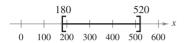
Finally, letting $C$ be the cost of refunds, you have $C = 8.95x$. So, the total cost of refunds for 100,000 units should fall within the interval given by

$$180(8.95) \le 8.95x \le 520(8.95)$$
$$\$1611 \le \quad C \quad \le \$4654. \qquad \text{Figure 0.12(c)}$$
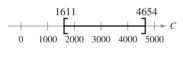
No, the refund budget will be less than $5000.



0.0018    0.0052

(a) Percent of defective units

180    520

(b) Number of defective units

1611    4654

(c) Cost of refunds

**FIGURE 0.12**

✓**CHECKPOINT 3**

Use the information in Example 3 to determine how much should be budgeted to cover refunds on 250,000 units. ■

In Example 3, the manufacturer should expect to spend between $1611 and $4654 for refunds. Of course, the safer budget figure for refunds would be the higher of these estimates. However, from a statistical point of view, the most representative estimate would be the average of these two extremes. Graphically, the average of two numbers is the **midpoint** of the interval with the two numbers as endpoints, as shown in Figure 0.13.



Midpoint $= \dfrac{1611 + 4654}{2} = 3132.5$

1611    4654
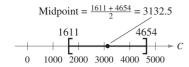
**FIGURE 0.13**

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**Midpoint of an Interval**

The **midpoint** of the interval with endpoints $a$ and $b$ is found by taking the average of the endpoints.

$$\text{Midpoint} = \frac{a + b}{2}$$

## Exercises 0.2

In Exercises 1–6, find (a) the directed distance from $a$ to $b$, (b) the directed distance from $b$ to $a$, and (c) the distance between $a$ and $b$.

**1.** $a = 126, b = 75$

**2.** $a = -126, b = -75$

**3.** $a = 9.34, b = -5.65$

**4.** $a = -2.05, b = 4.25$

**5.** $a = \frac{16}{5}, b = \frac{112}{75}$

**6.** $a = -\frac{18}{5}, b = \frac{61}{15}$

In Exercises 7–18, use absolute values to describe the given interval (or pair of intervals) on the real number line.

**7.** $[-2, 2]$

**8.** $(-3, 3)$

**9.** $(-\infty, -2) \cup (2, \infty)$

**10.** $(-\infty, -3] \cup [3, \infty)$

**11.** $[2, 8]$

**12.** $(-7, -1)$

**13.** $(-\infty, 0) \cup (4, \infty)$

**14.** $(-\infty, 20) \cup (24, \infty)$

**15.** All numbers *less than* three units from 5

**16.** All numbers *more than* five units from 2

**17.** $y$ is *at most* two units from $a$.

**18.** $y$ is *less than* $h$ units from $c$.

In Exercises 19–34, solve the inequality and sketch the graph of the solution on the real number line.

**19.** $|x| < 4$

**20.** $|2x| < 6$

**21.** $\left|\frac{x}{2}\right| > 3$

**22.** $|3x| > 12$

**23.** $|x - 5| < 2$

**24.** $|3x + 1| \geq 4$

**25.** $\left|\frac{x - 3}{2}\right| \geq 5$

**26.** $|2x + 1| < 5$

**27.** $|10 - x| > 4$

**28.** $|25 - x| \geq 20$

**29.** $|9 - 2x| < 1$

**30.** $\left|1 - \frac{2x}{3}\right| < 1$

**31.** $|x - a| \leq b, \ b > 0$

**32.** $|2x - a| \geq b, \ b > 0$

**33.** $\left|\frac{3x - a}{4}\right| < 2b, \ b > 0$

**34.** $\left|a - \frac{5x}{2}\right| > b, \ b > 0$

In Exercises 35–40, find the midpoint of the given interval.

**35.** $[8, 24]$

**36.** $[7.3, 12.7]$

**37.** $[-6.85, 9.35]$

**38.** $[-4.6, -1.3]$

**39.** $\left[-\frac{1}{2}, \frac{3}{4}\right]$

**40.** $\left[\frac{5}{6}, \frac{5}{2}\right]$

**41. Chemistry**   Copper has a melting point $M$ within 0.2°C of 1083.4°C. Use absolute values to write the range as an inequality.

**42. Stock Price**   A stock market analyst predicts that over the next year the price $p$ of a stock will not change from its current price of $33.15 by more than $2. Use absolute values to write this prediction as an inequality.

**43. Heights of a Population**   The heights $h$ of two-thirds of the members of a population satisfy the inequality

$$\left|\frac{h - 68.5}{2.7}\right| \leq 1$$

where $h$ is measured in inches. Determine the interval on the real number line in which these heights lie.

**44. Biology**   The American Kennel Club has developed guidelines for judging the features of various breeds of dogs. For collies, the guidelines specify that the weights for males satisfy the inequality

$$\left|\frac{w - 67.5}{7.5}\right| \leq 1$$

where $w$ is measured in pounds. Determine the interval on the real number line in which these weights lie.

**45. Production**   The estimated daily production $x$ at a refinery is given by

$$|x - 200{,}000| \leq 25{,}000$$

where $x$ is measured in barrels of oil. Determine the high and low production levels.

**46. Manufacturing**   The acceptable weights for a 20-ounce cereal box are given by $|x - 20| \leq 0.75$, where $x$ is measured in ounces. Determine the high and low weights for the cereal box.

**Budget Variance**   In Exercises 47–50, (a) use absolute value notation to represent the two intervals in which expenses must lie if they are to be within $500 and within 5% of the specified budget amount and (b) using the more stringent constraint, determine whether the given expense is at variance with the budget restriction.

| Item | Budget | Expense |
|---|---|---|
| **47.** Utilities | $4750.00 | $5116.37 |
| **48.** Insurance | $15,000.00 | $14,695.00 |
| **49.** Maintenance | $20,000.00 | $22,718.35 |
| **50.** Taxes | $7500.00 | $8691.00 |

**51. Quality Control**   In determining the reliability of a product, a manufacturer determines that it should expect $0.05\% \pm 0.01\%$ of the units to be defective. If the manufacturer offers a money-back guarantee on this product, how much should be budgeted to cover the refunds on 150,000 units? (Assume that the retail price is $195.99.)

## Section 0.3

# Exponents and Radicals

- Evaluate expressions involving exponents or radicals.
- Simplify expressions with exponents.
- Find the domains of algebraic expressions.

## Expressions Involving Exponents or Radicals

**Properties of Exponents**

1. Whole-number exponents: $\quad x^n = \underbrace{x \cdot x \cdot x \cdot \cdot \cdot x}_{n \text{ factors}}$

2. Zero exponent: $\quad x^0 = 1, \quad x \neq 0$

3. Negative exponents: $\quad x^{-n} = \dfrac{1}{x^n}, \quad x \neq 0$

4. Radicals (principal $n$th root): $\quad \sqrt[n]{x} = a \implies x = a^n$

5. Rational exponents $(1/n)$: $\quad x^{1/n} = \sqrt[n]{x}$

6. Rational exponents $(m/n)$: $\quad x^{m/n} = (x^{1/n})^m = \left(\sqrt[n]{x}\right)^m$

   $\quad x^{m/n} = (x^m)^{1/n} = \sqrt[n]{x^m}$

7. Special convention (square root): $\quad \sqrt[2]{x} = \sqrt{x}$

**STUDY TIP**

If $n$ is even, then the principal $n$th root is positive. For example, $\sqrt{4} = +2$ and $\sqrt[4]{81} = +3$.

### Example 1   Evaluating Expressions

|  | *Expression* | *x-Value* | *Substitution* |
|---|---|---|---|
| **a.** | $y = -2x^2$ | $x = 4$ | $y = -2(4^2) = -2(16) = -32$ |
| **b.** | $y = 3x^{-3}$ | $x = -1$ | $y = 3(-1)^{-3} = \dfrac{3}{(-1)^3} = \dfrac{3}{-1} = -3$ |
| **c.** | $y = (-x)^2$ | $x = \dfrac{1}{2}$ | $y = \left(-\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$ |
| **d.** | $y = \dfrac{2}{x^{-2}}$ | $x = 3$ | $y = \dfrac{2}{3^{-2}} = 2(3^2) = 18$ |

✓**CHECKPOINT 1**

Evaluate $y = 4x^{-2}$ for $x = 3$. ▪

### Example 2   Evaluating Expressions

|  | *Expression* | *x-Value* | *Substitution* |
|---|---|---|---|
| **a.** | $y = 2x^{1/2}$ | $x = 4$ | $y = 2\sqrt{4} = 2(2) = 4$ |
| **b.** | $y = \sqrt[3]{x^2}$ | $x = 8$ | $y = 8^{2/3} = (8^{1/3})^2 = 2^2 = 4$ |

✓**CHECKPOINT 2**

Evaluate $y = 4x^{1/3}$ for $x = 8$. ▪

## Operations with Exponents

**Operations with Exponents**

1. Multiplying like bases:      $x^n x^m = x^{n+m}$        Add exponents.

2. Dividing like bases:      $\dfrac{x^n}{x^m} = x^{n-m}$        Subtract exponents.

3. Removing parentheses:      $(xy)^n = x^n y^n$

   $$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

   $(x^n)^m = x^{nm}$

4. Special conventions:      $-x^n = -(x^n), \quad -x^n \neq (-x)^n$

   $cx^n = c(x^n), \quad cx^n \neq (cx)^n$

   $x^{n^m} = x^{(n^m)}, \quad x^{n^m} \neq (x^n)^m$

*Example 3*  **Simplifying Expressions with Exponents**

Simplify each expression.

**a.** $2x^2(x^3)$      **b.** $(3x)^2 \sqrt[3]{x}$      **c.** $\dfrac{3x^2}{(x^{1/2})^3}$

**d.** $\dfrac{5x^4}{(x^2)^3}$      **e.** $x^{-1}(2x^2)$      **f.** $\dfrac{-\sqrt{x}}{5x^{-1}}$

**SOLUTION**

**a.** $2x^2(x^3) = 2x^{2+3} = 2x^5$      $x^n x^m = x^{n+m}$

**b.** $(3x)^2 \sqrt[3]{x} = 9x^2 x^{1/3} = 9x^{2+(1/3)} = 9x^{7/3}$      $x^n x^m = x^{n+m}$

**c.** $\dfrac{3x^2}{(x^{1/2})^3} = 3\left(\dfrac{x^2}{x^{3/2}}\right) = 3x^{2-(3/2)} = 3x^{1/2}$      $(x^n)^m = x^{nm}, \dfrac{x^n}{x^m} = x^{n-m}$

**d.** $\dfrac{5x^4}{(x^2)^3} = \dfrac{5x^4}{x^6} = 5x^{4-6} = 5x^{-2} = \dfrac{5}{x^2}$      $(x^n)^m = x^{nm}, \dfrac{x^n}{x^m} = x^{n-m}$

**e.** $x^{-1}(2x^2) = 2x^{-1}x^2 = 2x^{2-1} = 2x$      $x^n x^m = x^{n+m}$

**f.** $\dfrac{-\sqrt{x}}{5x^{-1}} = -\dfrac{1}{5}\left(\dfrac{x^{1/2}}{x^{-1}}\right) = -\dfrac{1}{5}x^{(1/2)+1} = -\dfrac{1}{5}x^{3/2}$      $\dfrac{x^n}{x^m} = x^{n-m}$

✓**CHECKPOINT 3**

Simplify each expression.

**a.** $3x^2(x^4)$      **b.** $(2x)^3 \sqrt{x}$      **c.** $\dfrac{4x^2}{(x^{1/3})^2}$ ◼

Note in Example 3 that one characteristic of simplified expressions is the absence of negative exponents. Another characteristic of simplified expressions is that sums and differences are written in *factored form*. To do this, you can use the **Distributive Property.**

$$abx^n + acx^{n+m} = ax^n(b + cx^m)$$

Study the next example carefully to be sure that you understand the concepts involved in the factoring process.

**Example 4**   **Simplifying by Factoring**

Simplify each expression by factoring.

**a.** $2x^2 - x^3$    **b.** $2x^3 + x^2$    **c.** $2x^{1/2} + 4x^{5/2}$    **d.** $2x^{-1/2} + 3x^{5/2}$

**SOLUTION**

**a.** $2x^2 - x^3 = x^2(2 - x)$

**b.** $2x^3 + x^2 = x^2(2x + 1)$

**c.** $2x^{1/2} + 4x^{5/2} = 2x^{1/2}(1 + 2x^2)$

**d.** $2x^{-1/2} + 3x^{5/2} = x^{-1/2}(2 + 3x^3) = \dfrac{2 + 3x^3}{\sqrt{x}}$

✓ **CHECKPOINT 4**

Simplify each expression by factoring.

**a.** $x^3 - 2x$

**b.** $2x^{1/2} + 8x^{3/2}$ ∎

Many algebraic expressions obtained in calculus occur in unsimplified form. For instance, the two expressions shown in the following example are the result of an operation in calculus called **differentiation.** [The first is the derivative of $2(x + 1)^{3/2}(2x - 3)^{5/2}$, and the second is the derivative of $2(x + 1)^{1/2}(2x - 3)^{5/2}$.]

**STUDY TIP**

To check that the simplified expression is equivalent to the original expression, try substituting values for $x$ into each expression.

**Example 5**   **Simplifying by Factoring**

Simplify each expression by factoring.

**a.** $3(x + 1)^{1/2}(2x - 3)^{5/2} + 10(x + 1)^{3/2}(2x - 3)^{3/2}$

$$= (x + 1)^{1/2}(2x - 3)^{3/2}[3(2x - 3) + 10(x + 1)]$$

$$= (x + 1)^{1/2}(2x - 3)^{3/2}(6x - 9 + 10x + 10)$$

$$= (x + 1)^{1/2}(2x - 3)^{3/2}(16x + 1)$$

**b.** $(x + 1)^{-1/2}(2x - 3)^{5/2} + 10(x + 1)^{1/2}(2x - 3)^{3/2}$

$$= (x + 1)^{-1/2}(2x - 3)^{3/2}[(2x - 3) + 10(x + 1)]$$

$$= (x + 1)^{-1/2}(2x - 3)^{3/2}(2x - 3 + 10x + 10)$$

$$= (x + 1)^{-1/2}(2x - 3)^{3/2}(12x + 7)$$

$$= \dfrac{(2x - 3)^{3/2}(12x + 7)}{(x + 1)^{1/2}}$$

✓ **CHECKPOINT 5**

Simplify the expression by factoring.

$(x + 2)^{1/2}(3x - 1)^{3/2}$

$+ 4(x + 2)^{-1/2}(3x - 1)^{5/2}$ ∎

Example 6 shows some additional types of expressions that can occur in calculus. [The expression in Example 6(d) is an antiderivative of $(x + 1)^{2/3}(2x + 3)$, and the expression in Example 6(e) is the derivative of $(x + 2)^3/(x - 1)^3$.]

## TECHNOLOGY

(T) A graphing utility offers several ways to calculate rational exponents and radicals. You should be familiar with the *x*-squared key (x²). This key squares the value of an expression.

For rational exponents or exponents other than 2, use the (^) key.

For radical expressions, you can use the square root key (√), the cube root key (∛), or the *x*th root key (ˣ√). Consult your graphing utility user's guide for specific keystrokes you can use to evaluate rational exponents and radical expressions.

Use a graphing utility to evaluate each expression.

**a.** $(-8)^{2/3}$    **b.** $(16 - 5)^4$

**c.** $\sqrt{576}$    **d.** $\sqrt[3]{729}$

**e.** $\sqrt[4]{(16)^3}$

### *Example 6*   Factors Involving Quotients

Simplify each expression by factoring.

**a.** $\dfrac{3x^2 + x^4}{2x}$

**b.** $\dfrac{\sqrt{x} + x^{3/2}}{x}$

**c.** $(9x + 2)^{-1/3} + 18(9x + 2)$

**d.** $\dfrac{3}{5}(x + 1)^{5/3} + \dfrac{3}{4}(x + 1)^{8/3}$

**e.** $\dfrac{3(x + 2)^2(x - 1)^3 - 3(x + 2)^3(x - 1)^2}{[(x - 1)^3]^2}$

**SOLUTION**

**a.** $\dfrac{3x^2 + x^4}{2x} = \dfrac{x^2(3 + x^2)}{2x} = \dfrac{x^{2-1}(3 + x^2)}{2} = \dfrac{x(3 + x^2)}{2}$

**b.** $\dfrac{\sqrt{x} + x^{3/2}}{x} = \dfrac{x^{1/2}(1 + x)}{x} = \dfrac{1 + x}{x^{1 - (1/2)}} = \dfrac{1 + x}{\sqrt{x}}$

**c.** $(9x + 2)^{-1/3} + 18(9x + 2) = (9x + 2)^{-1/3}[1 + 18(9x + 2)^{4/3}]$

$$= \dfrac{1 + 18(9x + 2)^{4/3}}{\sqrt[3]{9x + 2}}$$

**d.** $\dfrac{3}{5}(x + 1)^{5/3} + \dfrac{3}{4}(x + 1)^{8/3} = \dfrac{12}{20}(x + 1)^{5/3} + \dfrac{15}{20}(x + 1)^{8/3}$

$$= \dfrac{3}{20}(x + 1)^{5/3}[4 + 5(x + 1)]$$

$$= \dfrac{3}{20}(x + 1)^{5/3}(4 + 5x + 5)$$

$$= \dfrac{3}{20}(x + 1)^{5/3}(5x + 9)$$

**e.** $\dfrac{3(x + 2)^2(x - 1)^3 - 3(x + 2)^3(x - 1)^2}{[(x - 1)^3]^2}$

$$= \dfrac{3(x + 2)^2(x - 1)^2[(x - 1) - (x + 2)]}{(x - 1)^6}$$

$$= \dfrac{3(x + 2)^2(x - 1 - x - 2)}{(x - 1)^{6-2}}$$

$$= \dfrac{-9(x + 2)^2}{(x - 1)^4}$$

## ✓ CHECKPOINT 6

Simplify the expression by factoring.

$$\dfrac{5x^3 + x^6}{3x} \quad \blacksquare$$

## Domain of an Algebraic Expression

When working with algebraic expressions involving $x$, you face the potential difficulty of substituting a value of $x$ for which the expression is not defined (does not produce a real number). For example, the expression $\sqrt{2x + 3}$ is *not defined* when $x = -2$ because $\sqrt{2(-2) + 3}$ is not a real number.

The set of all values for which an expression is defined is called its **domain.** So, the domain of $\sqrt{2x + 3}$ is the set of all values of $x$ such that $\sqrt{2x + 3}$ is a real number. In order for $\sqrt{2x + 3}$ to represent a real number, it is necessary that $2x + 3 \geq 0$. In other words, $\sqrt{2x + 3}$ is defined only for those values of $x$ that lie in the interval $\left[-\frac{3}{2}, \infty\right)$, as shown in Figure 0.14.

$\sqrt{2x + 3}$ is
not defined
for these $x$.

$\sqrt{2x + 3}$ is
defined
for these $x$.

$-\frac{3}{2}$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$      $x$
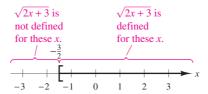
**FIGURE 0.14**

### *Example 7*   Finding the Domain of an Expression

Find the domain of each expression.

**a.** $\sqrt{3x - 2}$

**b.** $\dfrac{1}{\sqrt{3x - 2}}$

**c.** $\sqrt[3]{9x + 1}$

**SOLUTION**

**a.** The domain of $\sqrt{3x - 2}$ consists of all $x$ such that

$$3x - 2 \geq 0 \qquad \text{Expression must be nonnegative.}$$

which implies that $x \geq \frac{2}{3}$. So, the domain is $\left[\frac{2}{3}, \infty\right)$.

**b.** The domain of $1/\sqrt{3x - 2}$ is the same as the domain of $\sqrt{3x - 2}$, except that $1/\sqrt{3x - 2}$ is not defined when $3x - 2 = 0$. Because this occurs when $x = \frac{2}{3}$, the domain is $\left(\frac{2}{3}, \infty\right)$.

**c.** Because $\sqrt[3]{9x + 1}$ is defined for all real numbers, its domain is $(-\infty, \infty)$.

### ✓ CHECKPOINT 7

Find the domain of each expression.

**a.** $\sqrt{x - 2}$

**b.** $\dfrac{1}{\sqrt{x - 2}}$

**c.** $\sqrt[3]{x - 2}$  ■

## Exercises 0.3

In Exercises 1–20, evaluate the expression for the given value of $x$.

| Expression | x-Value | | Expression | x-Value |
|---|---|---|---|---|
| **1.** $-2x^3$ | $x = 3$ | | **2.** $\dfrac{x^2}{3}$ | $x = 6$ |
| **3.** $4x^{-3}$ | $x = 2$ | | **4.** $7x^{-2}$ | $x = 5$ |
| **5.** $\dfrac{1 + x^{-1}}{x^{-1}}$ | $x = 3$ | | **6.** $x - 4x^{-2}$ | $x = 3$ |
| **7.** $3x^2 - 4x^3$ | $x = -2$ | | **8.** $5(-x)^3$ | $x = 3$ |
| **9.** $6x^0 - (6x)^0$ | $x = 10$ | | **10.** $\dfrac{1}{(-x)^{-3}}$ | $x = 4$ |
| **11.** $\sqrt[3]{x^2}$ | $x = 27$ | | **12.** $\sqrt{x^3}$ | $x = \tfrac{1}{9}$ |
| **13.** $x^{-1/2}$ | $x = 4$ | | **14.** $x^{-3/4}$ | $x = 16$ |
| **15.** $x^{-2/5}$ | $x = -32$ | | **16.** $(x^{2/3})^3$ | $x = 10$ |
| **17.** $500x^{60}$ | $x = 1.01$ | | **18.** $\dfrac{10,000}{x^{120}}$ | $x = 1.075$ |
| **19.** $\sqrt[3]{x}$ | $x = -154$ | | **20.** $\sqrt[6]{x}$ | $x = 325$ |

In Exercises 21–30, simplify the expression.

**21.** $6y^{-2}(2y^4)^{-3}$

**22.** $z^{-3}(3z^4)$

**23.** $10(x^2)^2$

**24.** $(4x^3)^2$

**25.** $\dfrac{7x^2}{x^{-3}}$

**26.** $\dfrac{x^{-3}}{\sqrt{x}}$

**27.** $\dfrac{10(x + y)^3}{4(x + y)^{-2}}$

**28.** $\left(\dfrac{12s^2}{9s}\right)^3$

**29.** $\dfrac{3x\sqrt{x}}{x^{1/2}}$

**30.** $\left(\sqrt[3]{x^2}\right)^3$

In Exercises 31–36, simplify by removing all possible factors from the radical.

**31.** $\sqrt{8}$

**32.** $\sqrt[3]{\tfrac{16}{27}}$

**33.** $\sqrt[3]{54x^5}$

**34.** $\sqrt[4]{(3x^2y^3)^4}$

**35.** $\sqrt[3]{144x^9y^{-4}z^5}$

**36.** $\sqrt[4]{32xy^5z^{-8}}$

In Exercises 37–44, simplify each expression by factoring.

**37.** $3x^3 - 12x$

**38.** $8x^4 - 6x^2$

**39.** $2x^{5/2} + x^{-1/2}$

**40.** $5x^{3/2} - x^{-3/2}$

**41.** $3x(x + 1)^{3/2} - 6(x + 1)^{1/2}$

**42.** $2x(x - 1)^{5/2} - 4(x - 1)^{3/2}$

**43.** $\dfrac{(x + 1)(x - 1)^2 - (x - 1)^3}{(x + 1)^2}$

**44.** $(x^4 + 2)^3(x + 3)^{-1/2} + 4x^3(x^4 + 2)^2(x + 3)^{1/2}$

In Exercises 45–52, find the domain of the given expression.

**45.** $\sqrt{x - 4}$

**46.** $\sqrt{5 - 2x}$

**47.** $\sqrt{x^2 + 3}$

**48.** $\sqrt{4x^2 + 1}$

**49.** $\dfrac{1}{\sqrt[3]{x - 4}}$

**50.** $\dfrac{1}{\sqrt[3]{x + 4}}$

**51.** $\dfrac{\sqrt{x + 2}}{1 - x}$

**52.** $\dfrac{1}{\sqrt{2x + 3}} + \sqrt{6 - 4x}$

**(T) Compound Interest**   In Exercises 53–56, a certificate of deposit has a principal of $P$ and an annual percentage rate of $r$ (expressed as a decimal) compounded $n$ times per year. Enter the compound interest formula

$$A = P\left(1 + \frac{r}{n}\right)^N$$

into a graphing utility and use it to find the balance after $N$ compoundings.

**53.** $P = \$10,000,\quad r = 6.5\%,\quad n = 12,\quad N = 120$

**54.** $P = \$7000,\quad r = 5\%,\quad n = 365,\quad N = 1000$

**55.** $P = \$5000,\quad r = 5.5\%,\quad n = 4,\quad N = 60$

**56.** $P = \$8000,\quad r = 7\%,\quad n = 6,\quad N = 90$

**57. Period of a Pendulum**   The period of a pendulum is

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where $T$ is the period in seconds and $L$ is the length of the pendulum in feet. Find the period of a pendulum whose length is 4 feet.

**58. Annuity**   A balance $A$, after $n$ annual payments of $P$ dollars have been made into an annuity earning an annual percentage rate of $r$ compounded annually, is given by

$$A = P(1 + r) + P(1 + r)^2 + \cdots + P(1 + r)^n.$$

Rewrite this formula by completing the following factorization: $A = P(1 + r)(\quad)$.

**59. Extended Application**   To work an extended application analyzing the population per square mile of the United States, visit this text's website at *college.hmco.com*   *(Data Source: U.S. Census Bureau)*

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The symbol **(T)** indicates when to use graphing technology or a symbolic computer algebra system to solve a problem or an exercise. The solutions to other problems or exercises may also be facilitated by use of appropriate technology.

## Section 0.4

# Factoring Polynomials

■ Use special products and factorization techniques to factor polynomials.
■ Find the domains of radical expressions.
■ Use synthetic division to factor polynomials of degree three or more.
■ Use the Rational Zero Theorem to find the real zeros of polynomials.

### Factorization Techniques

The **Fundamental Theorem of Algebra** states that every $n$th-degree polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

has precisely $n$ **zeros.** (The zeros may be repeated or imaginary.) The problem of finding the zeros of a polynomial is equivalent to the problem of factoring the polynomial into linear factors.

---

**Special Products and Factorization Techniques**

*Quadratic Formula*

$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

*Example*

$$x^2 + 3x - 1 = 0 \implies x = \frac{-3 \pm \sqrt{13}}{2}$$

*Special Products*

$x^2 - a^2 = (x - a)(x + a)$

$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

$x^4 - a^4 = (x - a)(x + a)(x^2 + a^2)$

*Examples*

$x^2 - 9 = (x - 3)(x + 3)$

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

$x^3 + 64 = (x + 4)(x^2 - 4x + 16)$

$x^4 - 16 = (x - 2)(x + 2)(x^2 + 4)$

*Binomial Theorem*

$(x + a)^2 = x^2 + 2ax + a^2$

$(x - a)^2 = x^2 - 2ax + a^2$

$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$

$(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$

$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$

$(x - a)^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$

*Examples*

$(x + 3)^2 = x^2 + 6x + 9$

$(x^2 - 5)^2 = x^4 - 10x^2 + 25$

$(x + 2)^3 = x^3 + 6x^2 + 12x + 8$

$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$

$(x + 2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$

$(x - 4)^4 = x^4 - 16x^3 + 96x^2 - 256x + 256$

$$(x + a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} + \frac{n(n-1)(n-2)}{3!}a^3x^{n-3} + \cdots + na^{n-1}x + a^n{}^*$$

$$(x - a)^n = x^n - nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} - \frac{n(n-1)(n-2)}{3!}a^3x^{n-3} + \cdots \pm na^{n-1}x \mp a^n$$

*Factoring by Grouping*

$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d)$
$\qquad\qquad\qquad\qquad\qquad = (ax^2 + b)(cx + d)$

*Example*

$3x^3 - 2x^2 - 6x + 4 = x^2(3x - 2) - 2(3x - 2)$
$\qquad\qquad\qquad\qquad = (x^2 - 2)(3x - 2)$

---

\* The factorial symbol ! is defined as follows: $0! = 1$, $1! = 1$, $2! = 2 \cdot 1 = 2$,
$3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, and so on.

### *Example 1*  Applying the Quadratic Formula

Use the Quadratic Formula to find all real zeros of each polynomial.

**a.** $4x^2 + 6x + 1$    **b.** $x^2 + 6x + 9$    **c.** $2x^2 - 6x + 5$

**SOLUTION**

**a.** Using $a = 4$, $b = 6$, and $c = 1$, you can write

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 16}}{8}$$

$$= \frac{-6 \pm \sqrt{20}}{8}$$

$$= \frac{-6 \pm 2\sqrt{5}}{8}$$

$$= \frac{2\left(-3 \pm \sqrt{5}\right)}{2(4)}$$

$$= \frac{-3 \pm \sqrt{5}}{4}.$$

So, there are two real zeros:

$$x = \frac{-3 - \sqrt{5}}{4} \approx -1.309 \quad \text{and} \quad x = \frac{-3 + \sqrt{5}}{4} \approx -0.191.$$

**b.** In this case, $a = 1$, $b = 6$, and $c = 9$, and the Quadratic Formula yields

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 36}}{2} = -\frac{6}{2} = -3.$$

So, there is one (repeated) real zero: $x = -3$.

**c.** For this quadratic equation, $a = 2$, $b = -6$, and $c = 5$. So,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 40}}{4} = \frac{6 \pm \sqrt{-4}}{4}.$$

Because $\sqrt{-4}$ is imaginary, there are no real zeros.

**STUDY TIP**

Try solving Example 1(b) by factoring. Do you obtain the same answer?

### ✓ CHECKPOINT 1

Use the Quadratic Formula to find all real zeros of each polynomial.

**a.** $2x^2 + 4x + 1$    **b.** $x^2 - 8x + 16$    **c.** $2x^2 - x + 5$ ■

The zeros in Example 1(a) are irrational, and the zeros in Example 1(c) are imaginary. In both of these cases the quadratic is said to be **irreducible** because it cannot be factored into linear factors with rational coefficients. The next example shows how to find the zeros associated with *reducible* quadratics. In this example, factoring is used to find the zeros of each quadratic. Try using the Quadratic Formula to obtain the same zeros.

## *Example 2*   Factoring Quadratics

Find the zeros of each quadratic polynomial.

**a.** $x^2 - 5x + 6$   **b.** $x^2 - 6x + 9$   **c.** $2x^2 + 5x - 3$

**SOLUTION**

**a.** Because

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

the zeros are $x = 2$ and $x = 3$.

**b.** Because

$$x^2 - 6x + 9 = (x - 3)^2$$

the only zero is $x = 3$.

**c.** Because

$$2x^2 + 5x - 3 = (2x - 1)(x + 3)$$

the zeros are $x = \frac{1}{2}$ and $x = -3$.

**STUDY TIP**

The zeros of a polynomial in $x$ are the values of $x$ that make the polynomial zero. To find the zeros, factor the polynomial into linear factors and set each factor equal to zero. For instance, the zeros of $(x - 2)(x - 3)$ occur when $x - 2 = 0$ and $x - 3 = 0$.

### ✓ CHECKPOINT 2

Find the zeros of each quadratic polynomial.

**a.** $x^2 - 2x - 15$   **b.** $x^2 + 2x + 1$   **c.** $2x^2 - 7x + 6$ ■

## *Example 3*   Finding the Domain of a Radical Expression

Find the domain of $\sqrt{x^2 - 3x + 2}$.

**SOLUTION**   Because

$$x^2 - 3x + 2 = (x - 1)(x - 2)$$

you know that the zeros of the quadratic are $x = 1$ and $x = 2$. So, you need to test the sign of the quadratic in the three intervals $(-\infty, 1)$, $(1, 2)$, and $(2, \infty)$, as shown in Figure 0.15. After testing each of these intervals, you can see that the quadratic is negative in the center interval and positive in the outer two intervals. Moreover, because the quadratic is zero when $x = 1$ and $x = 2$, you can conclude that the domain of $\sqrt{x^2 - 3x + 2}$ is

$$(-\infty, 1] \cup [2, \infty). \qquad \text{Domain}$$
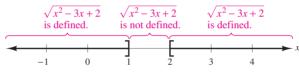
Values of $\sqrt{x^2 - 3x + 2}$

| $x$ | $\sqrt{x^2 - 3x + 2}$ |
|-----|------------------------|
| 0 | $\sqrt{2}$ |
| 1 | 0 |
| 1.5 | Undefined |
| 2 | 0 |
| 3 | $\sqrt{2}$ |

### ✓ CHECKPOINT 3

Find the domain of

$$\sqrt{x^2 + x - 2}. ■$$



**FIGURE 0.15**

## Factoring Polynomials of Degree Three or More
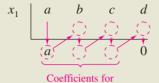
It can be difficult to find the zeros of polynomials of degree three or more. However, if one of the zeros of a polynomial is known, then you can use that zero to reduce the degree of the polynomial. For example, if you know that $x = 2$ is a zero of $x^3 - 4x^2 + 5x - 2$, then you know that $(x - 2)$ is a factor, and you can use long division to factor the polynomial as shown.

$$x^3 - 4x^2 + 5x - 2 = (x - 2)(x^2 - 2x + 1)$$
$$= (x - 2)(x - 1)(x - 1)$$

As an alternative to long division, many people prefer to use **synthetic division** to reduce the degree of a polynomial.

---

**Synthetic Division for a Cubic Polynomial**

Given: $x = x_1$ is a zero of $ax^3 + bx^2 + cx + d$.



Vertical pattern:
*Add terms.*

Diagonal pattern:
*Multiply by $x_1$.*

Coefficients for
quadratic factor

---

Performing synthetic division on the polynomial

$$x^3 - 4x^2 + 5x - 2$$

using the given zero, $x = 2$, produces the following.



$$(x - 2)(x^2 - 2x + 1) = x^3 - 4x^2 + 5x - 2$$

When you use synthetic division, remember to take *all* coefficients into account—*even if some of them are zero*. For instance, if you know that $x = -2$ is a zero of $x^3 + 3x + 14$, you can apply synthetic division as shown.



$$(x + 2)(x^2 - 2x + 7) = x^3 + 3x + 14$$

---

**STUDY TIP**

The algorithm for synthetic division given above works *only* for divisors of the form $x - x_1$. Remember that $x + x_1 = x - (-x_1)$.

## The Rational Zero Theorem

There is a systematic way to find the *rational* zeros of a polynomial. You can use the **Rational Zero Theorem** (also called the Rational Root Theorem).

---

**Rational Zero Theorem**

If a polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has integer coefficients, then every *rational* zero is of the form $x = p/q$, where $p$ is a factor of $a_0$, and $q$ is a factor of $a_n$.

---

### *Example 4*  Using the Rational Zero Theorem

Find all real zeros of the polynomial.

$$2x^3 + 3x^2 - 8x + 3$$

**SOLUTION**

$$2x^3 + 3x^2 - 8x + 3$$

Factors of constant term: $\pm 1, \pm 3$

Factors of leading coefficient: $\pm 1, \pm 2$

The possible rational zeros are the factors of the constant term divided by the factors of the leading coefficient.

$$1, -1, 3, -3, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}$$

By testing these possible zeros, you can see that $x = 1$ works.

$$2(1)^3 + 3(1)^2 - 8(1) + 3 = 2 + 3 - 8 + 3 = 0$$

Now, by synthetic division you have the following.

$$
\begin{array}{r|rrrr}
1 & 2 & 3 & -8 & 3 \\
  &   & 2 & 5 & -3 \\
\hline
  & 2 & 5 & -3 & 0
\end{array}
$$

$$(x - 1)(2x^2 + 5x - 3) = 2x^3 + 3x^2 - 8x + 3$$

Finally, by factoring the quadratic, $2x^2 + 5x - 3 = (2x - 1)(x + 3)$, you have

$$2x^3 + 3x^2 - 8x + 3 = (x - 1)(2x - 1)(x + 3)$$

and you can conclude that the zeros are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

### ✓ CHECKPOINT 4

Find all real zeros of the polynomial.

$$2x^3 - 3x^2 - 3x + 2 \ \blacksquare$$

---

**STUDY TIP**

In Example 4, you can check that the zeros are correct by substituting into the original polynomial.

*Check that x = 1 is a zero.*

$2(1)^3 + 3(1)^2 - 8(1) + 3$
$= 2 + 3 - 8 + 3$
$= 0$

*Check that $x = \frac{1}{2}$ is a zero.*

$2\left(\dfrac{1}{2}\right)^3 + 3\left(\dfrac{1}{2}\right)^2 - 8\left(\dfrac{1}{2}\right) + 3$
$= \dfrac{1}{4} + \dfrac{3}{4} - 4 + 3$
$= 0$

*Check that x = −3 is a zero.*

$2(-3)^3 + 3(-3)^2 - 8(-3) + 3$
$= -54 + 27 + 24 + 3$
$= 0$

## Exercises 0.4

In Exercises 1–8, use the Quadratic Formula to find all real zeros of the second-degree polynomial.

**1.** $6x^2 - 7x + 1$

**2.** $8x^2 - 2x - 1$

**3.** $4x^2 - 12x + 9$

**4.** $9x^2 + 12x + 4$

**5.** $y^2 + 4y + 1$

**6.** $y^2 + 5y - 2$

**7.** $2x^2 + 3x - 4$

**8.** $3x^2 - 8x - 4$

In Exercises 9–18, write the second-degree polynomial as the product of two linear factors.

**9.** $x^2 - 4x + 4$

**10.** $x^2 + 10x + 25$

**11.** $4x^2 + 4x + 1$

**12.** $9x^2 - 12x + 4$

**13.** $3x^2 - 4x + 1$

**14.** $2x^2 - x - 1$

**15.** $3x^2 - 5x + 2$

**16.** $x^2 - xy - 2y^2$

**17.** $x^2 - 4xy + 4y^2$

**18.** $a^2b^2 - 2abc + c^2$

In Exercises 19–34, completely factor the polynomial.

**19.** $81 - y^4$

**20.** $x^4 - 16$

**21.** $x^3 - 8$

**22.** $y^3 - 64$

**23.** $y^3 + 64$

**24.** $z^3 + 125$

**25.** $x^3 - y^3$

**26.** $(x - a)^3 + b^3$

**27.** $x^3 - 4x^2 - x + 4$

**28.** $x^3 - x^2 - x + 1$

**29.** $2x^3 - 3x^2 + 4x - 6$

**30.** $x^3 - 5x^2 - 5x + 25$

**31.** $2x^3 - 4x^2 - x + 2$

**32.** $x^3 - 7x^2 - 4x + 28$

**33.** $x^4 - 15x^2 - 16$

**34.** $2x^4 - 49x^2 - 25$

In Exercises 35–54, find all real zeros of the polynomial.

**35.** $x^2 - 5x$

**36.** $2x^2 - 3x$

**37.** $x^2 - 9$

**38.** $x^2 - 25$

**39.** $x^2 - 3$

**40.** $x^2 - 8$

**41.** $(x - 3)^2 - 9$

**42.** $(x + 1)^2 - 36$

**43.** $x^2 + x - 2$

**44.** $x^2 + 5x + 6$

**45.** $x^2 - 5x - 6$

**46.** $x^2 + x - 20$

**47.** $3x^2 + 5x + 2$

**48.** $2x^2 - x - 1$

**49.** $x^3 + 64$

**50.** $x^3 - 216$

**51.** $x^4 - 16$

**52.** $x^4 - 625$

**53.** $x^3 - x^2 - 4x + 4$

**54.** $2x^3 + x^2 + 6x + 3$

In Exercises 55–60, find the interval (or intervals) on which the given expression is defined.

**55.** $\sqrt{x^2 - 4}$

**56.** $\sqrt{4 - x^2}$

**57.** $\sqrt{x^2 - 7x + 12}$

**58.** $\sqrt{x^2 - 8x + 15}$

**59.** $\sqrt{5x^2 + 6x + 1}$

**60.** $\sqrt{3x^2 - 10x + 3}$

In Exercises 61–64, use synthetic division to complete the indicated factorization.

**61.** $x^3 - 3x^2 - 6x - 2 = (x + 1)(\quad)$

**62.** $x^3 - 2x^2 - x + 2 = (x - 2)(\quad)$

**63.** $2x^3 - x^2 - 2x + 1 = (x + 1)(\quad)$

**64.** $x^4 - 16x^3 + 96x^2 - 256x + 256 = (x - 4)(\quad)$

In Exercises 65–74, use the Rational Zero Theorem as an aid in finding all real zeros of the polynomial.

**65.** $x^3 - x^2 - 10x - 8$

**66.** $x^3 - 7x - 6$

**67.** $x^3 - 6x^2 + 11x - 6$

**68.** $x^3 + 2x^2 - 5x - 6$

**69.** $6x^3 - 11x^2 - 19x - 6$

**70.** $18x^3 - 9x^2 - 8x + 4$

**71.** $x^3 - 3x^2 - 3x - 4$

**72.** $2x^3 - x^2 - 13x - 6$

**73.** $4x^3 + 11x^2 + 5x - 2$

**74.** $3x^3 + 4x^2 - 13x + 6$

**75. Production Level**  The minimum average cost of producing $x$ units of a product occurs when the production level is set at the (positive) solution of

$$0.0003x^2 - 1200 = 0.$$

How many solutions does this equation have? Find and interpret the solution(s) in the context of the problem. What production level will minimize the average cost?

**76. Profit**  The profit $P$ from sales is given by

$$P = -200x^2 + 2000x - 3800$$

where $x$ is the number of units sold per day (in hundreds). Determine the interval for $x$ such that the profit will be greater than 1000.

Ⓑ **77. Chemistry: Finding Concentrations**  Use the Quadratic Formula to solve the expression

$$1.8 \times 10^{-5} = \frac{x^2}{1.0 \times 10^{-4} - x}$$

which is needed to determine the quantity of hydrogen ions ($[H^+]$) in a solution of $1.0 \times 10^{-4}$M acetic acid. Because $x$ represents a concentration of $[H^+]$, only positive values of $x$ are possible solutions. *(Source: Adapted from Zumdahl, Chemistry, Seventh Edition)*

**78. Finance**  After 2 years, an investment of \$1200 is made at an interest rate $r$, compounded annually, that will yield an amount of $A = 1200(1 + r)^2$. Determine the interest rate if $A = \$1300$.

The symbol Ⓑ indicates an exercise that contains material from textbooks in other disciplines.

## Section 0.5

# Fractions and Rationalization

- Add and subtract rational expressions.
- Simplify rational expressions involving radicals.
- Rationalize numerators and denominators of rational expressions.

## Operations with Fractions

In this section, you will review operations involving fractional expressions such as

$$\frac{2}{x}, \quad \frac{x^2 + 2x - 4}{x + 6}, \quad \text{and} \quad \frac{1}{\sqrt{x^2 + 1}}.$$

The first two expressions have polynomials as both numerator and denominator and are called **rational expressions.** A rational expression is **proper** if the degree of the numerator is less than the degree of the denominator. For example,

$$\frac{x}{x^2 + 1}$$

is proper. If the degree of the numerator is greater than or equal to the degree of the denominator, then the rational expression is **improper.** For example,

$$\frac{x^2}{x^2 + 1}, \quad \text{and} \quad \frac{x^3 + 2x + 1}{x + 1}$$

are both improper.

---

**Operations with Fractions**

**1.** Add fractions (find a common denominator):

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b}\left(\frac{d}{d}\right) + \frac{c}{d}\left(\frac{b}{b}\right) = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}, \quad b \neq 0, d \neq 0$$

**2.** Subtract fractions (find a common denominator):

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b}\left(\frac{d}{d}\right) - \frac{c}{d}\left(\frac{b}{b}\right) = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}, \quad b \neq 0, d \neq 0$$

**3.** Multiply fractions:

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}, \quad b \neq 0, d \neq 0$$

**4.** Divide fractions (invert and multiply):

$$\frac{a/b}{c/d} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}, \quad \frac{a/b}{c} = \frac{a/b}{c/1} = \left(\frac{a}{b}\right)\left(\frac{1}{c}\right) = \frac{a}{bc}, \quad b \neq 0,$$
$$c \neq 0, d \neq 0$$

**5.** Divide out like factors:

$$\frac{\cancel{a}b}{\cancel{a}c} = \frac{b}{c}, \quad \frac{ab + ac}{ad} = \frac{\cancel{a}(b + c)}{\cancel{a}d} = \frac{b + c}{d}, \quad a \neq 0, c \neq 0, d \neq 0$$

**Example 1**    **Adding and Subtracting Rational Expressions**

Perform each indicated operation and simplify.

**a.** $x + \dfrac{1}{x}$    **b.** $\dfrac{1}{x + 1} - \dfrac{2}{2x - 1}$

**SOLUTION**

**a.** $x + \dfrac{1}{x} = \dfrac{x^2}{x} + \dfrac{1}{x}$          <span style="color:magenta">Write with common denominator.</span>

$\qquad = \dfrac{x^2 + 1}{x}$          <span style="color:magenta">Add fractions.</span>

**b.** $\dfrac{1}{x + 1} - \dfrac{2}{2x - 1} = \dfrac{(2x - 1)}{(x + 1)(2x - 1)} - \dfrac{2(x + 1)}{(x + 1)(2x - 1)}$

$\qquad\qquad = \dfrac{2x - 1 - 2x - 2}{2x^2 + x - 1} = \dfrac{-3}{2x^2 + x - 1}$

✓ **CHECKPOINT 1**

Perform each indicated operation and simplify.

**a.** $x + \dfrac{2}{x}$    **b.** $\dfrac{2}{x + 1} - \dfrac{1}{2x + 1}$ ■

In adding (or subtracting) fractions whose denominators have no common factors, it is convenient to use the following pattern.

$$\frac{a}{b} + \frac{c}{d} = \frac{a \quad c}{b \quad d} = \frac{ad + bc}{bd}$$

For instance, in Example 1(b), you could have used this pattern as shown.

$$\frac{1}{x + 1} - \frac{2}{2x - 1} = \frac{(2x - 1) - 2(x + 1)}{(x + 1)(2x - 1)}$$

$$= \frac{2x - 1 - 2x - 2}{(x + 1)(2x - 1)} = \frac{-3}{2x^2 + x - 1}$$

In Example 1, the denominators of the rational expressions have no common factors. When the denominators do have common factors, it is best to find the least common denominator before adding or subtracting. For instance, when adding $1/x$ and $2/x^2$, you can recognize that the least common denominator is $x^2$ and write

$$\frac{1}{x} + \frac{2}{x^2} = \frac{x}{x^2} + \frac{2}{x^2}$$          <span style="color:magenta">Write with common denominator.</span>

$$= \frac{x + 2}{x^2}.$$          <span style="color:magenta">Add fractions.</span>

This is further demonstrated in Example 2.

Example 2   **Adding and Subtracting Rational Expressions**

Perform each indicated operation and simplify.

**a.** $\dfrac{x}{x^2 - 1} + \dfrac{3}{x + 1}$    **b.** $\dfrac{1}{2(x^2 + 2x)} - \dfrac{1}{4x}$

**SOLUTION**

**a.** Because $x^2 - 1 = (x + 1)(x - 1)$, the least common denominator is $x^2 - 1$.

$$\dfrac{x}{x^2 - 1} + \dfrac{3}{x + 1} = \dfrac{x}{(x - 1)(x + 1)} + \dfrac{3}{x + 1}$$   Factor.

$$= \dfrac{x}{(x - 1)(x + 1)} + \dfrac{3(x - 1)}{(x - 1)(x + 1)}$$   Write with common denominator.

$$= \dfrac{x + 3x - 3}{(x - 1)(x + 1)}$$   Add fractions.

$$= \dfrac{4x - 3}{x^2 - 1}$$   Simplify.

**b.** In this case, the least common denominator is $4x(x + 2)$.

$$\dfrac{1}{2(x^2 + 2x)} - \dfrac{1}{4x} = \dfrac{1}{2x(x + 2)} - \dfrac{1}{2(2x)}$$   Factor.

$$= \dfrac{2}{2(2x)(x + 2)} - \dfrac{x + 2}{2(2x)(x + 2)}$$   Write with common denominator.

$$= \dfrac{2 - x - 2}{4x(x + 2)}$$   Subtract fractions.

$$= \dfrac{-\cancel{x}}{4\cancel{x}(x + 2)}$$   Divide out like factor.

$$= \dfrac{-1}{4(x + 2)}, \quad x \neq 0$$   Simplify.

✓**CHECKPOINT 2**

Perform each indicated operation and simplify.

**a.** $\dfrac{x}{x^2 - 4} + \dfrac{2}{x - 2}$    **b.** $\dfrac{1}{3(x^2 + 2x)} - \dfrac{1}{3x}$ ◼

**STUDY TIP**

To add more than two fractions, you must find a denominator that is common to all the fractions. For instance, to add $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$, use a (least) common denominator of 30 and write

$$\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{5} = \dfrac{15}{30} + \dfrac{10}{30} + \dfrac{6}{30}$$   Write with common denominator.

$$= \dfrac{31}{30}.$$   Add fractions.

To add more than two rational expressions, use a similar procedure, as shown in Example 3. (Expressions such as those shown in this example are used in calculus to perform an integration technique called integration by partial fractions.)

### *Example 3*    **Adding More than Two Rational Expressions**

Perform each indicated addition of rational expressions.

**a.** $\dfrac{A}{x + 2} + \dfrac{B}{x - 3} + \dfrac{C}{x + 4}$

**b.** $\dfrac{A}{x + 2} + \dfrac{B}{(x + 2)^2} + \dfrac{C}{x - 1}$

**SOLUTION**

**a.** The least common denominator is $(x + 2)(x - 3)(x + 4)$.

$$\dfrac{A}{x + 2} + \dfrac{B}{x - 3} + \dfrac{C}{x + 4}$$

$$= \dfrac{A(x - 3)(x + 4) + B(x + 2)(x + 4) + C(x + 2)(x - 3)}{(x + 2)(x - 3)(x + 4)}$$

$$= \dfrac{A(x^2 + x - 12) + B(x^2 + 6x + 8) + C(x^2 - x - 6)}{(x + 2)(x - 3)(x + 4)}$$

$$= \dfrac{Ax^2 + Bx^2 + Cx^2 + Ax + 6Bx - Cx - 12A + 8B - 6C}{(x + 2)(x - 3)(x + 4)}$$

$$= \dfrac{(A + B + C)x^2 + (A + 6B - C)x + (-12A + 8B - 6C)}{(x + 2)(x - 3)(x + 4)}$$

**b.** Here the least common denominator is $(x + 2)^2(x - 1)$.

$$\dfrac{A}{x + 2} + \dfrac{B}{(x + 2)^2} + \dfrac{C}{x - 1}$$

$$= \dfrac{A(x + 2)(x - 1) + B(x - 1) + C(x + 2)^2}{(x + 2)^2(x - 1)}$$

$$= \dfrac{A(x^2 + x - 2) + B(x - 1) + C(x^2 + 4x + 4)}{(x + 2)^2(x - 1)}$$

$$= \dfrac{Ax^2 + Cx^2 + Ax + Bx + 4Cx - 2A - B + 4C}{(x + 2)^2(x - 1)}$$

$$= \dfrac{(A + C)x^2 + (A + B + 4C)x + (-2A - B + 4C)}{(x + 2)^2(x - 1)}$$

✓ **CHECKPOINT 3**

Perform each indicated addition of rational expressions.

**a.** $\dfrac{A}{x + 1} + \dfrac{B}{x - 1} + \dfrac{C}{x + 2}$

**b.** $\dfrac{A}{x + 1} + \dfrac{B}{(x + 1)^2} + \dfrac{C}{x - 2}$ ■

## Expressions Involving Radicals

In calculus, the operation of differentiation tends to produce "messy" expressions when applied to fractional expressions. This is especially true when the fractional expressions involve radicals. When differentiation is used, it is important to be able to simplify these expressions so that you can obtain more manageable forms. All of the expressions in Examples 4 and 5 are the results of differentiation. In each case, note how much *simpler* the simplified form is than the original form.

### *Example 4* Simplifying an Expression with Radicals

Simplify each expression.

**a.** $\dfrac{\sqrt{x+1} - \dfrac{x}{2\sqrt{x+1}}}{x+1}$ **b.** $\left(\dfrac{1}{x+\sqrt{x^2+1}}\right)\left(1 + \dfrac{2x}{2\sqrt{x^2+1}}\right)$

**SOLUTION**

**a.** $\dfrac{\sqrt{x+1} - \dfrac{x}{2\sqrt{x+1}}}{x+1} = \dfrac{\dfrac{2(x+1)}{2\sqrt{x+1}} - \dfrac{x}{2\sqrt{x+1}}}{x+1}$  Write with common denominator.

$= \dfrac{\dfrac{2x+2-x}{2\sqrt{x+1}}}{\dfrac{x+1}{1}}$  Subtract fractions.

$= \dfrac{x+2}{2\sqrt{x+1}}\left(\dfrac{1}{x+1}\right)$  To divide, invert and multiply

$= \dfrac{x+2}{2(x+1)^{3/2}}$  Multiply.

**b.** $\left(\dfrac{1}{x+\sqrt{x^2+1}}\right)\left(1 + \dfrac{2x}{2\sqrt{x^2+1}}\right)$

$= \left(\dfrac{1}{x+\sqrt{x^2+1}}\right)\left(1 + \dfrac{x}{\sqrt{x^2+1}}\right)$

$= \left(\dfrac{1}{x+\sqrt{x^2+1}}\right)\left(\dfrac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \dfrac{x}{\sqrt{x^2+1}}\right)$

$= \left(\dfrac{1}{x+\sqrt{x^2+1}}\right)\left(\dfrac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)$

$= \dfrac{1}{\sqrt{x^2+1}}$

### ✓ CHECKPOINT 4

Simplify each expression.

**a.** $\dfrac{\sqrt{x+2} - \dfrac{x}{4\sqrt{x+2}}}{x+2}$ **b.** $\left(\dfrac{1}{x+\sqrt{x^2+4}}\right)\left(1 + \dfrac{x}{\sqrt{x^2+4}}\right)$ ■

*Example 5* **Simplifying an Expression with Radicals**

Simplify the expression.

$$\frac{-x\left(\dfrac{2x}{2\sqrt{x^2+1}}\right)+\sqrt{x^2+1}}{x^2}+\left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)$$

**SOLUTION** From Example 4(b), you already know that the second part of this sum simplifies to $1/\sqrt{x^2+1}$. The first part simplifies as shown.

$$\frac{-x\left(\dfrac{2x}{2\sqrt{x^2+1}}\right)+\sqrt{x^2+1}}{x^2}=\frac{-x^2}{x^2\sqrt{x^2+1}}+\frac{\sqrt{x^2+1}}{x^2}$$

$$=\frac{-x^2}{x^2\sqrt{x^2+1}}+\frac{x^2+1}{x^2\sqrt{x^2+1}}$$

$$=\frac{-x^2+x^2+1}{x^2\sqrt{x^2+1}}$$

$$=\frac{1}{x^2\sqrt{x^2+1}}$$

So, the sum is

$$\frac{-x\left(\dfrac{2x}{2\sqrt{x^2+1}}\right)+\sqrt{x^2+1}}{x^2}+\left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)$$

$$=\frac{1}{x^2\sqrt{x^2+1}}+\frac{1}{\sqrt{x^2+1}}$$

$$=\frac{1}{x^2\sqrt{x^2+1}}+\frac{x^2}{x^2\sqrt{x^2+1}}$$

$$=\frac{x^2+1}{x^2\sqrt{x^2+1}}$$

$$=\frac{\sqrt{x^2+1}}{x^2}.$$

✓**CHECKPOINT 5**

Simplify the expression.

$$\frac{-x\left(\dfrac{3x}{3\sqrt{x^2+4}}\right)+\sqrt{x^2+4}}{x^2}+\left(\frac{1}{x+\sqrt{x^2+4}}\right)\left(1+\frac{3x}{3\sqrt{x^2+4}}\right)\ \blacksquare$$

**STUDY TIP**

To check that the simplified expression in Example 5 is equivalent to the original expression, try substituting values of $x$ into each expression. For instance, when you substitute $x=1$ into each expression, you obtain $\sqrt{2}$.

## Rationalization Techniques

In working with quotients involving radicals, it is often convenient to move the radical expression from the denominator to the numerator, or vice versa. For example, you can move $\sqrt{2}$ from the denominator to the numerator in the following quotient by multiplying by $\sqrt{2}/\sqrt{2}$.

*Radical in Denominator*     *Rationalize*     *Radical in Numerator*

$$\frac{1}{\sqrt{2}} \implies \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) \implies \frac{\sqrt{2}}{2}$$

This process is called **rationalizing the denominator.** A similar process is used to **rationalize the numerator.**

**STUDY TIP**

The success of the second and third rationalizing techniques stems from the following.

$$\left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right)$$
$$= a - b$$

---

**Rationalizing Techniques**

**1.** If the denominator is $\sqrt{a}$, multiply by $\dfrac{\sqrt{a}}{\sqrt{a}}$.

**2.** If the denominator is $\sqrt{a} - \sqrt{b}$, multiply by $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$.

**3.** If the denominator is $\sqrt{a} + \sqrt{b}$, multiply by $\dfrac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}}$.

The same guidelines apply to rationalizing numerators.

---

**Example 6**   **Rationalizing Denominators and Numerators**

Rationalize the denominator or numerator.

**a.** $\dfrac{3}{\sqrt{12}}$    **b.** $\dfrac{\sqrt{x+1}}{2}$    **c.** $\dfrac{1}{\sqrt{5} + \sqrt{2}}$    **d.** $\dfrac{1}{\sqrt{x} - \sqrt{x+1}}$

**SOLUTION**

**a.** $\dfrac{3}{\sqrt{12}} = \dfrac{3}{2\sqrt{3}} = \dfrac{3}{2\sqrt{3}}\left(\dfrac{\sqrt{3}}{\sqrt{3}}\right) = \dfrac{3\sqrt{3}}{2(3)} = \dfrac{\sqrt{3}}{2}$

**b.** $\dfrac{\sqrt{x+1}}{2} = \dfrac{\sqrt{x+1}}{2}\left(\dfrac{\sqrt{x+1}}{\sqrt{x+1}}\right) = \dfrac{x+1}{2\sqrt{x+1}}$

**c.** $\dfrac{1}{\sqrt{5} + \sqrt{2}} = \dfrac{1}{\sqrt{5} + \sqrt{2}}\left(\dfrac{\sqrt{5} - \sqrt{2}}{\sqrt{5} - \sqrt{2}}\right) = \dfrac{\sqrt{5} - \sqrt{2}}{5 - 2} = \dfrac{\sqrt{5} - \sqrt{2}}{3}$

**d.** $\dfrac{1}{\sqrt{x} - \sqrt{x+1}} = \dfrac{1}{\sqrt{x} - \sqrt{x+1}}\left(\dfrac{\sqrt{x} + \sqrt{x+1}}{\sqrt{x} + \sqrt{x+1}}\right)$

$$= \dfrac{\sqrt{x} + \sqrt{x+1}}{x - (x+1)}$$

$$= -\sqrt{x} - \sqrt{x+1}$$

**✓ CHECKPOINT 6**

Rationalize the denominator or numerator.

**a.** $\dfrac{5}{\sqrt{8}}$

**b.** $\dfrac{\sqrt{x+2}}{4}$

**c.** $\dfrac{1}{\sqrt{6} - \sqrt{3}}$

**d.** $\dfrac{1}{\sqrt{x} + \sqrt{x+2}}$ ∎

## Exercises 0.5

In Exercises 1–16, perform the indicated operations and simplify your answer.

**1.** $\dfrac{x}{x-2}+\dfrac{3}{x-2}$

**2.** $\dfrac{2x-1}{x+3}+\dfrac{1-x}{x+3}$

**3.** $\dfrac{2x}{x^2+2}-\dfrac{1-3x}{x^2+2}$

**4.** $\dfrac{5x+10}{2x-1}-\dfrac{2x+10}{2x-1}$

**5.** $\dfrac{2}{x^2-4}-\dfrac{1}{x-2}$

**6.** $\dfrac{x}{x^2+x-2}-\dfrac{1}{x+2}$

**7.** $\dfrac{5}{x-3}+\dfrac{3}{3-x}$

**8.** $\dfrac{x}{2-x}+\dfrac{2}{x-2}$

**9.** $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+2}$

**10.** $\dfrac{A}{x-5}+\dfrac{B}{x+5}+\dfrac{C}{(x+5)^2}$

**11.** $\dfrac{A}{x-6}+\dfrac{Bx+C}{x^2+3}$

**12.** $\dfrac{Ax+B}{x^2+2}+\dfrac{C}{x-4}$

**13.** $-\dfrac{2}{x}+\dfrac{1}{x^2+2}$

**14.** $\dfrac{2}{x+1}+\dfrac{1-x}{x^2-2x+3}$

**15.** $\dfrac{1}{x^2-x-2}-\dfrac{x}{x^2-5x+6}$

**16.** $\dfrac{x-1}{x^2+5x+4}+\dfrac{2}{x^2-x-2}+\dfrac{10}{x^2+2x-8}$

In Exercises 17–28, simplify each expression.

**17.** $\dfrac{-x}{(x+1)^{3/2}}+\dfrac{2}{(x+1)^{1/2}}$

**18.** $2\sqrt{x}(x-2)+\dfrac{(x-2)^2}{2\sqrt{x}}$

**19.** $\dfrac{2-t}{2\sqrt{1+t}}-\sqrt{1+t}$

**20.** $-\dfrac{\sqrt{x^2+1}}{x^2}+\dfrac{1}{\sqrt{x^2+1}}$

**21.** $\left(2x\sqrt{x^2+1}-\dfrac{x^3}{\sqrt{x^2+1}}\right)\div(x^2+1)$

**22.** $\left(\sqrt{x^3+1}-\dfrac{3x^3}{2\sqrt{x^3+1}}\right)\div(x^3+1)$

**23.** $\dfrac{(x^2+2)^{1/2}-x^2(x^2+2)^{-1/2}}{x^2}$

**24.** $\dfrac{x(x+1)^{-1/2}-(x+1)^{1/2}}{x^2}$

**25.** $\dfrac{\dfrac{\sqrt{x+1}}{\sqrt{x}}-\dfrac{\sqrt{x}}{\sqrt{x+1}}}{2(x+1)}$

**26.** $\dfrac{\dfrac{2x^2}{3(x^2-1)^{2/3}}-(x^2-1)^{1/3}}{x^2}$

**27.** $\dfrac{-x^2}{(2x+3)^{3/2}}+\dfrac{2x}{(2x+3)^{1/2}}$

**28.** $\dfrac{-x}{2(3+x^2)^{3/2}}+\dfrac{3}{(3+x^2)^{1/2}}$

In Exercises 29–44, rationalize the numerator or denominator and simplify.

**29.** $\dfrac{2}{\sqrt{10}}$

**30.** $\dfrac{3}{\sqrt{21}}$

**31.** $\dfrac{4x}{\sqrt{x-1}}$

**32.** $\dfrac{5y}{\sqrt{y+7}}$

**33.** $\dfrac{49(x-3)}{\sqrt{x^2-9}}$

**34.** $\dfrac{10(x+2)}{\sqrt{x^2-x-6}}$

**35.** $\dfrac{5}{\sqrt{14}-2}$

**36.** $\dfrac{13}{6+\sqrt{10}}$

**37.** $\dfrac{2x}{5-\sqrt{3}}$

**38.** $\dfrac{x}{\sqrt{2}+\sqrt{3}}$

**39.** $\dfrac{1}{\sqrt{6}+\sqrt{5}}$

**40.** $\dfrac{\sqrt{15}+3}{12}$

**41.** $\dfrac{2}{\sqrt{x}+\sqrt{x-2}}$

**42.** $\dfrac{10}{\sqrt{x}+\sqrt{x+5}}$

**43.** $\dfrac{\sqrt{x+2}-\sqrt{2}}{x}$

**44.** $\dfrac{\sqrt{x+1}-1}{x}$

In Exercises 45 and 46, perform the indicated operations and rationalize as needed.

**45.** $\dfrac{\dfrac{\sqrt{4-x^2}}{x^4}-\dfrac{2}{x^2\sqrt{4-x^2}}}{4-x^2}$

**46.** $\dfrac{\dfrac{\sqrt{x^2+1}}{x^2}-\dfrac{1}{x\sqrt{x^2+1}}}{x^2+1}$

**(T) 47. Installment Loan** The monthly payment $M$ for an installment loan is given by the formula

$$M=P\left[\dfrac{r/12}{1-\left(\dfrac{1}{(r/12)+1}\right)^N}\right]$$

where $P$ is the amount of the loan, $r$ is the annual percentage rate, and $N$ is the number of monthly payments. Enter the formula into a graphing utility, and use it to find the monthly payment for a loan of \$10,000 at an annual percentage rate of 7.5% ($r=0.075$) for 5 years ($N=60$ monthly payments).

**48. *MAKE A DECISION: INVENTORY*** A retailer has determined that the cost $C$ of ordering and storing $x$ units of a product is

$$C=6x+\dfrac{900{,}000}{x}.$$

(a) Write the expression for cost as a single fraction.

(b) Which order size should the retailer place: 240 units, 387 units, or 480 units? Explain your reasoning.

# Functions, Graphs, and Limits

# 1

A graph showing changes in a company's earnings and other financial indicators can depict the company's general financial trends over time. (See Section 1.2, Example 8.)

AP/Wide World Photos

## Applications

Functions and limit concepts have many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Health, Exercise 36, page 42
- Federal Education Spending, Exercise 70, page 55
- Profit Analysis, Exercise 93, page 67
- Make a Decision: Choosing a Job, Exercise 95, page 67
- Prescription Drugs, Exercise 63, page 80
- Consumer Awareness, Exercise 61, page 104

## Section 1.1

# The Cartesian Plane and the Distance Formula

- Plot points in a coordinate plane and read data presented graphically.
- Find the distance between two points in a coordinate plane.
- Find the midpoints of line segments connecting two points.
- Translate points in a coordinate plane.

## The Cartesian Plane
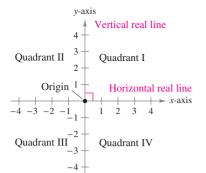


**FIGURE 1.1**    The Cartesian Plane



**FIGURE 1.2**
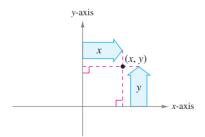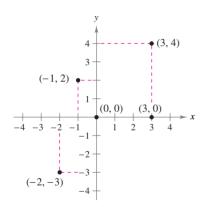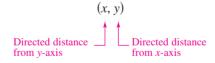
Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane called the **rectangular coordinate system,** or the **Cartesian plane,** after the French mathematician René Descartes (1596–1650).

The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure 1.1. The horizontal real number line is usually called the **x-axis,** and the vertical real number line is usually called the **y-axis.** The point of intersection of these two axes is the **origin,** and the two axes divide the plane into four parts called **quadrants.**

Each point in the plane corresponds to an **ordered pair** $(x, y)$ of real numbers $x$ and $y$, called **coordinates** of the point. The **x-coordinate** represents the directed distance from the $y$-axis to the point, and the **y-coordinate** represents the directed distance from the $x$-axis to the point, as shown in Figure 1.2.

$$(x, y)$$

Directed distance from $y$-axis ⌐ ⌐ Directed distance from $x$-axis

### STUDY TIP

The notation $(x, y)$ denotes both a point in the plane and an open interval on the real number line. The context will tell you which meaning is intended.

### *Example 1*   Plotting Points in the Cartesian Plane

Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$.

**SOLUTION**    To plot the point

$$(-1, 2)$$

$x$-coordinate ⌐     ⌐ $y$-coordinate

imagine a vertical line through $-1$ on the $x$-axis and a horizontal line through 2 on the $y$-axis. The intersection of these two lines is the point $(-1, 2)$. The other four points can be plotted in a similar way and are shown in Figure 1.3.



**FIGURE 1.3**

### ✓ CHECKPOINT 1

Plot the points $(-3, 2)$, $(4, -2)$, $(3, 1)$, $(0, -2)$, and $(-1, -2)$. ■

Using a rectangular coordinate system allows you to visualize relationships between two variables. In Example 2, notice how much your intuition is enhanced by the use of a graphical presentation.

**Amounts Spent on Snowmobiles**

**FIGURE 1.4**

### *Example 2*   Sketching a Scatter Plot ®

The amounts *A* (in millions of dollars) spent on snowmobiles in the United States from 1997 through 2006 are shown in the table, where *t* represents the year. Sketch a scatter plot of the data. *(Source: International Snowmobile Manufacturers Association)*

| *t* | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 |
|---|---|---|---|---|---|---|---|---|---|---|
| *A* | 1006 | 975 | 883 | 821 | 894 | 817 | 779 | 712 | 826 | 741 |

**SOLUTION**    To sketch a *scatter plot* of the data given in the table, you simply represent each pair of values by an ordered pair $(t, A)$, and plot the resulting points, as shown in Figure 1.4. For instance, the first pair of values is represented by the ordered pair (1997, 1006). Note that the break in the *t*-axis indicates that the numbers between 0 and 1996 have been omitted.

### STUDY TIP

In Example 2, you could let $t = 1$ represent the year 1997. In that case, the horizontal axis would not have been broken, and the tick marks would have been labeled 1 through 10 (instead of 1997 through 2006).

### ✓ CHECKPOINT 2
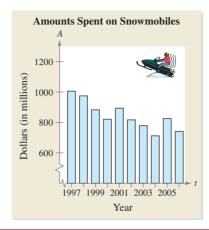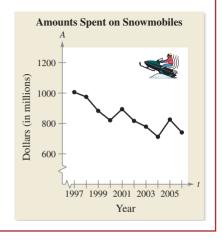
From 1995 through 2004, the enrollments *E* (in millions) of students in U.S. public colleges are shown, where *t* represents the year. Sketch a scatter plot of the data. *(Source: U.S. National Center for Education Statistics)*

| *t* | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 |
|---|---|---|---|---|---|---|---|---|---|---|
| *E* | 11.1 | 11.1 | 11.2 | 11.1 | 11.3 | 11.8 | 12.2 | 12.8 | 12.9 | 13.0 |

### TECHNOLOGY

Ⓣ    The scatter plot in Example 2 is only one way to represent the given data graphically. Two other techniques are shown at the right. The first is a *bar graph* and the second is a *line graph*. All three graphical representations were created with a computer. If you have access to computer graphing software, try using it to represent graphically the data given in Example 2.

The symbol ® indicates an example that uses or is derived from real-life data.

$$a^2 + b^2 = c^2$$
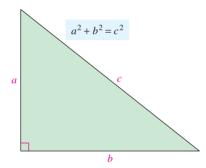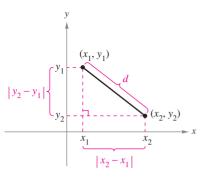
**FIGURE 1.5**   Pythagorean Theorem

**FIGURE 1.6**   Distance Between Two Points
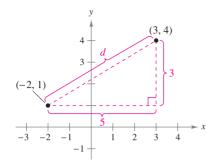
**FIGURE 1.7**

## The Distance Formula

Recall from the Pythagorean Theorem that, for a right triangle with hypotenuse of length $c$ and sides of lengths $a$ and $b$, you have

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean Theorem}$$

as shown in Figure 1.5. (The converse is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.)

Suppose you want to determine the distance $d$ between two points $(x_1, y_1)$ and $(x_2, y_2)$ in the plane. With these two points, a right triangle can be formed, as shown in Figure 1.6. The length of the vertical side of the triangle is

$$|y_2 - y_1|$$

and the length of the horizontal side is

$$|x_2 - x_1|.$$

By the Pythagorean Theorem, you can write

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This result is the **Distance Formula.**

### The Distance Formula

The distance $d$ between the points $(x_1, y_1)$ and $(x_2, y_2)$ in the plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

### *Example 3*   Finding a Distance

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

**SOLUTION**   Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$. Then apply the Distance Formula as shown.

$$\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &&\text{Distance Formula} \\
&= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} &&\text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\
&= \sqrt{(5)^2 + (3)^2} &&\text{Simplify.} \\
&= \sqrt{34} \\
&\approx 5.83 &&\text{Use a calculator.}
\end{aligned}$$

Note in Figure 1.7 that a distance of 5.83 looks about right.

### ✓ **CHECKPOINT 3**

Find the distance between the points $(-2, 1)$ and $(2, 4)$. ■

**FIGURE 1.8**

*Example 4*   **Verifying a Right Triangle**

Use the Distance Formula to show that the points $(2, 1)$, $(4, 0)$, and $(5, 7)$ are vertices of a right triangle.

**SOLUTION**   The three points are plotted in Figure 1.8. Using the Distance Formula, you can find the lengths of the three sides as shown below.

$$d_1 = \sqrt{(5 - 2)^2 + (7 - 1)^2} = \sqrt{9 + 36} = \sqrt{45}$$
$$d_2 = \sqrt{(4 - 2)^2 + (0 - 1)^2} = \sqrt{4 + 1} = \sqrt{5}$$
$$d_3 = \sqrt{(5 - 4)^2 + (7 - 0)^2} = \sqrt{1 + 49} = \sqrt{50}$$

Because

$$d_1^2 + d_2^2 = 45 + 5 = 50 = d_3^2$$

you can apply the converse of the Pythagorean Theorem to conclude that the triangle must be a right triangle.

✓**CHECKPOINT 4**

Use the Distance Formula to show that the points $(2, -1)$, $(5, 5)$, and $(6, -3)$ are vertices of a right triangle. ■

The figures provided with Examples 3 and 4 were not really essential to the solution. *Nevertheless*, we strongly recommend that you develop the habit of including sketches with your solutions—even if they are not required.

*Example 5*   **Finding the Length of a Pass**   Ⓡ

In a football game, a quarterback throws a pass from the 5-yard line, 20 yards from the sideline. The pass is caught by a wide receiver on the 45-yard line, 50 yards from the same sideline, as shown in Figure 1.9. How long was the pass?

**SOLUTION**   You can find the length of the pass by finding the distance between the points $(20, 5)$ and $(50, 45)$.

$$d = \sqrt{(50 - 20)^2 + (45 - 5)^2} \qquad \text{Distance Formula}$$
$$= \sqrt{900 + 1600}$$
$$= 50 \qquad\qquad\qquad\qquad \text{Simplify.}$$

So, the pass was 50 yards long.



**FIGURE 1.9**

✓**CHECKPOINT 5**

A quarterback throws a pass from the 10-yard line, 10 yards from the sideline. The pass is caught by a wide receiver on the 30-yard line, 25 yards from the same sideline. How long was the pass? ■

**STUDY TIP**

In Example 5, the scale along the goal line showing distance from the sideline does not normally appear on a football field. However, when you use coordinate geometry to solve real-life problems, you are free to place the coordinate system in any way that is convenient to the solution of the problem.

## The Midpoint Formula

To find the **midpoint** of the line segment that joins two points in a coordinate plane, you can simply find the average values of the respective coordinates of the two endpoints.

---

**The Midpoint Formula**

The midpoint of the segment joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$\text{Midpoint} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

---



**FIGURE 1.10**

*Example 6*   **Finding a Segment's Midpoint**

Find the midpoint of the line segment joining the points $(-5, -3)$ and $(9, 3)$, as shown in Figure 1.10.

**SOLUTION**   Let $(x_1, y_1) = (-5, -3)$ and $(x_2, y_2) = (9, 3)$.

$$\text{Midpoint} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{-5 + 9}{2}, \frac{-3 + 3}{2} \right) = (2, 0)$$
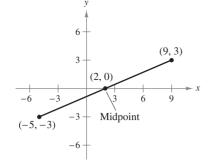
✓ **CHECKPOINT 6**

Find the midpoint of the line segment joining $(-6, 2)$ and $(2, 8)$. ■



**FIGURE 1.11**

*Example 7*   **Estimating Annual Sales**   ®
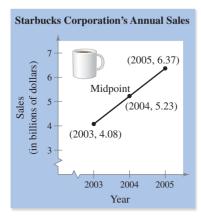
Starbucks Corporation had annual sales of $4.08 billion in 2003 and $6.37 billion in 2005. Without knowing any additional information, what would you estimate the 2004 sales to have been?   *(Source: Starbucks Corp.)*

**SOLUTION**   One solution to the problem is to assume that sales followed a linear pattern. With this assumption, you can estimate the 2004 sales by finding the midpoint of the segment connecting the points (2003, 4.08) and (2005, 6.37).

$$\text{Midpoint} = \left( \frac{2003 + 2005}{2}, \frac{4.08 + 6.37}{2} \right) \approx (2004, 5.23)$$

So, you would estimate the 2004 sales to have been about $5.23 billion, as shown in Figure 1.11. (The actual 2004 sales were $5.29 billion.)

✓ **CHECKPOINT 7**

Whirlpool Corporation had annual sales of $12.18 billion in 2003 and $14.32 billion in 2005. What would you estimate the 2004 annual sales to have been? *(Source: Whirlpool Corp.)* ■

## Translating Points in the Plane

*Example 8*   **Translating Points in the Plane**
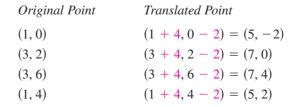
Figure 1.12(a) shows the vertices of a parallelogram. Find the vertices of the parallelogram after it has been translated two units down and four units to the right.

**SOLUTION**   To translate each vertex two units down, subtract 2 from each $y$-coordinate. To translate each vertex four units to the right, add 4 to each $x$-coordinate.

| Original Point | Translated Point |
| --- | --- |
| $(1, 0)$ | $(1 + 4, 0 - 2) = (5, -2)$ |
| $(3, 2)$ | $(3 + 4, 2 - 2) = (7, 0)$ |
| $(3, 6)$ | $(3 + 4, 6 - 2) = (7, 4)$ |
| $(1, 4)$ | $(1 + 4, 4 - 2) = (5, 2)$ |

The translated parallelogram is shown in Figure 1.12(b).



(a)                                          (b)
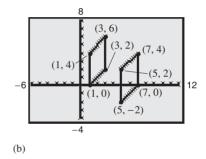
**FIGURE 1.12**

✓ **CHECKPOINT 8**

Find the vertices of the parallelogram in Example 8 after it has been translated two units to the left and four units down. ◼

Walt Disney/The Kobal Collection

Many movies now use extensive computer graphics, much of which consists of transformations of points in two- and three-dimensional space. The photo above shows a character from *Pirates of the Caribbean: Dead Man's Chest*. The movie's animators used computer graphics to design the scenery, characters, motion, and even the lighting throughout much of the film.

---

**CONCEPT CHECK**

1. What is the $y$-coordinate of any point on the $x$-axis? What is the $x$-coordinate of any point on the $y$-axis?

2. Describe the signs of the $x$- and $y$-coordinates of points that lie in the first and second quadrants.

3. To divide a line segment into four equal parts, how many times is the Midpoint Formula used?

4. When finding the distance between two points, does it matter which point is chosen as $(x_1, y_1)$? Explain.

In Exercises 1–6, simplify each expression.

**1.** $\sqrt{(3-6)^2 + [1-(-5)]^2}$

**2.** $\sqrt{(-2-0)^2 + [-7-(-3)]^2}$

**3.** $\dfrac{5 + (-4)}{2}$

**4.** $\dfrac{-3 + (-1)}{2}$

**5.** $\sqrt{27} + \sqrt{12}$

**6.** $\sqrt{8} - \sqrt{18}$

In Exercises 7–10, solve for $x$ or $y$.

**7.** $\sqrt{(3-x)^2 + (7-4)^2} = \sqrt{45}$

**8.** $\sqrt{(6-2)^2 + (-2-y)^2} = \sqrt{52}$

**9.** $\dfrac{x + (-5)}{2} = 7$

**10.** $\dfrac{-7 + y}{2} = -3$

---

# Exercises 1.1

In Exercises 1 and 2, plot the points in the Cartesian plane.

**1.** $(-5, 3), (1, -1), (-2, -4), (2, 0), (1, -6)$

**2.** $(0, -4), (5, 1), (-3, 5), (2, -2), (-6, -1)$

In Exercises 3–12, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

**3.** $(3, 1), (5, 5)$

**4.** $(-3, 2), (3, -2)$

**5.** $\left(\frac{1}{2}, 1\right), \left(-\frac{3}{2}, -5\right)$

**6.** $\left(\frac{2}{3}, -\frac{1}{3}\right), \left(\frac{5}{6}, 1\right)$

**7.** $(2, 2), (4, 14)$

**8.** $(-3, 7), (1, -1)$

**9.** $\left(1, \sqrt{3}\right), (-1, 1)$

**10.** $(-2, 0), \left(0, \sqrt{2}\right)$

**11.** $(0, -4.8), (0.5, 6)$
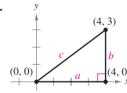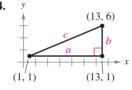
**12.** $(5.2, 6.4), (-2.7, 1.8)$
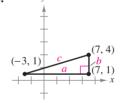
In Exercises 13–16, (a) find the length of each side of the right triangle and (b) show that these lengths satisfy the Pythagorean Theorem.

**13.**



**14.**
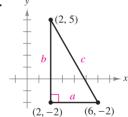


**15.**



**16.**



In Exercises 17–20, show that the points form the vertices of the given figure. (A rhombus is a quadrilateral whose sides have the same length.)

| Vertices | Figure |
|---|---|
| **17.** $(0, 1), (3, 7), (4, -1)$ | Right triangle |
| **18.** $(1, -3), (3, 2), (-2, 4)$ | Isosceles triangle |
| **19.** $(0, 0), (1, 2), (2, 1), (3, 3)$ | Rhombus |
| **20.** $(0, 1), (3, 7), (4, 4), (1, -2)$ | Parallelogram |

In Exercises 21 and 22, find $x$ such that the distance between the points is 5.
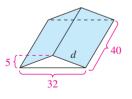
**21.** $(1, 0), (x, -4)$

**22.** $(2, -1), (x, 2)$

In Exercises 23 and 24, find $y$ such that the distance between the points is 8.

**23.** $(0, 0), (3, y)$

**24.** $(5, 1), (5, y)$

---

The answers to the odd-numbered and selected even-numbered exercises are given in the back of the text. Worked-out solutions to the odd-numbered exercises are given in the *Student Solutions Guide*.

**25. Building Dimensions**  The base and height of the trusses for the roof of a house are 32 feet and 5 feet, respectively (see figure).

(a) Find the distance from the eaves to the peak of the roof.

(b) The length of the house is 40 feet. Use the result of part (a) to find the number of square feet of roofing.
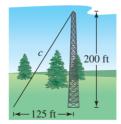
Figure for 25                    Figure for 26

**26. Wire Length**  A guy wire is stretched from a broadcasting tower at a point 200 feet above the ground to an anchor 125 feet from the base (see figure). How long is the wire?

(T) In Exercises 27 and 28, use a graphing utility to graph a scatter plot, a bar graph, or a line graph to represent the data. Describe any trends that appear.

**27. Consumer Trends**  The numbers (in millions) of basic cable television subscribers in the United States for 1996 through 2005 are shown in the table.  *(Source: National Cable & Telecommunications Association)*

| Year | 1996 | 1997 | 1998 | 1999 | 2000 |
|------|------|------|------|------|------|
| Subscribers | 62.3 | 63.6 | 64.7 | 65.5 | 66.3 |

| Year | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|
| Subscribers | 66.7 | 66.5 | 66.0 | 65.7 | 65.3 |

**28. Consumer Trends**  The numbers (in millions) of cellular telephone subscribers in the United States for 1996 through 2005 are shown in the table.  *(Source: Cellular Telecommunications & Internet Association)*

| Year | 1996 | 1997 | 1998 | 1999 | 2000 |
|------|------|------|------|------|------|
| Subscribers | 44.0 | 55.3 | 69.2 | 86.0 | 109.5 |

| Year | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|
| Subscribers | 128.4 | 140.8 | 158.7 | 182.1 | 207.9 |

**Dow Jones Industrial Average**  In Exercises 29 and 30, use the figure below showing the Dow Jones Industrial Average for common stocks.  *(Source: Dow Jones, Inc.)*

**29.** Estimate the Dow Jones Industrial Average for each date.

(a) March 2005                    (b) December 2005

(c) May 2006                      (d) January 2007

**30.** Estimate the percent increase or decrease in the Dow Jones Industrial Average (a) from March 2005 to November 2005 and (b) from May 2006 to February 2007.

Figure for 29 and 30

**Construction**  In Exercises 31 and 32, use the figure, which shows the median sales prices of existing one-family homes sold (in thousands of dollars) in the United States from 1990 through 2005.  *(Source: National Association of Realtors)*

**31.** Estimate the median sales price of existing one-family homes for each year.

(a) 1990    (b) 1992    (c) 1997    (d) 2005

**32.** Estimate the percent increases in the value of existing one-family homes (a) from 1993 to 1994 and (b) from 2003 to 2004.
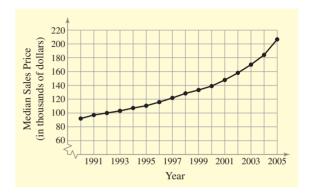
Figure for 31 and 32

The symbol (T) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by use of appropriate technology.

**Research Project**    In Exercises 33 and 34, (a) use the Midpoint Formula to estimate the revenue and profit of the company in 2003. (b) Then use your school's library, the Internet, or some other reference source to find the actual revenue and profit for 2003. (c) Did the revenue and profit increase in a linear pattern from 2001 to 2005? Explain your reasoning. (d) What were the company's expenses during each of the given years? (e) How would you rate the company's growth from 2001 to 2005?    *(Source: The Walt Disney Company and CVS Corporation)*

**33.** The Walt Disney Company

| Year | 2001 | 2003 | 2005 |
|---|---|---|---|
| Revenue (millions of $) | 25,269 | | 31,944 |
| Profit (millions of $) | 2058.0 | | 2729.0 |

**34.** CVS Corporation

| Year | 2001 | 2003 | 2005 |
|---|---|---|---|
| Revenue (millions of $) | 22,241 | | 37,006 |
| Profit (millions of $) | 638.0 | | 1172.1 |

Ⓑ **35. Economics**    The table shows the numbers of ear infections treated by doctors at HMO clinics of three different sizes: small, medium, and large.
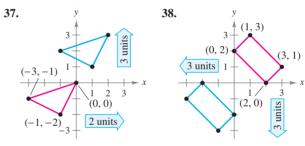
| Number of doctors | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Cases per small clinic | 0 | 20 | 28 | 35 | 40 |
| Cases per medium clinic | 0 | 30 | 42 | 53 | 60 |
| Cases per large clinic | 0 | 35 | 49 | 62 | 70 |

(a) Show the relationship between doctors and treated ear infections using *three* curves, where the number of doctors is on the horizontal axis and the number of ear infections treated is on the vertical axis.

(b) Compare the three relationships.    *(Source: Adapted from Taylor, Economics, Fifth Edition)*

**36. Health**    The percents of adults who are considered drinkers or smokers are shown in the table. Drinkers were defined as those who had five or more drinks in 1 day at least once during a recent year. Smokers were defined as those who smoked more than 100 cigarettes in their lifetime and smoked daily or semi-daily.    *(Source: National Health Interview Survey)*

| Year | 2001 | 2002 | 2003 | 2004 | 2005 |
|---|---|---|---|---|---|
| Drinkers | 20.0 | 19.9 | 19.1 | 19.1 | 19.5 |
| Smokers | 22.7 | 22.4 | 21.6 | 20.9 | 20.9 |

(a) Sketch a line graph of each data set.

(b) Describe any trends that appear.

Ⓣ **Computer Graphics**    In Exercises 37 and 38, the red figure is translated to a new position in the plane to form the blue figure. (a) Find the vertices of the transformed figure. (b) Then use a graphing utility to draw both figures.

**37.**



**38.**



**39.** Use the Midpoint Formula repeatedly to find the three points that divide the segment joining $(x_1, y_1)$ and $(x_2, y_2)$ into four equal parts.

**40.** Show that $\left(\frac{1}{3}[2x_1 + x_2], \frac{1}{3}[2y_1 + y_2]\right)$ is one of the points of trisection of the line segment joining $(x_1, y_1)$ and $(x_2, y_2)$. Then, find the second point of trisection by finding the midpoint of the segment joining

$$\left(\frac{1}{3}[2x_1 + x_2], \frac{1}{3}[2y_1 + y_2]\right) \text{ and } (x_2, y_2).$$

**41.** Use Exercise 39 to find the points that divide the line segment joining the given points into four equal parts.

(a) $(1, -2), (4, -1)$     (b) $(-2, -3), (0, 0)$

**42.** Use Exercise 40 to find the points of trisection of the line segment joining the given points.

(a) $(1, -2), (4, 1)$     (b) $(-2, -3), (0, 0)$

---

The symbol Ⓑ indicates an exercise that contains material from textbooks in other disciplines.

## Section 1.2

# Graphs of Equations

- Sketch graphs of equations by hand.
- Find the *x*- and *y*-intercepts of graphs of equations.
- Write the standard forms of equations of circles.
- Find the points of intersection of two graphs.
- Use mathematical models to model and solve real-life problems.

## The Graph of an Equation

In Section 1.1, you used a coordinate system to represent graphically the relationship between two quantities. There, the graphical picture consisted of a collection of points in a coordinate plane (see Example 2 in Section 1.1).

Frequently, a relationship between two quantities is expressed as an equation. For instance, degrees on the Fahrenheit scale are related to degrees on the Celsius scale by the equation $F = \frac{9}{5}C + 32$. In this section, you will study some basic procedures for sketching the graphs of such equations. The **graph** of an equation is the set of all points that are solutions of the equation.

### *Example 1*    Sketching the Graph of an Equation

Sketch the graph of $y = 7 - 3x$.
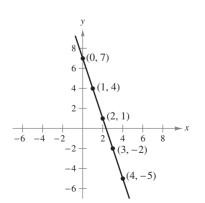
**SOLUTION**    The simplest way to sketch the graph of an equation is the *point-plotting method*. With this method, you construct a table of values that consists of several solution points of the equation, as shown in the table below. For instance, when $x = 0$

$$y = 7 - 3(0) = 7$$

which implies that $(0, 7)$ is a solution point of the graph.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y = 7 - 3x$ | 7 | 4 | 1 | $-2$ | $-5$ |

From the table, it follows that $(0, 7)$, $(1, 4)$, $(2, 1)$, $(3, -2)$, and $(4, -5)$ are solution points of the equation. After plotting these points, you can see that they appear to lie on a line, as shown in Figure 1.13. The graph of the equation is the line that passes through the five plotted points.



**FIGURE 1.13**    Solution Points for $y = 7 - 3x$

✓ **CHECKPOINT 1**

Sketch the graph of $y = 2x + 1$. ■

### STUDY TIP

Even though we refer to the sketch shown in Figure 1.13 as the graph of $y = 7 - 3x$, it actually represents only a *portion* of the graph. The entire graph is a line that would extend off the page.

*Example 2*   **Sketching the Graph of an Equation**

Sketch the graph of $y = x^2 - 2$.

**SOLUTION**   Begin by constructing a table of values, as shown below.

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|-----|------|------|---|---|---|---|
| $y = x^2 - 2$ | 2 | $-1$ | $-2$ | $-1$ | 2 | 7 |

Next, plot the points given in the table, as shown in Figure 1.14(a). Finally, connect the points with a smooth curve, as shown in Figure 1.14(b).



(a)                                    (b)
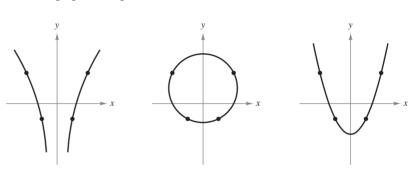
**FIGURE 1.14**

✓ **CHECKPOINT 2**
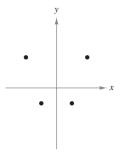
Sketch the graph of $y = x^2 - 4$. ■

The point-plotting technique demonstrated in Examples 1 and 2 is easy to use, but it does have some shortcomings. With too few solution points, you can badly misrepresent the graph of a given equation. For instance, how would you connect the four points in Figure 1.15? Without further information, any one of the three graphs in Figure 1.16 would be reasonable.
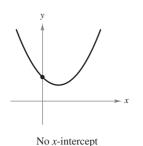


**FIGURE 1.15**
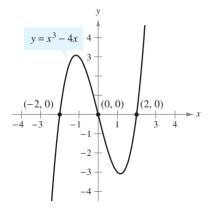


**FIGURE 1.16**

---

**STUDY TIP**

The graph shown in Example 2 is a **parabola.** The graph of any second-degree equation of the form

$$y = ax^2 + bx + c, \quad a \neq 0$$

has a similar shape. If $a > 0$, the parabola opens upward, as in Figure 1.14(b), and if $a < 0$, the parabola opens downward.
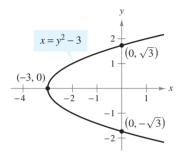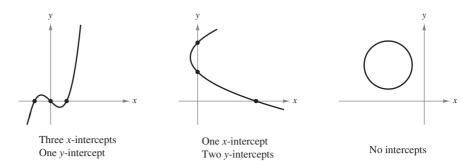
## Intercepts of a Graph

It is often easy to determine the solution points that have zero as either the $x$-coordinate or the $y$-coordinate. These points are called **intercepts** because they are the points at which the graph intersects the $x$- or $y$-axis.

Some texts denote the $x$-intercept as the $x$-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, we will use the term *intercept* to mean either the point or the coordinate.

A graph may have no intercepts or several intercepts, as shown in Figure 1.17.



No $x$-intercept
One $y$-intercept

Three $x$-intercepts
One $y$-intercept

One $x$-intercept
Two $y$-intercepts

No intercepts

**FIGURE 1.17**



$y = x^3 - 4x$

$(-2, 0)$  $(0, 0)$  $(2, 0)$

**FIGURE 1.18**

### Finding Intercepts

1. To find **$x$-intercepts,** let $y$ be zero and solve the equation for $x$.

2. To find **$y$-intercepts,** let $x$ be zero and solve the equation for $y$.

*Example 3*  **Finding $x$- and $y$-Intercepts**

Find the $x$- and $y$-intercepts of the graph of each equation.

**a.** $y = x^3 - 4x$     **b.** $x = y^2 - 3$

**SOLUTION**

**a.** Let $y = 0$. Then $0 = x(x^2 - 4) = x(x + 2)(x - 2)$ has solutions $x = 0$ and $x = \pm 2$. Let $x = 0$. Then $y = (0)^3 - 4(0) = 0$.

$x$-intercepts: $(0, 0), (2, 0), (-2, 0)$     $y$-intercept: $(0, 0)$     See Figure 1.18.

**b.** Let $y = 0$. Then $x = (0)^2 - 3 = -3$. Let $x = 0$. Then $y^2 - 3 = 0$ has solutions $y = \pm\sqrt{3}$.

$x$-intercept: $(-3, 0)$     $y$-intercepts: $(0, \sqrt{3}), (0, -\sqrt{3})$     See Figure 1.19.



$x = y^2 - 3$

$(0, \sqrt{3})$

$(-3, 0)$

$(0, -\sqrt{3})$

**FIGURE 1.19**

✓**CHECKPOINT 3**

Find the $x$- and $y$-intercepts of the graph of each equation.

**a.** $y = x^2 - 2x - 3$     **b.** $y^2 - 4 = x$  ▪

**TECHNOLOGY**
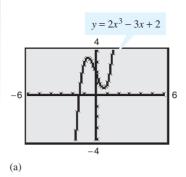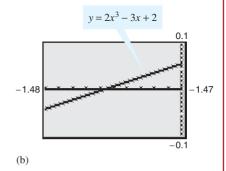
## Zooming in to Find Intercepts

(T) You can use the *zoom* feature of a graphing utility to approximate the *x*-intercepts of a graph. Suppose you want to approximate the *x*-intercept(s) of the graph of

$$y = 2x^3 - 3x + 2.$$

Begin by graphing the equation, as shown below in part (a). From the viewing window shown, the graph appears to have only one *x*-intercept. This intercept lies between $-2$ and $-1$. By zooming in on the intercept, you can improve the approximation, as shown in part (b). To three decimal places, the solution is $x \approx -1.476$.

(a)



(b)
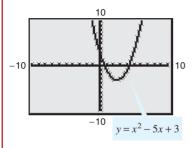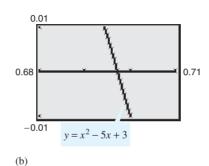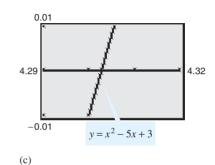
Here are some suggestions for using the *zoom* feature.

1. With each successive zoom-in, adjust the *x*-scale so that the viewing window shows at least one tick mark on each side of the *x*-intercept.

2. The error in your approximation will be less than the distance between two scale marks.

3. The *trace* feature can usually be used to add one more decimal place of accuracy without changing the viewing window.

Part (a) below shows the graph of $y = x^2 - 5x + 3$. Parts (b) and (c) show "zoom-in views" of the two intercepts. From these views, you can approximate the *x*-intercepts to be $x \approx 0.697$ and $x \approx 4.303$.
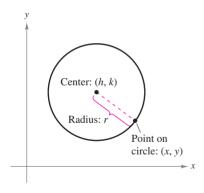


(a)



(b)



(c)

*Specific calculator keystroke instructions for operations in this and other technology boxes can be found at *college.hmco.com/info/larsonapplied*.
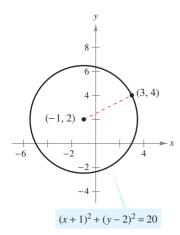
## Circles

Throughout this course, you will learn to recognize several types of graphs from their equations. For instance, you should recognize that the graph of a second-degree equation of the form

$$y = ax^2 + bx + c, \quad a \neq 0$$

is a parabola (see Example 2). Another easily recognized graph is that of a **circle.**

Consider the circle shown in Figure 1.20. A point $(x, y)$ is on the circle if and only if its distance from the center $(h, k)$ is $r$. By the Distance Formula,

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

By squaring both sides of this equation, you obtain the **standard form of the equation of a circle.**



**FIGURE 1.20**

---

**Standard Form of the Equation of a Circle**

The point $(x, y)$ lies on the circle of **radius** $r$ and **center** $(h, k)$ if and only if

$$(x - h)^2 + (y - k)^2 = r^2.$$

---

From this result, you can see that the standard form of the equation of a circle *with its center at the origin*, $(h, k) = (0, 0)$, is simply

$$x^2 + y^2 = r^2. \qquad \text{Circle with center at origin}$$

### *Example 4* Finding the Equation of a Circle

The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure 1.21. Find the standard form of the equation of this circle.

**SOLUTION** The radius of the circle is the distance between $(-1, 2)$ and $(3, 4)$.

$$
\begin{aligned}
r &= \sqrt{[3 - (-1)]^2 + (4 - 2)^2} && \text{Distance Formula} \\
&= \sqrt{16 + 4} && \text{Simplify.} \\
&= \sqrt{20} && \text{Radius}
\end{aligned}
$$

Using $(h, k) = (-1, 2)$, the standard form of the equation of the circle is

$$
\begin{aligned}
(x - h)^2 + (y - k)^2 &= r^2 \\
[x - (-1)]^2 + (y - 2)^2 &= \left(\sqrt{20}\right)^2 && \text{Substitute for } h, k, \text{ and } r. \\
(x + 1)^2 + (y - 2)^2 &= 20. && \text{Write in standard form.}
\end{aligned}
$$



**FIGURE 1.21**

### ✓ CHECKPOINT 4

The point $(1, 5)$ lies on a circle whose center is at $(-2, 1)$. Find the standard form of the equation of this circle. ■
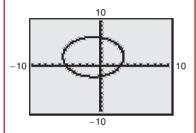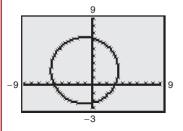
**General Form of the Equation of a Circle**

$$Ax^2 + Ay^2 + Dx + Ey + F = 0, \quad A \neq 0$$

To change from general form to standard form, you can use a process called **completing the square,** as demonstrated in Example 5.

### Example 5   Completing the Square

Sketch the graph of the circle whose general equation is
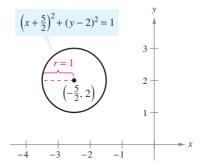
$$4x^2 + 4y^2 + 20x - 16y + 37 = 0.$$

**SOLUTION**   First divide by 4 so that the coefficients of $x^2$ and $y^2$ are both 1.

$$4x^2 + 4y^2 + 20x - 16y + 37 = 0 \qquad \text{Write original equation.}$$

$$x^2 + y^2 + 5x - 4y + \tfrac{37}{4} = 0 \qquad \text{Divide each side by 4.}$$

$$(x^2 + 5x + \boxed{\phantom{xx}}) + (y^2 - 4y + \boxed{\phantom{xx}}) = -\tfrac{37}{4} \qquad \text{Group terms.}$$

$$\left(x^2 + 5x + \tfrac{25}{4}\right) + (y^2 - 4y + 4) = -\tfrac{37}{4} + \tfrac{25}{4} + 4 \qquad \text{Complete the square.}$$

$$\underbrace{\qquad}_{(\text{Half})^2} \qquad \underbrace{\qquad}_{(\text{Half})^2}$$

$$\left(x + \tfrac{5}{2}\right)^2 + (y - 2)^2 = 1 \qquad \text{Write in standard form.}$$

From the standard form, you can see that the circle is centered at $\left(-\tfrac{5}{2}, 2\right)$ and has a radius of 1, as shown in Figure 1.22.



**FIGURE 1.22**

The general equation $Ax^2 + Ay^2 + Dx + Ey + F = 0$ may not always represent a circle. In fact, such an equation will have no solution points if the procedure of completing the square yields the impossible result

$$(x - h)^2 + (y - k)^2 = \text{negative number.} \qquad \text{No solution points}$$

### ✓ CHECKPOINT 5

Write the equation of the circle $x^2 + y^2 - 4x + 2y + 1 = 0$ in standard form and sketch its graph. ■
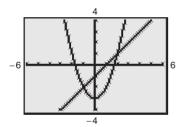
**FIGURE 1.23**
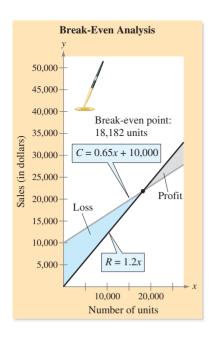
**FIGURE 1.24**

## Points of Intersection

A **point of intersection** of two graphs is an ordered pair that is a solution point of both graphs. For instance, Figure 1.23 shows that the graphs of

$$y = x^2 - 3 \quad \text{and} \quad y = x - 1$$

have two points of intersection: $(2, 1)$ and $(-1, -2)$. To find the points analytically, set the two *y*-values equal to each other and solve the equation

$$x^2 - 3 = x - 1$$

for *x*.

A common business application that involves points of intersection is **break-even analysis.** The marketing of a new product typically requires an initial investment. When sufficient units have been sold so that the total revenue has offset the total cost, the sale of the product has reached the **break-even point.** The **total cost** of producing *x* units of a product is denoted by *C*, and the **total revenue** from the sale of *x* units of the product is denoted by *R*. So, you can find the break-even point by setting the cost *C* equal to the revenue *R*, and solving for *x*.

*Example 6*
**MAKE A DECISION**    Finding a Break-Even Point    ®

A business manufactures a product at a cost of $0.65 per unit and sells the product for $1.20 per unit. The company's initial investment to produce the product was $10,000. Will the company break even if it sells 18,000 units? How many units must the company sell to break even?

**SOLUTION**    The total cost of producing *x* units of the product is given by

$$C = 0.65x + 10,000. \qquad \textcolor{red}{\text{Cost equation}}$$

The total revenue from the sale of *x* units is given by

$$R = 1.2x. \qquad \textcolor{red}{\text{Revenue equation}}$$

To find the break-even point, set the cost equal to the revenue and solve for *x*.

$$R = C \qquad \textcolor{red}{\text{Set revenue equal to cost.}}$$
$$1.2x = 0.65x + 10,000 \qquad \textcolor{red}{\text{Substitute for } R \text{ and } C.}$$
$$0.55x = 10,000 \qquad \textcolor{red}{\text{Subtract } 0.65x \text{ from each side.}}$$
$$x = \frac{10,000}{0.55} \qquad \textcolor{red}{\text{Divide each side by 0.55.}}$$
$$x \approx 18,182 \qquad \textcolor{red}{\text{Use a calculator.}}$$

No, the company will not break even if it sells 18,000 units. The company must sell 18,182 units before it breaks even. This result is shown graphically in Figure 1.24.

✓**CHECKPOINT 6**

How many units must the company in Example 6 sell to break even if the selling price is $1.45 per unit? ■
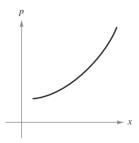
**FIGURE 1.25**    Supply Curve
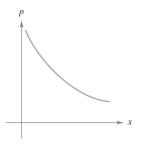


**FIGURE 1.26**    Demand Curve
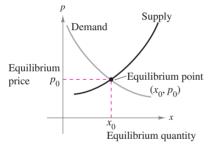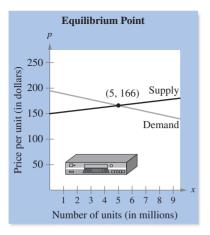


**FIGURE 1.27**    Equilibrium Point



**FIGURE 1.28**

Two types of applications that economists use to analyze a market are supply and demand equations. A **supply equation** shows the relationship between the unit price $p$ of a product and the quantity supplied $x$. The graph of a supply equation is called a **supply curve.** (See Figure 1.25.) A typical supply curve rises because producers of a product want to sell more units if the unit price is higher.

A **demand equation** shows the relationship between the unit price $p$ of a product and the quantity demanded $x$. The graph of a demand equation is called a **demand curve.** (See Figure 1.26.) A typical demand curve tends to show a decrease in the quantity demanded with each increase in price.

In an ideal situation, with no other factors present to influence the market, the production level should stabilize at the point of intersection of the graphs of the supply and demand equations. This point is called the **equilibrium point.** The $x$-coordinate of the equilibrium point is called the **equilibrium quantity** and the $p$-coordinate is called the **equilibrium price.** (See Figure 1.27.) You can find the equilibrium point by setting the demand equation equal to the supply equation and solving for $x$.

### *Example 7*  Finding the Equilibrium Point ®

The demand and supply equations for a DVD player are given by

$$p = 195 - 5.8x \qquad \text{Demand equation}$$
$$p = 150 + 3.2x \qquad \text{Supply equation}$$

where $p$ is the price in dollars and $x$ represents the number of units in millions. Find the equilibrium point for this market.

**SOLUTION**    Begin by setting the demand equation equal to the supply equation.

$$195 - 5.8x = 150 + 3.2x \qquad \text{Set equations equal to each other.}$$
$$45 - 5.8x = 3.2x \qquad \text{Subtract 150 from each side.}$$
$$45 = 9x \qquad \text{Add } 5.8x \text{ to each side.}$$
$$5 = x \qquad \text{Divide each side by 9.}$$

So, the equilibrium point occurs when the demand and supply are each five million units. (See Figure 1.28.) The price that corresponds to this $x$-value is obtained by substituting $x = 5$ into either of the original equations. For instance, substituting into the demand equation produces

$$p = 195 - 5.8(5) = 195 - 29 = \$166.$$

Substitute $x = 5$ into the supply equation to see that you obtain the same price.

### ✓CHECKPOINT 7

The demand and supply equations for a calculator are $p = 136 - 3.5x$ and $p = 112 + 2.5x$, respectively, where $p$ is the price in dollars and $x$ represents the number of units in millions. Find the equilibrium point for this market. ■

## Mathematical Models

In this text, you will see many examples of the use of equations as **mathematical models** of real-life phenomena. In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals— accuracy and simplicity.

*Example 8* **Using Mathematical Models** Ⓡ

The table shows the annual sales (in millions of dollars) for Dillard's and Kohl's for 2001 to 2005. In the summer of 2006, the publication *Value Line* listed the projected 2006 sales for the companies as $7625 million and $15,400 million, respectively. How do you think these projections were obtained? *(Source: Dillard's Inc. and Kohl's Corp.)*

| Year | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|
| $t$ | 1 | 2 | 3 | 4 | 5 |
| Dillard's | 8155 | 7911 | 7599 | 7529 | 7560 |
| Kohl's | 7489 | 9120 | 10,282 | 11,701 | 13,402 |



**Annual Sales**

FIGURE 1.29
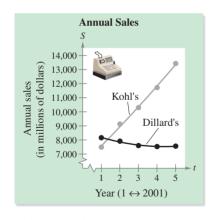
**SOLUTION** The projections were obtained by using past sales to predict future sales. The past sales were modeled by equations that were found by a statistical procedure called least squares regression analysis.

$$S = 56.57t^2 - 496.6t + 8618, \quad 1 \le t \le 5 \qquad \text{Dillard's}$$
$$S = 28.36t^2 + 1270.6t + 6275, \quad 1 \le t \le 5 \qquad \text{Kohl's}$$

Using $t = 6$ to represent 2006, you can predict the 2006 sales to be

$$S = 56.57(6)^2 - 496.6(6) + 8618 \approx 7675 \qquad \text{Dillard's}$$
$$S = 28.36(6)^2 + 1270.6(6) + 6275 \approx 14,920 \qquad \text{Kohl's}$$

These two projections are close to those projected by *Value Line*. The graphs of the two models are shown in Figure 1.29.

✓**CHECKPOINT 8**

The table shows the annual sales (in millions of dollars) for Dollar General for 1999 through 2005. In the summer of 2005, the publication *Value Line* listed projected 2006 sales for Dollar General as $9300 million. How does this projection compare with the projection obtained using the model below? *(Source: Dollar General Corp.)*

$$S = 16.246t^2 + 390.53t - 951.2, \quad 9 \le t \le 15$$
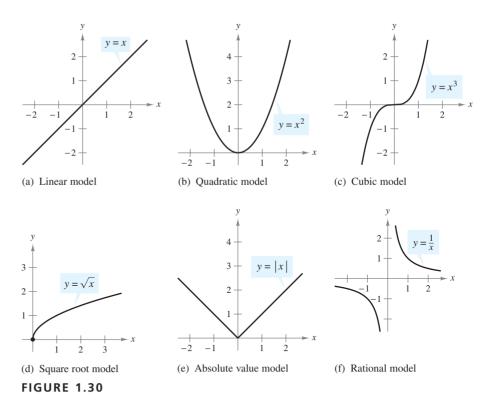
| Year | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|------|------|
| $t$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| Sales | 3888.0 | 4550.6 | 5322.9 | 6100.4 | 6872.0 | 7660.9 | 8582.2 |

## STUDY TIP

To test the accuracy of a model, you can compare the actual data with the values given by the model. For instance, the table below compares the actual Kohl's sales with those given by the model found in Example 8.

| Year | 2001 | 2002 | 2003 | 2004 | 2005 |
|---|---|---|---|---|---|
| Actual | 7489 | 9120 | 10,282 | 11,701 | 13,402 |
| Model | 7574.0 | 8929.6 | 10,342.0 | 11,811.2 | 13,337.0 |

Much of your study of calculus will center around the behavior of the graphs of mathematical models. Figure 1.30 shows the graphs of six basic algebraic equations. Familiarity with these graphs will help you in the creation and use of mathematical models.



(a) Linear model

(b) Quadratic model

(c) Cubic model

(d) Square root model

(e) Absolute value model

(f) Rational model

**FIGURE 1.30**

## CONCEPT CHECK

1. What does the graph of an equation represent?
2. Describe how to find the *x*- and *y*-intercepts of the graph of an equation.
3. How can you check that an ordered pair is a point of intersection of two graphs?
4. Can the graph of an equation have more than one *y*-intercept?

**Skills Review 1.2**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 0.4.

In Exercises 1–6, solve for $y$.

**1.** $5y - 12 = x$

**2.** $-y = 15 - x$

**3.** $x^3 y + 2y = 1$

**4.** $x^2 + x - y^2 - 6 = 0$

**5.** $(x - 2)^2 + (y + 1)^2 = 9$

**6.** $(x + 6)^2 + (y - 5)^2 = 81$

In Exercises 7–10, complete the square to write the expression as a perfect square trinomial.

**7.** $x^2 - 4x + \phantom{xx}$

**8.** $x^2 + 6x + \phantom{xx}$

**9.** $x^2 - 5x + \phantom{xx}$

**10.** $x^2 + 3x + \phantom{xx}$

In Exercises 11–14, factor the expression.

**11.** $x^2 - 3x + 2$

**12.** $x^2 + 5x + 6$

**13.** $y^2 - 3y + \frac{9}{4}$

**14.** $y^2 - 7y + \frac{49}{4}$

## Exercises 1.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, determine whether the points are solution points of the given equation.

**1.** $2x - y - 3 = 0$

  (a) $(1, 2)$    (b) $(1, -1)$    (c) $(4, 5)$

**2.** $7x + 4y - 6 = 0$

  (a) $(6, -9)$    (b) $(-5, 10)$    (c) $\left(\frac{1}{2}, \frac{5}{8}\right)$

**3.** $x^2 + y^2 = 4$

  (a) $\left(1, -\sqrt{3}\right)$    (b) $\left(\frac{1}{2}, -1\right)$    (c) $\left(\frac{3}{2}, \frac{7}{2}\right)$

**4.** $x^2 y + x^2 - 5y = 0$

  (a) $\left(0, \frac{1}{5}\right)$    (b) $(2, 4)$    (c) $(-2, -4)$

In Exercises 5–10, match the equation with its graph. Use a graphing utility, set for a square setting, to confirm your result. [The graphs are labeled (a)–(f).]

**5.** $y = x - 2$

**6.** $y = -\frac{1}{2}x + 2$

**7.** $y = x^2 + 2x$

**8.** $y = \sqrt{9 - x^2}$

**9.** $y = |x| - 2$

**10.** $y = x^3 - x$

(a)



(b)



(c)



(d)



(e)



(f)
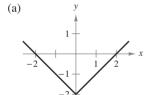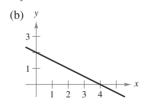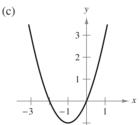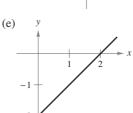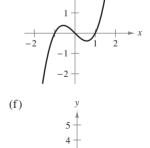


In Exercises 11–20, find the $x$- and $y$-intercepts of the graph of the equation.

**11.** $2x - y - 3 = 0$

**12.** $4x - 2y - 5 = 0$

**13.** $y = x^2 + x - 2$

**14.** $y = x^2 - 4x + 3$

**15.** $y = \sqrt{4 - x^2}$

**16.** $y^2 = x^3 - 4x$

**17.** $y = \dfrac{x^2 - 4}{x - 2}$

**18.** $y = \dfrac{x^2 + 3x}{2x}$

**19.** $x^2 y - x^2 + 4y = 0$

**20.** $2x^2 y + 8y - x^2 = 1$

In Exercises 21–36, sketch the graph of the equation and label the intercepts. Use a graphing utility to verify your results.

**21.** $y = 2x + 3$

**22.** $y = -3x + 2$

**23.** $y = x^2 - 3$

**24.** $y = x^2 + 6$

**25.** $y = (x - 1)^2$

**26.** $y = (5 - x)^2$

**27.** $y = x^3 + 2$

**28.** $y = 1 - x^3$

**29.** $y = -\sqrt{x - 1}$

**30.** $y = \sqrt{x + 1}$

**31.** $y = |x + 1|$

**32.** $y = -|x - 2|$

**33.** $y = 1/(x - 3)$

**34.** $y = 1/(x + 1)$

**35.** $x = y^2 - 4$

**36.** $x = 4 - y^2$

In Exercises 37–44, write the general form of the equation of the circle.

**37.** Center: $(0, 0)$; radius: 4    **38.** Center: $(0, 0)$; radius: 5

**39.** Center: $(2, -1)$; radius: 3

**40.** Center: $(-4, 3)$; radius: 2

**41.** Center: $(-1, 2)$; solution point: $(0, 0)$

**42.** Center: $(3, -2)$; solution point: $(-1, 1)$

**43.** Endpoints of a diameter: $(0, 0), (6, 8)$

**44.** Endpoints of a diameter: $(-4, -1), (4, 1)$

(T) In Exercises 45–52, complete the square to write the equation of the circle in standard form. Then use a graphing utility to graph the circle.

**45.** $x^2 + y^2 - 2x + 6y + 6 = 0$

**46.** $x^2 + y^2 - 2x + 6y - 15 = 0$

**47.** $x^2 + y^2 - 4x - 2y + 1 = 0$

**48.** $x^2 + y^2 - 4x + 2y + 3 = 0$

**49.** $2x^2 + 2y^2 - 2x - 2y - 3 = 0$

**50.** $4x^2 + 4y^2 - 4x + 2y - 1 = 0$

**51.** $4x^2 + 4y^2 + 12x - 24y + 41 = 0$

**52.** $3x^2 + 3y^2 - 6y - 1 = 0$

In Exercises 53–60, find the points of intersection (if any) of the graphs of the equations. Use a graphing utility to check your results.

**53.** $x + y = 2, 2x - y = 1$

**54.** $x + y = 7, 3x - 2y = 11$

**55.** $x^2 + y^2 = 25, 2x + y = 10$

**56.** $x^2 - y = -2, x - y = -4$

**57.** $y = x^3, y = 2x$

**58.** $y = \sqrt{x}, y = x$

**59.** $y = x^4 - 2x^2 + 1, y = 1 - x^2$

**60.** $y = x^3 - 2x^2 + x - 1, y = -x^2 + 3x - 1$

**61. Break-Even Analysis**  You are setting up a part-time business with an initial investment of $15,000. The unit cost of the product is $11.80, and the selling price is $19.30.

(a) Find equations for the total cost $C$ and total revenue $R$ for $x$ units.

(b) Find the break-even point by finding the point of intersection of the cost and revenue equations.

(c) How many units would yield a profit of $1000?

**62. Break-Even Analysis**  A 2006 Chevrolet Impala costs $23,730 with a gasoline engine. A 2006 Toyota Camry costs $25,900 with a hybrid engine. The Impala gets 20 miles per gallon of gasoline and the Camry gets 34 miles per gallon of gasoline. Assume that the price of gasoline is $2.909. *(Source: Consumer Reports, March and August 2006)*

(a) Show that the cost $C_g$ of driving the Chevrolet Impala $x$ miles is

$$C_g = 23{,}730 + 2.909x/20$$

and the cost $C_h$ of driving the Toyota Camry $x$ miles is

$$C_h = 25{,}900 + 2.909x/34$$

(b) Find the break-even point. That is, find the mileage at which the hybrid-powered Toyota Camry becomes more economical than the gasoline-powered Chevrolet Impala.

(T) **Break-Even Analysis**  In Exercises 63–66, find the sales necessary to break even for the given cost and revenue equations. (Round your answer up to the nearest whole unit.) Use a graphing utility to graph the equations and then find the break-even point.

**63.** $C = 0.85x + 35{,}000, R = 1.55x$

**64.** $C = 6x + 500{,}000, R = 35x$

**65.** $C = 8650x + 250{,}000, R = 9950x$

**66.** $C = 2.5x + 10{,}000, R = 4.9x$

**67. Supply and Demand**  The demand and supply equations for an electronic organizer are given by

$p = 180 - 4x$        Demand equation

$p = 75 + 3x$        Supply equation

where $p$ is the price in dollars and $x$ represents the number of units, in thousands. Find the equilibrium point for this market.

**68. Supply and Demand**  The demand and supply equations for a portable CD player are given by

$p = 190 - 15x$        Demand equation

$p = 75 + 8x$        Supply equation

where $p$ is the price in dollars and $x$ represents the number of units, in hundreds of thousands. Find the equilibrium point for this market.

**69. Textbook Spending**   The amounts of money $y$ (in millions of dollars) spent on college textbooks in the United States in the years 2000 through 2005 are shown in the table.   *(Source: Book Industry Study Group, Inc.)*

| Year | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|------|
| Expense | 4265 | 4571 | 4899 | 5086 | 5479 | 5703 |

A mathematical model for the data is given by $y = 0.796t^3 - 8.65t^2 + 312.9t + 4268$, where $t$ represents the year, with $t = 0$ corresponding to 2000.

(a) Compare the actual expenses with those given by the model. How well does the model fit the data? Explain your reasoning.

(b) Use the model to predict the expenses in 2013.

**70. Federal Education Spending**   The federal outlays $y$ (in billions of dollars) for elementary, secondary, and vocational education in the United States in the years 2001 through 2005 are shown in the table.   *(Source: U.S. Office of Management and Budget)*

| Year | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|------|
| Outlay | 20.6 | 22.9 | 25.9 | 31.5 | 34.4 | 38.3 |

A mathematical model for the data is given by

$y = 0.136t^2 + 3.00t + 20.2$

where $t$ represents the year, with $t = 0$ corresponding to 2000.

(a) Compare the actual outlays with those given by the model. How well does the model fit the data? Explain.

(b) Use the model to predict the outlays in 2012.

**71. *MAKE A DECISION: WEEKLY SALARY***   A mathematical model for the average weekly salary $y$ of a person in finance and insurance is given by the equation $y = -0.77t^2 + 27.3t + 587$, where $t$ represents the year, with $t = 0$ corresponding to 2000.   *(Source: U.S. Bureau of Labor Statistics)*

(a) Use the model to complete the table.

| Year | 2000 | 2001 | 2002 | 2003 | 2004 | 2007 |
|------|------|------|------|------|------|------|
| Salary | | | | | | |

(b) This model was created using actual data from 2000 through 2005. How accurate do you think the model is in predicting the 2007 average weekly salary? Explain your reasoning.

(c) What does this model predict the average weekly salary to be in 2009? Do you think this prediction is valid?

**72. *MAKE A DECISION: KIDNEY TRANSPLANTS***   A mathematical model for the numbers of kidney transplants performed in the United States in the years 2001 through 2005 is given by $y = 30.57t^2 + 381.4t + 13,852$, where $y$ is the number of transplants and $t$ represents the year, with $t = 1$ corresponding to 2001.   *(Source: United Network for Organ Sharing)*

(T)(S) (a) Use a graphing utility or a spreadsheet to complete the table.

| Year | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|
| Transplants | | | | | |

(b) Use your school's library, the Internet, or some other reference source to find the actual numbers of kidney transplants for the years 2001 through 2005. Compare the actual numbers with those given by the model. How well does the model fit the data? Explain your reasoning.

(c) Using this model, what is the prediction for the number of transplants in the year 2011? Do you think this prediction is valid? What factors could affect this model's accuracy?

(T) **73.** Use a graphing utility to graph the equation $y = cx + 1$ for $c = 1, 2, 3, 4$, and $5$. Then make a conjecture about the $x$-coefficient and the graph of the equation.

**74. Break-Even Point**   Define the break-even point for a business marketing a new product. Give examples of a linear cost equation and a linear revenue equation for which the break-even point is 10,000 units.

(T) In Exercises 75–80, use a graphing utility to graph the equation and approximate the $x$- and $y$-intercepts of the graph.

**75.** $y = 0.24x^2 + 1.32x + 5.36$

**76.** $y = -0.56x^2 - 5.34x + 6.25$

**77.** $y = \sqrt{0.3x^2 - 4.3x + 5.7}$

**78.** $y = \sqrt{-1.21x^2 + 2.34x + 5.6}$

**79.** $y = \dfrac{0.2x^2 + 1}{0.1x + 2.4}$      **80.** $y = \dfrac{0.4x - 5.3}{0.4x^2 + 5.3}$

**81. Extended Application**   To work an extended application analyzing the numbers of workers in the farm work force in the United States from 1955 through 2005, visit this text's website at *college.hmco.com*. *(Data Source: U.S. Bureau of Labor Statistics)*

The symbol (S) indicates an exercise in which you are instructed to use a spreadsheet.

# Section 1.3

# Lines in the Plane and Slope
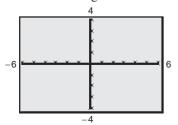
- Use the slope-intercept form of a linear equation to sketch graphs.
- Find slopes of lines passing through two points.
- Use the point-slope form to write equations of lines.
- Find equations of parallel and perpendicular lines.
- Use linear equations to model and solve real-life problems.

## Using Slope

The simplest mathematical model for relating two variables is the **linear equation** $y = mx + b$. The equation is called *linear* because its graph is a line. (In this text, the term *line* is used to mean *straight line*.) By letting $x = 0$, you can see that the line crosses the $y$-axis at $y = b$, as shown in Figure 1.31. In other words, the $y$-intercept is $(0, b)$. The steepness or slope of the line is $m$.

$$y = mx + b$$

Slope ⌐    ⌐ $y$-intercept

The **slope** of a line is the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right, as shown in Figure 1.31.



Positive slope, line rises.    Negative slope, line falls.

**FIGURE 1.31**

A linear equation that is written in the form $y = mx + b$ is said to be written in **slope-intercept form.**

---

**The Slope-Intercept Form of the Equation of a Line**

The graph of the equation

$$y = mx + b$$

is a line whose slope is $m$ and whose $y$-intercept is $(0, b)$.

---

### TECHNOLOGY

On most graphing utilities, the display screen is two-thirds as high as it is wide. On such screens, you can obtain a graph with a true geometric perspective by using a **square setting**—one in which

$$\frac{Y_{max} - Y_{min}}{X_{max} - X_{min}} = \frac{2}{3}.$$

One such setting is shown below. Notice that the $x$ and $y$ tick marks are equally spaced on a square setting, but not on a standard setting.



### DISCOVERY

Use a graphing utility to compare the slopes of the lines $y = mx$, where $m = 0.5, 1, 2,$ and $4$. Which line rises most quickly? Now, let $m = -0.5,$ $-1, -2,$ and $-4$. Which line falls most quickly? Let $m = 0.01, 0.001,$ and $0.0001$. What is the slope of a horizontal line? Use a square setting to obtain a true geometric perspective.

**FIGURE 1.32**   When the line is vertical, the slope is undefined.

Once you have determined the slope and the y-intercept of a line, it is a relatively simple matter to sketch its graph.

In the following example, note that none of the lines is vertical. A vertical line has an equation of the form

$$x = a. \qquad \text{Vertical line}$$

Because such an equation cannot be written in the form $y = mx + b$, it follows that the slope of a vertical line is undefined, as indicated in Figure 1.32.

### Example 1   Graphing a Linear Equation

Sketch the graph of each linear equation.

**a.** $y = 2x + 1$

**b.** $y = 2$

**c.** $x + y = 2$

**SOLUTION**

**a.** Because $b = 1$, the y-intercept is $(0, 1)$. Moreover, because the slope is $m = 2$, the line *rises* two units for each unit the line moves to the right, as shown in Figure 1.33(a).

**b.** By writing this equation in the form $y = (0)x + 2$, you can see that the y-intercept is $(0, 2)$ and the slope is zero. A zero slope implies that the line is horizontal—that is, it doesn't rise *or* fall, as shown in Figure 1.33(b).

**c.** By writing this equation in slope-intercept form

$$x + y = 2 \qquad \text{Write original equation.}$$
$$y = -x + 2 \qquad \text{Subtract } x \text{ from each side.}$$
$$y = (-1)x + 2 \qquad \text{Write in slope-intercept form.}$$
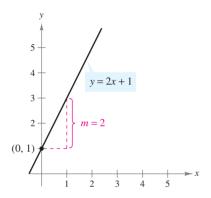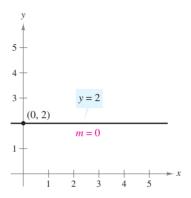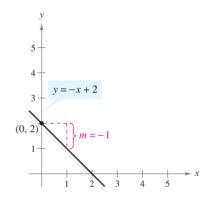
you can see that the y-intercept is $(0, 2)$. Moreover, because the slope is $m = -1$, this line *falls* one unit for each unit the line moves to the right, as shown in Figure 1.33(c).

✓**CHECKPOINT 1**

Sketch the graph of each linear equation.

**a.** $y = 4x - 2$

**b.** $x = 1$ ◼



(a) When $m$ is positive, the line rises.



(b) When $m$ is zero, the line is horizontal.



(c) When $m$ is negative, the line falls.

**FIGURE 1.33**

In real-life problems, the slope of a line can be interpreted as either a *ratio* or a *rate*. If the *x*-axis and *y*-axis have the same unit of measure, then the slope has no units and is a **ratio.** If the *x*-axis and *y*-axis have different units of measure, then the slope is a **rate** or **rate of change.**

### *Example 2*
**MAKE A DECISION**  **Using Slope as a Ratio**  ®
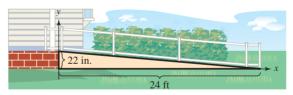
The maximum recommended slope of a wheelchair ramp is $\frac{1}{12} \approx 0.083$. A business is installing a wheelchair ramp that rises 22 inches over a horizontal length of 24 feet, as shown in Figure 1.34. Is the ramp steeper than recommended? *(Source: American Disabilities Act Handbook)*

**SOLUTION**  The horizontal length of the ramp is 24 feet or $12(24) = 288$ inches. So, the slope of the ramp is

$$\text{Slope} = \frac{\text{vertical change}}{\text{horizontal change}}$$

$$= \frac{22 \text{ in.}}{288 \text{ in.}}$$

$$\approx 0.076.$$

So, the slope is not steeper than recommended.

✓ **CHECKPOINT 2**

If the ramp in Example 2 rises 27 inches over a horizontal length of 26 feet, is it steeper than recommended? ∎



**22 in.**

**24 ft**

**FIGURE 1.34**



**Production Cost**

$C = 25x + 3500$

Marginal cost: $m = \$25$

Fixed cost: $\$3500$

Cost (in dollars)

Number of units

**FIGURE 1.35**

### *Example 3*  **Using Slope as a Rate of Change**  ®

A manufacturing company determines that the total cost in dollars of producing *x* units of a product is $C = 25x + 3500$. Describe the practical significance of the *y*-intercept and slope of the line given by this equation.
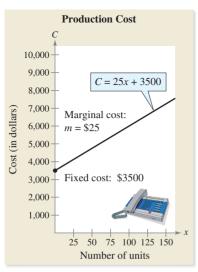
**SOLUTION**  The *y*-intercept $(0, 3500)$ tells you that the cost of producing zero units is $3500. This is the **fixed cost** of production—it includes costs that must be paid regardless of the number of units produced. The slope of $m = 25$ tells you that the cost of producing each unit is $25, as shown in Figure 1.35. Economists call the cost per unit the **marginal cost.** If the production increases by one unit, then the "margin" or extra amount of cost is $25.

✓ **CHECKPOINT 3**
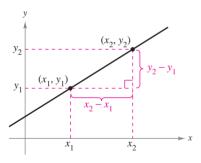
A small business purchases a copier and determines that the value of the copier *t* years after its purchase is $V = -175t + 875$. Describe the practical significance of the *y*-intercept and slope of the line given by this equation. ∎

## Finding the Slope of a Line



$y$

$(x_2, y_2)$

$y_2$

$(x_1, y_1)$

$y_2 - y_1$

$y_1$

$x_2 - x_1$

$x_1$   $x_2$   $x$

**FIGURE 1.36**

Given an equation of a nonvertical line, you can find its slope by writing the equation in slope-intercept form. If you are not given an equation, you can still find the slope of a line. For instance, suppose you want to find the slope of the line passing through the points $(x_1, y_1)$ and $(x_2, y_2)$, as shown in Figure 1.36. As you move from left to right along this line, a change of $(y_2 - y_1)$ units in the vertical direction corresponds to a change of $(x_2 - x_1)$ units in the horizontal direction. These two changes are denoted by the symbols

$$\Delta y = y_2 - y_1 = \text{the change in } y$$

and

$$\Delta x = x_2 - x_1 = \text{the change in } x.$$

(The symbol $\Delta$ is the Greek capital letter delta, and the symbols $\Delta y$ and $\Delta x$ are read as "delta $y$" and "delta $x$.") The ratio of $\Delta y$ to $\Delta x$ represents the slope of the line that passes through the points $(x_1, y_1)$ and $(x_2, y_2)$.

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Be sure you see that $\Delta x$ represents a single number, not the product of two numbers ($\Delta$ and $x$). The same is true for $\Delta y$.

---

**The Slope of a Line Passing Through Two Points**

The **slope** $m$ of the line passing through $(x_1, y_1)$ and $(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_1 \neq x_2$.

---

When this formula is used for slope, the *order of subtraction* is important. Given two points on a line, you are free to label either one of them as $(x_1, y_1)$ and the other as $(x_2, y_2)$. However, once you have done this, you must form the numerator and denominator using the same order of subtraction.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \frac{y_1 - y_2}{x_1 - x_2} \qquad m = \frac{y_2 - y_1}{x_1 - x_2}$$

Correct                    Correct                    Incorrect

For instance, the slope of the line passing through the points $(3, 4)$ and $(5, 7)$ can be calculated as

$$m = \frac{7 - 4}{5 - 3} = \frac{3}{2}$$

or

$$m = \frac{4 - 7}{3 - 5} = \frac{-3}{-2} = \frac{3}{2}.$$

*Example 4*  **Finding the Slope of a Line**

Find the slope of the line passing through each pair of points.

**a.** $(-2, 0)$ and $(3, 1)$     **b.** $(-1, 2)$ and $(2, 2)$

**c.** $(0, 4)$ and $(1, -1)$     **d.** $(3, 4)$ and $(3, 1)$

**SOLUTION**

**a.** Letting $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (3, 1)$, you obtain a slope of

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{3 - (-2)} = \frac{1}{5} \qquad \longleftarrow \text{Difference in } y\text{-values} \\ \longleftarrow \text{Difference in } x\text{-values}$$

as shown in Figure 1.37(a).
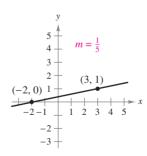
**b.** The slope of the line passing through $(-1, 2)$ and $(2, 2)$ is

$$m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0. \qquad \text{See Figure 1.37(b).}$$
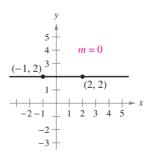
**c.** The slope of the line passing through $(0, 4)$ and $(1, -1)$ is
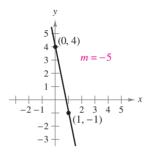
$$m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5. \qquad \text{See Figure 1.37(c).}$$
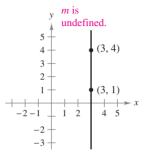
**d.** The slope of the vertical line passing through $(3, 4)$ and $(3, 1)$ is not defined because division by zero is undefined. [See Figure 1.37(d).]

**DISCOVERY**

The line in Example 4(b) is a horizontal line. Find an equation for this line. The line in Example 4(d) is a vertical line. Find an equation for this line.



(a) Positive slope, line rises.

(b) Zero slope, line is horizontal.

(c) Negative slope, line falls.

(d) Vertical line, undefined slope.

**FIGURE 1.37**

✓**CHECKPOINT 4**

Find the slope of the line passing through each pair of points.

**a.** $(-3, 2)$ and $(5, 18)$

**b.** $(-2, 1)$ and $(-4, 2)$ ■

## Writing Linear Equations

If $(x_1, y_1)$ is a point lying on a nonvertical line of slope $m$ and $(x, y)$ is *any other* point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation, involving the variables $x$ and $y$, can be rewritten in the form $y - y_1 = m(x - x_1)$, which is the **point-slope form** of the equation of a line.

---

**Point-Slope Form of the Equation of a Line**

The equation of the line with slope $m$ passing through the point $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1).$$

---

The point-slope form is most useful for *finding* the equation of a nonvertical line. You should remember this formula—it is used throughout the text.



**FIGURE 1.38**
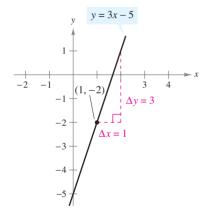
### Example 5   Using the Point-Slope Form

Find the equation of the line that has a slope of 3 and passes through the point $(1, -2)$.

**SOLUTION**   Use the point-slope form with $m = 3$ and $(x_1, y_1) = (1, -2)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) &&\text{Point-slope form} \\ y - (-2) &= 3(x - 1) &&\text{Substitute for } m, x_1, \text{ and } y_1. \\ y + 2 &= 3x - 3 &&\text{Simplify.} \\ y &= 3x - 5 &&\text{Write in slope-intercept form.} \end{aligned}$$

The slope-intercept form of the equation of the line is $y = 3x - 5$. The graph of this line is shown in Figure 1.38.

### ✓ CHECKPOINT 5

Find the equation of the line that has a slope of 2 and passes through the point $(-1, 2)$. ■

The point-slope form can be used to find an equation of the line passing through points $(x_1, y_1)$ and $(x_2, y_2)$. To do this, first find the slope of the line

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2$$

and then use the point-slope form to obtain the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \qquad \text{Two-point form}$$
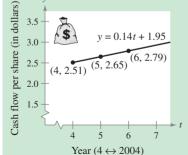
This is sometimes called the **two-point form** of the equation of a line.

**STUDY TIP**

The two-point form of a line is similar to the slope-intercept form. What is the slope of a line given in two-point form
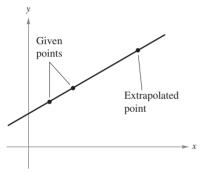
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)?$$
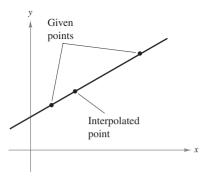
**Cash Flow**

**FIGURE 1.39**

---

*Example 6* **Predicting Cash Flow Per Share** Ⓡ

The cash flow per share for Ruby Tuesday, Inc. was \$2.51 in 2004 and \$2.65 in 2005. Using only this information, write a linear equation that gives the cash flow per share in terms of the year. Then predict the cash flow for 2006. *(Source: Ruby Tuesday, Inc.)*

**SOLUTION** Let $t = 4$ represent 2004. Then the two given values are represented by the data points $(4, 2.51)$ and $(5, 2.65)$. The slope of the line through these points is

$$m = \frac{2.65 - 2.51}{5 - 4} = 0.14$$

Using the point-slope form, you can find the equation that relates the cash flow $y$ and the year $t$ to be $y = 0.14t + 1.95$. According to this equation, the cash flow in 2006 was \$2.79, as shown in Figure 1.39. (In this case, the prediction is fairly good—the actual cash flow in 2006 was \$2.96.)

✓ **CHECKPOINT 6**

The cash flow per share for Energizer Holdings, Inc. was \$5.22 in 2004 and \$6.01 in 2005. Write a linear equation that gives the cash flow per share in terms of the year. Let $t = 4$ represent 2004. Then predict the sales per share for 2006. *(Source: Energizer Holdings, Inc.)* ■

The prediction method illustrated in Example 6 is called **linear extrapolation.** Note in Figure 1.40(a) that an extrapolated point does not lie between the given points. When the estimated point lies between two given points, as shown in Figure 1.40(b), the procedure is called **linear interpolation.**

Because the slope of a vertical line is not defined, its equation cannot be written in slope-intercept form. However, every line has an equation that can be written in the **general form**

$$Ax + By + C = 0 \qquad \text{General form}$$

where $A$ and $B$ are not both zero. For instance, the vertical line given by $x = a$ can be represented by the general form $x - a = 0$. The five most common forms of equations of lines are summarized below.



(a) Linear Extrapolation



(b) Linear Interpolation

**FIGURE 1.40**

---

**Equations of Lines**

**1.** General form:        $Ax + By + C = 0$

**2.** Vertical line:        $x = a$
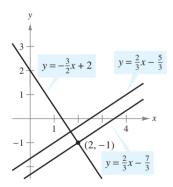
**3.** Horizontal line:      $y = b$

**4.** Slope-intercept form:  $y = mx + b$

**5.** Point-slope form:     $y - y_1 = m(x - x_1)$

Ⓣ On a graphing utility, lines will not appear to have the correct slope unless you use a viewing window that has a "square setting." For instance, try graphing the lines in Example 7 using the standard setting $-10 \le x \le 10$ and $-10 \le y \le 10$. Then reset the viewing window with the square setting $-9 \le x \le 9$ and $-6 \le y \le 6$. On which setting do the lines $y = \frac{2}{3}x - \frac{5}{3}$ and $y = -\frac{3}{2}x + 2$ appear to be perpendicular?



$y = -\frac{3}{2}x + 2$   $y = \frac{2}{3}x - \frac{5}{3}$

$(2, -1)$

$y = \frac{2}{3}x - \frac{7}{3}$

**FIGURE 1.41**

✓ **CHECKPOINT 7**

Find equations of the lines that pass through the point $(2, 1)$ and are

**a.** parallel to the line $2x - 4y = 5$.

**b.** perpendicular to the line $2x - 4y = 5$. ■

## Parallel and Perpendicular Lines

Slope can be used to decide whether two nonvertical lines in a plane are parallel, perpendicular, or neither.

**Parallel and Perpendicular Lines**

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal. That is, $m_1 = m_2$.

2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is, $m_1 = -1/m_2$.

*Example 7*   **Finding Parallel and Perpendicular Lines**

Find equations of the lines that pass through the point $(2, -1)$ and are

**a.** parallel to the line $2x - 3y = 5$.

**b.** perpendicular to the line $2x - 3y = 5$.

**SOLUTION**   By writing the given equation in slope-intercept form

| | |
|---|---|
| $2x - 3y = 5$ | Write original equation. |
| $-3y = -2x + 5$ | Subtract $2x$ from each side. |
| $y = \frac{2}{3}x - \frac{5}{3}$ | Write in slope-intercept form. |

you can see that it has a slope of $m = \frac{2}{3}$, as shown in Figure 1.41.

**a.** Any line parallel to the given line must also have a slope of $\frac{2}{3}$. So, the line through $(2, -1)$ that is parallel to the given line has the following equation.

| | |
|---|---|
| $y - (-1) = \frac{2}{3}(x - 2)$ | Write in point-slope form. |
| $3(y + 1) = 2(x - 2)$ | Multiply each side by 3. |
| $3y + 3 = 2x - 4$ | Distributive Property |
| $2x - 3y - 7 = 0$ | Write in general form. |
| $y = \frac{2}{3}x - \frac{7}{3}$ | Write in slope-intercept form. |

**b.** Any line perpendicular to the given line must have a slope of

$$\frac{-1}{\frac{2}{3}} \text{ or } -\frac{3}{2}.$$

So, the line through $(2, -1)$ that is perpendicular to the given line has the following equation.
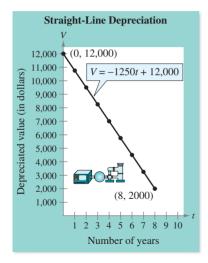
| | |
|---|---|
| $y - (-1) = -\frac{3}{2}(x - 2)$ | Write in point-slope form. |
| $2(y + 1) = -3(x - 2)$ | Multiply each side by 2. |
| $2y + 2 = -3x + 6$ | Distributive Property |
| $3x + 2y - 4 = 0$ | Write in general form. |
| $y = -\frac{3}{2}x + 2$ | Write in slope-intercept form. |

## Extended Application: Linear Depreciation

Most business expenses can be deducted the same year they occur. One exception to this is the cost of property that has a useful life of more than 1 year, such as buildings, cars, or equipment. Such costs must be **depreciated** over the useful life of the property. If the *same amount* is depreciated each year, the procedure is called **linear depreciation** or **straight-line depreciation.** The *book value* is the difference between the original value and the total amount of depreciation accumulated to date.

✓ **CHECKPOINT 8**

Write a linear equation for the machine in Example 8 if the salvage value at the end of 8 years is $1000. ■

### Example 8    Depreciating Equipment    ®

Your company has purchased a $12,000 machine that has a useful life of 8 years. The salvage value at the end of 8 years is $2000. Write a linear equation that describes the book value of the machine each year.

**SOLUTION**    Let $V$ represent the value of the machine at the end of year $t$. You can represent the initial value of the machine by the ordered pair $(0, 12,000)$ and the salvage value of the machine by the ordered pair $(8, 2000)$. The slope of the line is

$$m = \frac{2000 - 12,000}{8 - 0} = -\$1250 \qquad m = \frac{V_2 - V_1}{t_2 - t_1}$$

which represents the annual depreciation in *dollars per year*. Using the point-slope form, you can write the equation of the line as shown.

$$V - 12,000 = -1250(t - 0) \qquad \text{Write in point-slope form.}$$
$$V = -1250t + 12,000 \qquad \text{Write in slope-intercept form.}$$

The table shows the book value of the machine at the end of each year.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $V$ | 12,000 | 10,750 | 9500 | 8250 | 7000 | 5750 | 4500 | 3250 | 2000 |



**Straight-Line Depreciation**

$V = -1250t + 12,000$

(0, 12,000)

(8, 2000)

**FIGURE 1.42**

The graph of this equation is shown in Figure 1.42.

### CONCEPT CHECK

1. In the form $y = mx + b$, what does the $m$ represent? What does the $b$ represent?

2. Can any pair of points on a line be used to calculate the slope of the line? Explain.

3. The slopes of two lines are $-4$ and $\frac{5}{2}$. Which is steeper? Explain your reasoning.

4. Is it possible for two lines with positive slopes to be perpendicular? Why or why not?

| **Skills Review 1.3** | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 0.5. |

In Exercises 1 and 2, simplify the expression.

**1.** $\dfrac{5 - (-2)}{-3 - 4}$

**2.** $\dfrac{-7 - (-10)}{4 - 1}$

**3.** Evaluate $-\dfrac{1}{m}$ when $m = -3$.

**4.** Evaluate $-\dfrac{1}{m}$ when $m = \dfrac{6}{7}$.

In Exercises 5–10, solve for $y$ in terms of $x$.

**5.** $-4x + y = 7$

**6.** $3x - y = 7$

**7.** $y - 2 = 3(x - 4)$

**8.** $y - (-5) = -1[x - (-2)]$

**9.** $y - (-3) = \dfrac{4 - (-3)}{2 - 1}(x - 2)$
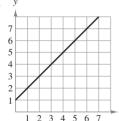
**10.** $y - 1 = \dfrac{-3 - 1}{-7 - (-1)}[x - (-1)]$

## Exercises 1.3
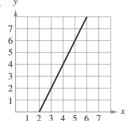
In Exercises 1–4, estimate the slope of the line.

**1.**


**2.**


**3.**


**4.**
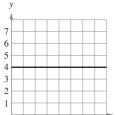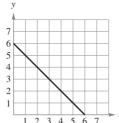

In Exercises 5–18, plot the points and find the slope of the line passing through the pair of points.

**5.** $(0, -3), (9, 0)$

**6.** $(-2, 0), (1, 4)$

**7.** $(3, -4), (5, 2)$

**8.** $(1, 2), (-2, 2)$

**9.** $\left(\frac{1}{2}, 2\right), (6, 2)$

**10.** $\left(\frac{11}{3}, -2\right), \left(\frac{11}{3}, -10\right)$

**11.** $(-8, -3), (-8, -5)$

**12.** $(2, -1), (-2, -5)$

**13.** $(-2, 1), (4, -3)$

**14.** $(3, -5), (-2, -5)$

**15.** $\left(\frac{1}{4}, -2\right), \left(-\frac{3}{8}, 1\right)$

**16.** $\left(-\frac{3}{2}, -5\right), \left(\frac{5}{6}, 4\right)$

**17.** $\left(\frac{2}{3}, \frac{5}{2}\right), \left(\frac{1}{4}, -\frac{5}{6}\right)$

**18.** $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

In Exercises 19–26, use the point on the line and the slope of the line to find three additional points through which the line passes. (There are many correct answers.)

|  | *Point* | *Slope* |  | *Point* | *Slope* |
|---|---|---|---|---|---|
| **19.** | $(2, 1)$ | $m = 0$ | **20.** | $(-3, -1)$ | $m = 0$ |
| **21.** | $(6, -4)$ | $m = \frac{2}{3}$ | **22.** | $(-1, -6)$ | $m = -\frac{1}{2}$ |
| **23.** | $(1, 7)$ | $m = -3$ | **24.** | $(7, -2)$ | $m = 2$ |
| **25.** | $(-8, 1)$ | $m$ is undefined. |  |  |  |
| **26.** | $(-3, 4)$ | $m$ is undefined. |  |  |  |

In Exercises 27–36, find the slope and $y$-intercept (if possible) of the equation of the line.

**27.** $x + 5y = 20$

**28.** $2x + y = 40$

**29.** $7x + 6y = 30$

**30.** $2x + 3y = 9$

**31.** $3x - y = 15$

**32.** $2x - 3y = 24$

**33.** $x = 4$

**34.** $x + 5 = 0$

**35.** $y - 4 = 0$

**36.** $y + 1 = 0$

In Exercises 37–48, write an equation of the line that passes through the points. Then use the equation to sketch the line.

**37.** $(4, 3), (0, -5)$

**38.** $(-3, -4), (1, 4)$

**39.** $(0, 0), (-1, 3)$

**40.** $(-3, 6), (1, 2)$

**41.** $(2, 3), (2, -2)$

**42.** $(6, 1), (10, 1)$

**43.** $(3, -1), (-2, -1)$

**44.** $(2, 5), (2, -10)$

**45.** $\left(-\frac{1}{3}, 1\right), \left(-\frac{2}{3}, \frac{5}{6}\right)$

**46.** $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

**47.** $\left(-\frac{1}{2}, 4\right), \left(\frac{1}{2}, 8\right)$

**48.** $(4, -1), \left(\frac{1}{4}, -5\right)$

(T) In Exercises 49–58, write an equation of the line that passes through the given point and has the given slope. Then use a graphing utility to graph the line.

| *Point* | *Slope* | | *Point* | *Slope* |
|---|---|---|---|---|
| **49.** $(0, 3)$ | $m = \frac{3}{4}$ | | **50.** $(0, 0)$ | $m = \frac{2}{3}$ |
| **51.** $(-1, 2)$ | $m$ is undefined. | | | |
| **52.** $(0, 4)$ | $m$ is undefined. | | | |
| **53.** $(-2, 7)$ | $m = 0$ | | **54.** $(-2, 4)$ | $m = 0$ |
| **55.** $(0, -2)$ | $m = -4$ | | **56.** $(-1, -4)$ | $m = -2$ |
| **57.** $\left(0, \frac{2}{3}\right)$ | $m = \frac{3}{4}$ | | **58.** $\left(0, -\frac{2}{3}\right)$ | $m = \frac{1}{6}$ |

In Exercises 59–62, explain how to use the concept of slope to determine whether the three points are collinear.

**59.** $(-2, 1), (-1, 0), (2, -2)$    **60.** $(0, 4), (7, -6), (-5, 11)$

**61.** $(-2, -1), (0, 3), (2, 7)$    **62.** $(4, 1), (-2, -2), (8, 3)$

**63.** Write an equation of the vertical line with $x$-intercept at 3.

**64.** Write an equation of the horizontal line through $(0, -5)$.

**65.** Write an equation of the line with $y$-intercept at $-10$ and parallel to all horizontal lines.

**66.** Write an equation of the line with $x$-intercept at $-5$ and parallel to all vertical lines.

(T) In Exercises 67–74, write equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line. Then use a graphing utility to graph all three equations in the same viewing window.

| *Point* | *Line* | | *Point* | *Line* |
|---|---|---|---|---|
| **67.** $(-3, 2)$ | $x + y = 7$ | | **68.** $(2, 1)$ | $4x - 2y = 3$ |
| **69.** $\left(-\frac{2}{3}, \frac{7}{8}\right)$ | $3x + 4y = 7$ | | **70.** $\left(\frac{7}{8}, \frac{3}{4}\right)$ | $5x + 3y = 0$ |
| **71.** $(-1, 0)$ | $y + 3 = 0$ | | **72.** $(2, 5)$ | $y + 4 = 0$ |
| **73.** $(1, 1)$ | $x - 2 = 0$ | | **74.** $(12, -3)$ | $x + 4 = 0$ |

In Exercises 75–84, sketch the graph of the equation. Use a graphing utility to verify your result.

**75.** $y = -2$    **76.** $y = -4$

**77.** $2x - y - 3 = 0$    **78.** $x + 2y + 6 = 0$

**79.** $y = -2x + 1$    **80.** $4x + 5y = 20$

**81.** $3x + 5y + 15 = 0$    **82.** $-5x + 2y - 20 = 0$

**83.** $y + 2 = -4(x + 1)$    **84.** $y - 1 = 3(x + 4)$

**85. Temperature Conversion**   Write a linear equation that expresses the relationship between the temperature in degrees Celsius $C$ and degrees Fahrenheit $F$. Use the fact that water freezes at $0°C$ ($32°F$) and boils at $100°C$ ($212°F$).

(B) **86. Chemistry**   Use the result of Exercise 85 to answer the following:

(a) A person has a temperature of $100.4°F$. What is this temperature on the Celsius scale?

(b) If the temperature in a room is $72°F$, what is this temperature on the Celsius scale?

*(Source: Adapted from Zumdahl, Chemistry, Seventh Edition)*

**87. Population**   The resident population of South Carolina (in thousands) was 4024 in 2000 and 4255 in 2005. Assume that the relationship between the population $y$ and the year $t$ is linear. Let $t = 0$ represent 2000.   *(Source: U.S. Census Bureau)*

(a) Write a linear model for the data. What is the slope and what does it tell you about the population?

(b) Estimate the population in 2002.

(c) Use your model to estimate the population in 2004.

(d) Use your school's library, the Internet, or some other reference source to find the actual populations in 2002 and 2004. How close were your estimates?

(e) Do you think your model could be used to predict the population in 2009? Explain.

**88. Union Negotiation**   You are on a negotiating panel in a union hearing for a large corporation. The union is asking for a base pay of $11.25 per hour plus an additional piecework rate of $1.05 per unit produced. The corporation is offering a base pay of $8.85 per hour plus a piecework rate of $1.30.

(a) Write a linear equation for the hourly wages $W$ in terms of $x$, the number of units produced per hour, for each pay schedule.

(T) (b) Use a graphing utility to graph each linear equation and find the point of intersection.

(c) Interpret the meaning of the point of intersection of the graphs. How would you use this information to advise the corporation and the union?

(T) **89. Linear Depreciation**   A small business purchases a piece of equipment for $1025. After 5 years the equipment will be outdated, having no value.

(a) Write a linear equation giving the value $y$ of the equipment in terms of the time $t$ in years, $0 \le t \le 5$.

(b) Use a graphing utility to graph the equation.

(c) Move the cursor along the graph and estimate (to two-decimal-place accuracy) the value of the equipment when $t = 3$.

(d) Move the cursor along the graph and estimate (to two-decimal-place accuracy) the time when the value of the equipment will be $600.

**90. Linear Depreciation**   A company constructs a warehouse for $1,725,000. The warehouse has an estimated useful life of 25 years, after which its value is expected to be $100,000. Write a linear equation giving the value $y$ of the warehouse during its 25 years of useful life. (Let $t$ represent the time in years.)

**91. Personal Income**   Personal income (in billions of dollars) in the United States was 7802 in 1999 and 10,239 in 2005. Assume that the relationship between the personal income $Y$ and the time $t$ (in years) is linear. Let $t = 0$ correspond to 1990.   *(Source: U.S. Bureau of Economic Analysis)*

(a) Write a linear model for the data.

(b) *Linear Interpolation*   Estimate the personal income in 2001.

(c) *Linear Extrapolation*   Estimate the personal income in 2007.

(d) Use your school's library, the Internet, or some other reference source to find the actual personal income in 2001 and 2007. How close were your estimates?

**92. Consumer Awareness**   A real estate office handles an apartment complex with 50 units. When the rent is $480 per month, all 50 units are occupied. When the rent is $525, however, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent $p$ and the demand $x$ is linear. (The term *demand* refers to the number of occupied units.)

(a) Write a linear equation expressing $x$ in terms of $p$.

(b) *Linear Extrapolation*   Predict the number of occupied units when the rent is set at $555.

(c) *Linear Interpolation*   Predict the number of occupied units when the rent is set at $495.

**93. Profit Analysis**   A business manufactures a product at a cost of $50 per unit and sells the product for $120 per unit. The company's initial investment to produce the product is $350,000. The company estimates it can sell 13,000 units.

(a) Write a linear equation giving the total cost $C$ of producing $x$ units.

(b) Write an equation for the revenue $R$ derived from selling $x$ units.

(c) Use the formula for profit, $P = R - C$, to write an equation for the profit derived from $x$ units produced and sold.

(d) If the estimated sales of 13,000 units occurs, what is the company's profit or loss?

(e) How many units must the company sell to break even?

**94. Profit**   You are a contractor and have purchased a piece of equipment for $26,500. The equipment costs an average of $5.25 per hour for fuel and maintenance, and the operator is paid $12.50 per hour.

(a) Write a linear equation giving the total cost $C$ of operating the equipment for $t$ hours.

(b) You charge your customers $28 per hour of machine use. Write an equation for the revenue $R$ derived from $t$ hours of use.

(c) Use the formula for profit, $P = R - C$, to write an equation for the profit derived from $t$ hours of use.

(d) Find the number of hours you must operate the equipment before you break even.

**95. MAKE A DECISION: CHOOSING A JOB**   As a salesperson, you receive a monthly salary of $2000, plus a commission of 7% of sales. You are offered a new job at $2300 per month, plus a commission of 5% of sales.

(a) Write a linear equation for your current monthly wage $W$ in terms of your monthly sales $S$.

(b) Write a linear equation for the monthly wage $W$ of your job offer in terms of the monthly sales $S$.

(T) (c) Use a graphing utility to graph both equations in the same viewing window. Find the point of intersection. What does it signify?

(d) You think you can sell $20,000 worth of a product per month. Should you change jobs? Explain.

(T) **96. Plasma TV Sales**   Plasma televisions were first introduced for sale to the public in 1997. The data below gives the expected international plasma TV sales $S$ (in millions of units) for selected years from 2005 through 2011, where $t = 5$ represents 2005.   *(Source: Matsushita Electric Industrial Co., Ltd (Panasonic))*

| $t$ | 5 | 6 | 7 | 8 | 11 |
|---|---|---|---|---|---|
| $S$ | 2.85 | 5.70 | 10.0 | 14.8 | 25.0 |

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data.

(c) What is the rate of change in the expected sales for plasma televisions from 2005 through 2011?

(d) Use your model to estimate the expected sales for plasma televisions in 2009 and 2010.

(T) In Exercises 97–106, use a graphing utility to graph the cost equation. Determine the maximum production level $x$, given that the cost $C$ cannot exceed $100,000.

**97.** $C = 23,500 + 3100x$

**98.** $C = 30,000 + 575x$

**99.** $C = 18,375 + 1150x$

**100.** $C = 24,900 + 1785x$

**101.** $C = 75,500 + 89x$

**102.** $C = 83,620 + 67x$

**103.** $C = 32,000 + 650x$

**104.** $C = 53,500 + 495x$

**105.** $C = 50,000 + 0.25x$

**106.** $C = 75,500 + 1.50x$

## Mid-Chapter Quiz

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

**1.** $(3, -2), (-3, 1)$     **2.** $\left(\frac{1}{4}, -\frac{3}{2}\right), \left(\frac{1}{2}, 2\right)$     **3.** $(0, -4), \left(\sqrt{3}, 0\right)$

**4.** Use the Distance Formula to show that the points $(4, 0)$, $(2, 1)$, and $(-1, -5)$ are vertices of a right triangle.

**5.** The resident population of Missouri (in thousands) was 5719 in 2003 and 5800 in 2005. Use the Midpoint Formula to estimate the population in 2004.   *(Source: U.S. Census Bureau)*

In Exercises 6–8, sketch the graph of the equation and label the intercepts.

**6.** $y = 5x + 2$     **7.** $y = x^2 + x - 6$     **8.** $y = |x - 3|$

In Exercises 9 and 10, write the general form of the equation of the circle.

**9.** Center: $(-1, 0)$; radius: $\sqrt{37}$

**10.** Center: $(2, -2)$; solution point: $(-1, 2)$

(T) In Exercises 11 and 12, write the equation of the circle in standard form. Then use a graphing utility to graph the circle.

**11.** $x^2 + y^2 + 8x - 6y + 16 = 0$

**12.** $4x^2 + 4y^2 - 8x + 4y - 11 = 0$

**13.** A business manufactures a product at a cost of $4.55 per unit and sells the product for $7.19 per unit. The company's initial investment to produce the product was $12,500. How many units must the company sell to break even?

In Exercises 14–16, write an equation of the line that passes through the points. Then use the equation to sketch the line.

**14.** $(1, -1), (-4, 5)$     **15.** $(-2, 3), (-2, 2)$     **16.** $\left(\frac{5}{2}, 2\right), (0, 2)$

**17.** Find equations of the lines that pass through the point $(3, -5)$ and are

    (a) parallel to the line $x + 4y = -2$.

    (b) perpendicular to the line $x + 4y = -2$.

**18.** A company had sales of $1,330,000 in 2005 and $1,800,000 in 2007. If the company's sales follow a linear growth pattern, predict the sales in 2006 and in 2009.

**19.** **Reimbursed Expenses**   A company reimburses its sales representatives $175 per day for lodging and meals, plus $0.42 per mile driven. Write a linear equation giving the daily cost $C$ in terms of $x$, the number of miles driven.

**20.** **Annual Salary**   Your annual salary was $28,300 in 2004 and $31,700 in 2006. Assume your salary can be modeled by a linear equation.

    (a) Write a linear equation giving your salary $S$ in terms of the year. Let $t = 4$ represent 2004.

    (b) Use the linear model to predict your salary in 2010.

# Functions

- Decide whether relations between two variables are functions.
- Find the domains and ranges of functions.
- Use function notation and evaluate functions.
- Combine functions to create other functions.
- Find inverse functions algebraically.

## Functions

In many common relationships between two variables, the value of one of the variables depends on the value of the other variable. For example, the sales tax on an item depends on its selling price, the distance an object moves in a given amount of time depends on its speed, the price of mailing a package with an overnight delivery service depends on the package's weight, and the area of a circle depends on its radius.

Consider the relationship between the area of a circle and its radius. This relationship can be expressed by the equation

$$A = \pi r^2.$$

In this equation, the value of $A$ depends on the choice of $r$. Because of this, $A$ is the **dependent variable** and $r$ is the **independent variable.**

Most of the relationships that you will study in this course have the property that for a given value of the independent variable, there corresponds exactly one value of the dependent variable. Such a relationship is a **function.**



**FIGURE 1.43**

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**Definition of Function**

A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.
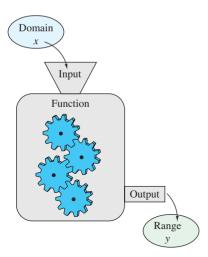
The **domain** of the function is the set of all values of the independent variable for which the function is defined. The **range** of the function is the set of all values taken on by the dependent variable.

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In Figure 1.43, notice that you can think of a function as a machine that inputs values of the independent variable and outputs values of the dependent variable.

Although functions can be described by various means such as tables, graphs, and diagrams, they are most often specified by formulas or equations. For instance, the equation

$$y = 4x^2 + 3$$

describes $y$ as a function of $x$. For this function, $x$ is the independent variable and $y$ is the dependent variable.

*Example 1*  **Deciding Whether Relations Are Functions**

Which of the equations below define $y$ as a function of $x$?

**a.** $x + y = 1$        **b.** $x^2 + y^2 = 1$

**c.** $x^2 + y = 1$        **d.** $x + y^2 = 1$

**SOLUTION**    To decide whether an equation defines a function, it is helpful to isolate the dependent variable on the left side. For instance, to decide whether the equation $x + y = 1$ defines $y$ as a function of $x$, write the equation in the form

$$y = 1 - x.$$

From this form, you can see that for any value of $x$, there is exactly one value of $y$. So, $y$ is a function of $x$.

| *Original Equation* | *Rewritten Equation* | *Test: Is y a function of x?* |
|---|---|---|
| **a.** $x + y = 1$ | $y = 1 - x$ | Yes, each value of $x$ determines exactly one value of $y$. |
| **b.** $x^2 + y^2 = 1$ | $y = \pm\sqrt{1 - x^2}$ | No, some values of $x$ determine two values of $y$. |
| **c.** $x^2 + y = 1$ | $y = 1 - x^2$ | Yes, each value of $x$ determines exactly one value of $y$. |
| **d.** $x + y^2 = 1$ | $y = \pm\sqrt{1 - x}$ | No, some values of $x$ determine two values of $y$. |
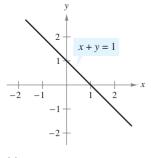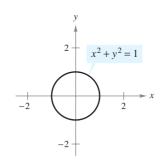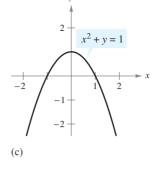
Note that the equations that assign two values ($\pm$) to the dependent variable for a given value of the independent variable do not define functions of $x$. For instance, in part (b), when $x = 0$, the equation $y = \pm\sqrt{1 - x^2}$ indicates that $y = +1$ or $y = -1$. Figure 1.44 shows the graphs of the four equations.



(a)    (b)    (c)    (d)

**FIGURE 1.44**
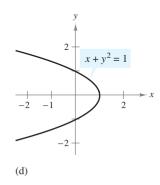
✓ **CHECKPOINT 1**

Which of the equations below define $y$ as a function of $x$? Explain your answer.

**a.** $x - y = 1$    **b.** $x^2 + y^2 = 4$    **c.** $y^2 + x = 2$    **d.** $x^2 - y = 0$ ■

## The Graph of a Function

When the graph of a function is sketched, the standard convention is to let the horizontal axis represent the independent variable. When this convention is used, the test described in Example 1 has a nice graphical interpretation called the **Vertical Line Test.** This test states that if every vertical line intersects the graph of an equation at most once, then the equation defines $y$ as a function of $x$. For instance, in Figure 1.44, the graphs in parts (a) and (c) pass the Vertical Line Test, but those in parts (b) and (d) do not.

The domain of a function may be described explicitly, or it may be *implied* by an equation used to define the function. For example, the function given by

$$y = \frac{1}{x^2 - 4}$$

has an implied domain that consists of all real $x$ except $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined.

Another type of implied domain is that used to avoid even roots of negative numbers, as indicated in Example 2.

**Example 2** **Finding the Domain and Range of a Function**

Find the domain and range of each function.
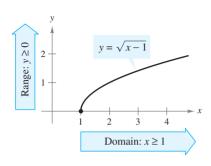
**a.** $y = \sqrt{x - 1}$    **b.** $y = \begin{cases} 1 - x, & x < 1 \\ \sqrt{x - 1}, & x \geq 1 \end{cases}$

**SOLUTION**
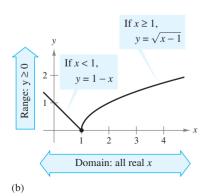
**a.** Because $\sqrt{x - 1}$ is not defined for $x - 1 < 0$ (that is, for $x < 1$), it follows that the domain of the function is the interval $x \geq 1$ or $[1, \infty)$. To find the range, observe that $\sqrt{x - 1}$ is never negative. Moreover, as $x$ takes on the various values in the domain, $y$ takes on all nonnegative values. So, the range is the interval $y \geq 0$ or $[0, \infty)$. The graph of the function, shown in Figure 1.45(a), confirms these results.

**b.** Because this function is defined for $x < 1$ *and* for $x \geq 1$, the domain is the entire set of real numbers. This function is called a **piecewise-defined function** because it is defined by two or more equations over a specified domain. When $x \geq 1$, the function behaves as in part (a). For $x < 1$, the value of $1 - x$ is positive, and therefore the range of the function is $y \geq 0$ or $[0, \infty)$, as shown in Figure 1.45(b).

(a)

(b)

**FIGURE 1.45**

A function is **one-to-one** if to each value of the dependent variable in the range there corresponds exactly one value of the independent variable. For instance, the function in Example 2(a) is one-to-one, whereas the function in Example 2(b) is not one-to-one.

Geometrically, a function is one-to-one if every horizontal line intersects the graph of the function at most once. This geometrical interpretation is the **Horizontal Line Test** for one-to-one functions. So, a graph that represents a one-to-one function must satisfy *both* the Vertical Line Test and the Horizontal Line Test.

✓ **CHECKPOINT 2**

Find the domain and range of each function.

**a.** $y = \sqrt{x + 1}$

**b.** $y = \begin{cases} x^2, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$ ■

## Function Notation

When using an equation to define a function, you generally isolate the dependent variable on the left. For instance, writing the equation $x + 2y = 1$ as

$$y = \frac{1 - x}{2}$$

indicates that $y$ is the dependent variable. In **function notation,** this equation has the form

$$f(x) = \frac{1 - x}{2}. \qquad \text{Function notation}$$

The independent variable is $x$, and the name of the function is "$f$." The symbol $f(x)$ is read as "$f$ of $x$," and it denotes the value of the dependent variable. For instance, the value of $f$ when $x = 3$ is

$$f(3) = \frac{1 - (3)}{2} = \frac{-2}{2} = -1.$$

The value $f(3)$ is called a **function value,** and it lies in the range of $f$. This means that the point $(3, f(3))$ lies on the graph of $f$. One of the advantages of function notation is that it allows you to be less wordy. For instance, instead of asking "What is the value of $y$ when $x = 3$?" you can ask "What is $f(3)$?"

### Example 3  Evaluating a Function

Find the values of the function $f(x) = 2x^2 - 4x + 1$ when $x$ is $-1$, $0$, and $2$. Is $f$ one-to-one?

**SOLUTION**    When $x = -1$, the value of $f$ is

$$f(-1) = 2(-1)^2 - 4(-1) + 1 = 2 + 4 + 1 = 7.$$
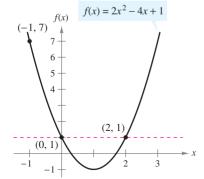
When $x = 0$, the value of $f$ is

$$f(0) = 2(0)^2 - 4(0) + 1 = 0 - 0 + 1 = 1.$$

When $x = 2$, the value of $f$ is

$$f(2) = 2(2)^2 - 4(2) + 1 = 8 - 8 + 1 = 1.$$

Because two different values of $x$ yield the same value of $f(x)$, the function is *not* one-to-one, as shown in Figure 1.46.

$f(x) = 2x^2 - 4x + 1$

**FIGURE 1.46**

### ✓ CHECKPOINT 3

Find the values of $f(x) = x^2 - 5x + 1$ when $x$ is $0$, $1$, and $4$. Is $f$ one-to-one?  ■

### STUDY TIP

You can use the Horizontal Line Test to determine whether the function in Example 3 is one-to-one. Because the line $y = 1$ intersects the graph of the function twice, the function is *not* one-to-one.

Example 3 suggests that the role of the variable $x$ in the equation

$$f(x) = 2x^2 - 4x + 1$$

is simply that of a placeholder. Informally, $f$ could be defined by the equation

$$f(\quad) = 2(\quad)^2 - 4(\quad) + 1.$$

To evaluate $f(-2)$, simply place $-2$ in each set of parentheses.

$$f(-2) = 2(-2)^2 - 4(-2) + 1 = 8 + 8 + 1 = 17$$

The ratio in Example 4(b) is called a *difference quotient*. In Section 2.1, you will see that it has special significance in calculus.

### *Example 4*   Evaluating a Function

Let $f(x) = x^2 - 4x + 7$, and find

**a.** $f(x + \Delta x)$     **b.** $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$.

**SOLUTION**

**a.** To evaluate $f$ at $x + \Delta x$, substitute $x + \Delta x$ for $x$ in the original function, as shown.

$$f(x + \Delta x) = (x + \Delta x)^2 - 4(x + \Delta x) + 7$$
$$= x^2 + 2x\,\Delta x + (\Delta x)^2 - 4x - 4\,\Delta x + 7$$

**b.** Using the result of part (a), you can write

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \frac{[(x + \Delta x)^2 - 4(x + \Delta x) + 7] - [x^2 - 4x + 7]}{\Delta x}$$
$$= \frac{x^2 + 2x\,\Delta x + (\Delta x)^2 - 4x - 4\,\Delta x + 7 - x^2 + 4x - 7}{\Delta x}$$
$$= \frac{2x\,\Delta x + (\Delta x)^2 - 4\,\Delta x}{\Delta x}$$
$$= 2x + \Delta x - 4, \quad \Delta x \neq 0.$$

✓ **CHECKPOINT 4**

Let $f(x) = x^2 - 2x + 3$, and find (a) $f(x + \Delta x)$ and (b) $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$. ■

Although $f$ is often used as a convenient function name and $x$ as the independent variable, you can use other symbols. For instance, the following equations all define the same function.

$$f(x) = x^2 - 4x + 7$$
$$f(t) = t^2 - 4t + 7$$
$$g(s) = s^2 - 4s + 7$$

**FIGURE 1.47**

## Combinations of Functions

Two functions can be combined in various ways to create new functions. For instance, if $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the following functions.

$$f(x) + g(x) = (2x - 3) + (x^2 + 1) = x^2 + 2x - 2 \qquad \text{Sum}$$
$$f(x) - g(x) = (2x - 3) - (x^2 + 1) = -x^2 + 2x - 4 \qquad \text{Difference}$$
$$f(x)g(x) = (2x - 3)(x^2 + 1) = 2x^3 - 3x^2 + 2x - 3 \qquad \text{Product}$$
$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 + 1} \qquad \text{Quotient}$$

You can combine two functions in yet another way called a **composition.** The resulting function is a **composite function.**

---

**Definition of Composite Function**

The function given by $(f \circ g)(x) = f(g(x))$ is the **composite** of $f$ with $g$. The **domain** of $(f \circ g)$ is the set of all $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$, as indicated in Figure 1.47.

---

The composite of $f$ with $g$ may not be equal to the composite of $g$ with $f$, as shown in the next example.

### *Example 5*  Forming Composite Functions

Let $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, and find

**a.** $f(g(x))$     **b.** $g(f(x))$.

**SOLUTION**

**a.** The composite of $f$ with $g$ is given by

$$f(g(x)) = 2(g(x)) - 3 \qquad \text{Evaluate } f \text{ at } g(x).$$
$$= 2(x^2 + 1) - 3 \qquad \text{Substitute } x^2 + 1 \text{ for } g(x).$$
$$= 2x^2 - 1. \qquad \text{Simplify.}$$

**b.** The composite of $g$ with $f$ is given by

$$g(f(x)) = (f(x))^2 + 1 \qquad \text{Evaluate } g \text{ at } f(x).$$
$$= (2x - 3)^2 + 1 \qquad \text{Substitute } 2x - 3 \text{ for } f(x).$$
$$= 4x^2 - 12x + 10. \qquad \text{Simplify.}$$

**✓CHECKPOINT 5**

Let $f(x) = 2x + 1$ and $g(x) = x^2 + 2$, and find

**a.** $f(g(x))$     **b.** $g(f(x))$. ■

**STUDY TIP**

The results of $f(g(x))$ and $g(f(x))$ are different in Example 5. You can verify this by substituting specific values of $x$ into each function and comparing the results.
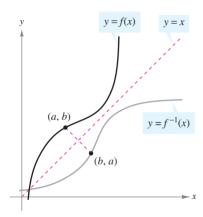
## Inverse Functions

Informally, the inverse function of $f$ is another function $g$ that "undoes" what $f$ has done.

$$x \xrightarrow{\ f\ } f(x) \xrightarrow{\ g\ } g(f(x)) = x$$

### STUDY TIP

Don't be confused by the use of the superscript $-1$ to denote the inverse function $f^{-1}$. In this text, whenever $f^{-1}$ is written, it *always* refers to the inverse function of $f$ and *not* to the reciprocal of $f(x)$.

**Definition of Inverse Function**

Let $f$ and $g$ be two functions such that

$$f(g(x)) = x \text{ for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \text{ for each } x \text{ in the domain of } f.$$

Under these conditions, the function $g$ is the **inverse function** of $f$. The function $g$ is denoted by $f^{-1}$, which is read as "$f$-inverse." So,

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

The domain of $f$ must be equal to the range of $f^{-1}$, and the range of $f$ must be equal to the domain of $f^{-1}$.



**FIGURE 1.48**   The graph of $f^{-1}$ is a reflection of the graph of $f$ in the line $y = x$.

### *Example 6*   Finding Inverse Functions

Several functions and their inverse functions are shown below. In each case, note that the inverse function "undoes" the original function. For instance, to undo multiplication by 2, you should divide by 2.

**a.** $f(x) = 2x$          $f^{-1}(x) = \frac{1}{2}x$

**b.** $f(x) = \frac{1}{3}x$          $f^{-1}(x) = 3x$

**c.** $f(x) = x + 4$        $f^{-1}(x) = x - 4$

**d.** $f(x) = 2x - 5$      $f^{-1}(x) = \frac{1}{2}(x + 5)$

**e.** $f(x) = x^3$          $f^{-1}(x) = \sqrt[3]{x}$

**f.** $f(x) = \dfrac{1}{x}$        $f^{-1}(x) = \dfrac{1}{x}$

### STUDY TIP

You can verify that the functions in Example 6 are inverse functions by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. For Example 6(a), you obtain the following.

$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$

### ✓CHECKPOINT 6

Informally find the inverse function of each function.

**a.** $f(x) = \frac{1}{5}x$      **b.** $f(x) = 3x + 2$  ■

The graphs of $f$ and $f^{-1}$ are mirror images of each other (with respect to the line $y = x$), as shown in Figure 1.48. Try using a graphing utility to confirm this for each of the functions given in Example 6.

The functions in Example 6 are simple enough so that their inverse functions can be found by inspection. The next example demonstrates a strategy for finding the inverse functions of more complicated functions.

### Example 7    Finding an Inverse Function

Find the inverse function of $f(x) = \sqrt{2x - 3}$.

**SOLUTION**    Begin by replacing $f(x)$ with $y$. Then, interchange $x$ and $y$ and solve for $y$.
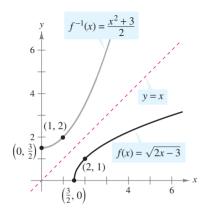
$$f(x) = \sqrt{2x - 3} \qquad \text{Write original function.}$$

$$y = \sqrt{2x - 3} \qquad \text{Replace } f(x) \text{ with } y.$$

$$x = \sqrt{2y - 3} \qquad \text{Interchange } x \text{ and } y.$$

$$x^2 = 2y - 3 \qquad \text{Square each side.}$$

$$x^2 + 3 = 2y \qquad \text{Add 3 to each side.}$$

$$\frac{x^2 + 3}{2} = y \qquad \text{Divide each side by 2.}$$

So, the inverse function has the form

$$f^{-1}(\boxed{\phantom{x}}) = \frac{(\boxed{\phantom{x}})^2 + 3}{2}.$$

Using $x$ as the independent variable, you can write

$$f^{-1}(x) = \frac{x^2 + 3}{2}, \quad x \geq 0.$$

In Figure 1.49, note that the domain of $f^{-1}$ coincides with the range of $f$.



**FIGURE 1.49**

### ✓ CHECKPOINT 7

Find the inverse function of $f(x) = x^2 + 2$ for $x \geq 0$. ■

After you have found an inverse function, you should check your results. You can check your results *graphically* by observing that the graphs of $f$ and $f^{-1}$ are reflections of each other in the line $y = x$. You can check your results *algebraically* by evaluating $f(f^{-1}(x))$ and $f^{-1}(f(x))$—both should be equal to $x$.

*Check that $f(f^{-1}(x)) = x$*

$$f(f^{-1}(x)) = f\left(\frac{x^2 + 3}{2}\right)$$

$$= \sqrt{2\left(\frac{x^2 + 3}{2}\right) - 3}$$

$$= \sqrt{x^2}$$

$$= x, \quad x \geq 0$$

*Check that $f^{-1}(f(x)) = x$*

$$f^{-1}(f(x)) = f^{-1}\left(\sqrt{2x - 3}\right)$$

$$= \frac{\left(\sqrt{2x - 3}\right)^2 + 3}{2}$$

$$= \frac{2x}{2}$$

$$= x, \quad x \geq \frac{3}{2}$$

Not every function has an inverse function. In fact, for a function to have an inverse function, it must be one-to-one.

### Example 8 A Function That Has No Inverse Function

Show that the function

$$f(x) = x^2 - 1$$

has no inverse function. (Assume that the domain of $f$ is the set of all real numbers.)

**SOLUTION** Begin by sketching the graph of $f$, as shown in Figure 1.50. Note that

$$f(2) = (2)^2 - 1 = 3$$

and
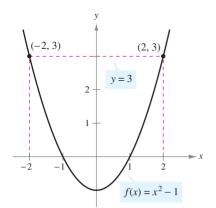
$$f(-2) = (-2)^2 - 1 = 3.$$

So, $f$ does not pass the Horizontal Line Test, which implies that it is not one-to-one, and therefore has no inverse function. The same conclusion can be obtained by trying to find the inverse function of $f$ algebraically.

| | |
|---|---|
| $f(x) = x^2 - 1$ | Write original function. |
| $y = x^2 - 1$ | Replace $f(x)$ with $y$. |
| $x = y^2 - 1$ | Interchange $x$ and $y$. |
| $x + 1 = y^2$ | Add 1 to each side. |
| $\pm\sqrt{x + 1} = y$ | Take square root of each side. |

The last equation does not define $y$ as a function of $x$, and so $f$ has no inverse function.



**FIGURE 1.50** *f* is not one-to-one and has no inverse function.

### ✓ CHECKPOINT 8

Show that the function

$$f(x) = x^2 + 4$$

has no inverse function. ■

---

### CONCEPT CHECK

1. Explain the difference between a relation and a function.
2. In your own words, explain the meanings of *domain* and *range*.
3. Is every relation a function? Explain.
4. Describe how to find the inverse of a function given by an equation in *x* and *y*.

**Skills Review 1.4** The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 0.5.

In Exercises 1–6, simplify the expression.

**1.** $5(-1)^2 - 6(-1) + 9$

**2.** $(-2)^3 + 7(-2)^2 - 10$

**3.** $(x - 2)^2 + 5x - 10$

**4.** $(3 - x) + (x + 3)^3$

**5.** $\dfrac{1}{1 - (1 - x)}$

**6.** $1 + \dfrac{x - 1}{x}$

In Exercises 7–12, solve for $y$ in terms of $x$.

**7.** $2x + y - 6 = 11$

**8.** $5y - 6x^2 - 1 = 0$

**9.** $(y - 3)^2 = 5 + (x + 1)^2$

**10.** $y^2 - 4x^2 = 2$

**11.** $x = \dfrac{2y - 1}{4}$

**12.** $x = \sqrt[3]{2y - 1}$

## Exercises 1.4

In Exercises 1–8, decide whether the equation defines $y$ as a function of $x$.

**1.** $x^2 + y^2 = 4$

**2.** $x + y^2 = 4$

**3.** $\frac{1}{2}x - 6y = -3$
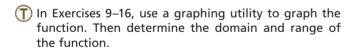
**4.** $3x - 2y + 5 = 0$

**5.** $x^2 + y = 4$

**6.** $x^2 + y^2 + 2x = 0$

**7.** $y = |x + 2|$

**8.** $x^2y - x^2 + 4y = 0$

(T) In Exercises 9–16, use a graphing utility to graph the function. Then determine the domain and range of the function.

**9.** $f(x) = 2x^2 - 5x + 1$

**10.** $f(x) = 5x^3 + 6x^2 - 1$

**11.** $f(x) = \dfrac{|x|}{x}$

**12.** $f(x) = \sqrt{9 - x^2}$

**13.** $f(x) = \dfrac{x}{\sqrt{x - 4}}$
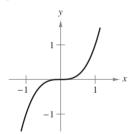
**14.** $f(x) = \begin{cases} 3x + 2, & x < 0 \\ 2 - x, & x \geq 0 \end{cases}$

**15.** $f(x) = \dfrac{x - 2}{x + 4}$
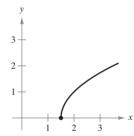
**16.** $f(x) = \dfrac{x^2}{1 - x}$

In Exercises 17–20, find the domain and range of the function. Use interval notation to write your result.

**17.** $f(x) = x^3$
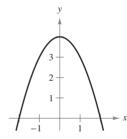
**18.** $f(x) = \sqrt{2x - 3}$
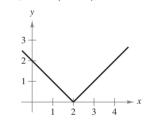




**19.** $f(x) = 4 - x^2$

**20.** $f(x) = |x - 2|$





In Exercises 21–24, evaluate the function at the specified values of the independent variable. Simplify the result.

**21.** $f(x) = 3x - 2$

(a) $f(0)$     (b) $f(x - 1)$     (c) $f(x + \Delta x)$

**22.** $f(x) = x^2 - 4x + 1$

(a) $f(-1)$     (b) $f(c + 2)$     (c) $f(x + \Delta x)$

**23.** $g(x) = 1/x$

(a) $g\left(\frac{1}{4}\right)$     (b) $g(x + 4)$     (c) $g(x + \Delta x) - g(x)$

**24.** $f(x) = |x| + 4$

(a) $f(-2)$     (b) $f(x + 2)$     (c) $f(x + \Delta x) - f(x)$

In Exercises 25–30, evaluate the difference quotient and simplify the result.

**25.** $f(x) = x^2 - 5x + 2$

$$\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$$

**26.** $h(x) = x^2 + x + 3$

$$\dfrac{h(2 + \Delta x) - h(2)}{\Delta x}$$

**27.** $g(x) = \sqrt{x+1}$

$$\dfrac{g(x+\Delta x) - g(x)}{\Delta x}$$

**28.** $f(x) = \dfrac{1}{\sqrt{x}}$

$$\dfrac{f(x) - f(2)}{x - 2}$$

**29.** $f(x) = \dfrac{1}{x-2}$

$$\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$$

**30.** $f(x) = \dfrac{1}{x+4}$
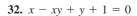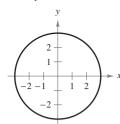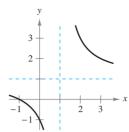
$$\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$$

In Exercises 31–34, use the Vertical Line Test to determine whether y is a function of x.

**31.** $x^2 + y^2 = 9$

**32.** $x - xy + y + 1 = 0$



**33.** $x^2 = xy - 1$

**34.** $x = |y|$
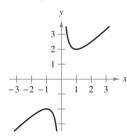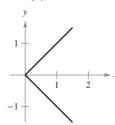


In Exercises 35–38, find (a) f(x) + g(x), (b) f(x) · g(x), (c) f(x)/g(x), (d) f(g(x)), and (e) g(f(x)), if defined.

**35.** $f(x) = 2x - 5$

$g(x) = 5$

**36.** $f(x) = x^2 + 5$

$g(x) = \sqrt{1-x}$

**37.** $f(x) = x^2 + 1$

$g(x) = x - 1$

**38.** $f(x) = \dfrac{x}{x+1}$

$g(x) = x^3$

**39.** Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, find the composite functions.

(a) $f(g(1))$     (b) $g(f(1))$

(c) $g(f(0))$     (d) $f(g(-4))$

(e) $f(g(x))$     (f) $g(f(x))$

**40.** Given $f(x) = 1/x$ and $g(x) = x^2 - 1$, find the composite functions.

(a) $f(g(2))$     (b) $g(f(2))$

(c) $f(g(1/\sqrt{2}))$     (d) $g(f(1/\sqrt{2}))$

(e) $f(g(x))$     (f) $g(f(x))$

In Exercises 41–44, show that f and g are inverse functions by showing that $f(g(x)) = x$ and $g(f(x)) = x$. Then sketch the graphs of f and g on the same coordinate axes.

**41.** $f(x) = 5x + 1$,     $g(x) = \dfrac{x-1}{5}$

**42.** $f(x) = \dfrac{1}{x}$,     $g(x) = \dfrac{1}{x}$

**43.** $f(x) = 9 - x^2$, $x \ge 0$,     $g(x) = \sqrt{9-x}$, $x \le 9$

**44.** $f(x) = 1 - x^3$,     $g(x) = \sqrt[3]{1-x}$

Ⓣ In Exercises 45–52, find the inverse function of f. Then use a graphing utility to graph f and $f^{-1}$ on the same coordinate axes.

**45.** $f(x) = 2x - 3$

**46.** $f(x) = 7 - x$

**47.** $f(x) = x^5$

**48.** $f(x) = x^3$

**49.** $f(x) = \sqrt{9 - x^2}$, $0 \le x \le 3$

**50.** $f(x) = \sqrt{x^2 - 4}$, $x \ge 2$

**51.** $f(x) = x^{2/3}$, $x \ge 0$

**52.** $f(x) = x^{3/5}$

Ⓣ In Exercises 53–58, use a graphing utility to graph the function. Then use the Horizontal Line Test to determine whether the function is one-to-one. If it is, find its inverse function.

**53.** $f(x) = 3 - 7x$

**54.** $f(x) = \sqrt{x-2}$

**55.** $f(x) = x^2$

**56.** $f(x) = x^4$

**57.** $f(x) = |x + 3|$

**58.** $f(x) = -5$

**59.** Use the graph of $f(x) = \sqrt{x}$ below to sketch the graph of each function.

(a) $y = \sqrt{x} + 2$
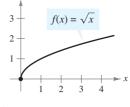
(b) $y = -\sqrt{x}$

(c) $y = \sqrt{x-2}$

(d) $y = \sqrt{x+3}$

(e) $y = \sqrt{x-4}$

(f) $y = 2\sqrt{x}$



**60.** Use the graph of $f(x) = |x|$ below to sketch the graph of each function.
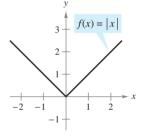
(a) $y = |x| + 3$

(b) $y = -\frac{1}{2}|x|$
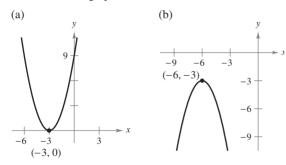
(c) $y = |x - 2|$

(d) $y = |x + 1| - 1$
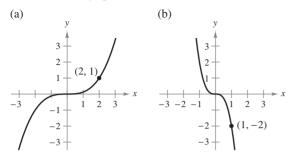
(e) $y = 2|x|$

**61.** Use the graph of $f(x) = x^2$ to write an equation for each function whose graph is shown.

(a)



(b)
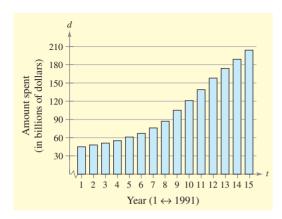


**62.** Use the graph of $f(x) = x^3$ to write an equation for each function whose graph is shown.

(a)



(b)



**63. Prescription Drugs** The amounts $d$ (in billions of dollars) spent on prescription drugs in the United States from 1991 through 2005 (see figure) can be approximated by the model

$$d(t) = \begin{cases} y = 0.68t^2 - 0.3t + 45, & 1 \le t \le 8 \\ y = 16.7t - 45, & 9 \le t \le 15 \end{cases}$$

where $t$ represents the year, with $t = 1$ corresponding to 1991. *(Source: U.S. Centers for Medicare & Medicaid Services)*



(T) (a) Use a graphing utility to graph the function.

(b) Find the amounts spent on prescription drugs in 1997, 2000, and 2004.

**64. Real Estate** Express the value $V$ of a real estate firm in terms of $x$, the number of acres of property owned. Each acre is valued at $2500 and other company assets total $750,000.

(T) **65. Owning a Business** You own two restaurants. From 2001 through 2007, the sales $R_1$ (in thousands of dollars) for one restaurant can be modeled by

$$R_1 = 690 - 8t - 0.8t^2, \quad t = 1, 2, 3, 4, 5, 6, 7$$

where $t = 1$ represents 2001. During the same seven-year period, the sales $R_2$ (in thousands of dollars) for the second restaurant can be modeled by

$$R_2 = 458 + 0.78t, \quad t = 1, 2, 3, 4, 5, 6, 7.$$

Write a function that represents the total sales for the two restaurants. Use a graphing utility to graph the total sales function.

**66. Cost** The inventor of a new game believes that the variable cost for producing the game is $1.95 per unit. The fixed cost is $6000.

(a) Express the total cost $C$ as a function of $x$, the number of games sold.

(b) Find a formula for the average cost per unit $\overline{C} = C/x$.

(c) The selling price for each game is $4.95. How many units must be sold before the average cost per unit falls below the selling price?

**67. Demand** The demand function for a commodity is
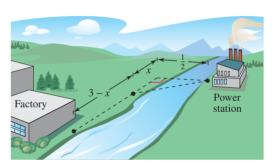
$$p = \frac{14.75}{1 + 0.01x}, \quad x \ge 0$$

where $p$ is the price per unit and $x$ is the number of units sold.

(a) Find $x$ as a function of $p$.

(b) Find the number of units sold when the price is $10.

**68. Cost** A power station is on one side of a river that is $\frac{1}{2}$ mile wide. A factory is 3 miles downstream on the other side of the river (see figure). It costs $15/ft to run the power lines on land and $20/ft to run them under water. Express the cost $C$ of running the lines from the power station to the factory as a function of $x$.

**69. Cost**   The weekly cost of producing $x$ units in a manufacturing process is given by the function

$$C(x) = 70x + 375.$$

The number of units produced in $t$ hours is given by $x(t) = 40t$. Find and interpret $C(x(t))$.

**(T) 70. Market Equilibrium**   The supply function for a product relates the number of units $x$ that producers are willing to supply for a given price per unit $p$. The supply and demand functions for a market are

$$p = \frac{2}{5}x + 4 \qquad \text{Supply}$$

$$p = -\frac{16}{25}x + 30. \qquad \text{Demand}$$

(a) Use a graphing utility to graph the supply and demand functions in the same viewing window.

(b) Use the *trace* feature of the graphing utility to find the *equilibrium point* for the market.

(c) For what values of $x$ does the demand exceed the supply?

(d) For what values of $x$ does the supply exceed the demand?

**71. Profit**   A manufacturer charges $90 per unit for units that cost $60 to produce. To encourage large orders from distributors, the manufacturer will reduce the price by $0.01 per unit for each unit in excess of 100 units. (For example, an order of 101 units would have a price of $89.99 per unit, and an order of 102 units would have a price of $89.98 per unit.) This price reduction is discontinued when the price per unit drops to $75.

(a) Express the price per unit $p$ as a function of the order size $x$.

(b) Express the profit $P$ as a function of the order size $x$.

**72. Cost, Revenue, and Profit**   A company invests $98,000 for equipment to produce a new product. Each unit of the product costs $12.30 and is sold for $17.98. Let $x$ be the number of units produced and sold.

(a) Write the total cost $C$ as a function of $x$.

(b) Write the revenue $R$ as a function of $x$.

(c) Write the profit $P$ as a function of $x$.

**73. *MAKE A DECISION: REVENUE***   For groups of 80 or more people, a charter bus company determines the rate $r$ (in dollars per person) according to the formula

$$r = 15 - 0.05(n - 80), \quad n \geq 80$$

where $n$ is the number of people.

(a) Express the revenue $R$ for the bus company as a function of $n$.

(b) Complete the table.

| $n$ | 100 | 125 | 150 | 175 | 200 | 225 | 250 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $R$ |     |     |     |     |     |     |     |

(c) Criticize the formula for the rate. Would you use this formula? Explain your reasoning.

**(T) 74. Medicine**   The temperature of a patient after being given a fever-reducing drug is given by

$$F(t) = 98 + \frac{3}{t + 1}$$

where $F$ is the temperature in degrees Fahrenheit and $t$ is the time in hours since the drug was administered. Use a graphing utility to graph the function. Be sure to choose an appropriate viewing window. For what values of $t$ do you think this function would be valid? Explain.

**(T)** In Exercises 75–80, use a graphing utility to graph the function. Then use the *zoom* and *trace* features to find the zeros of the function. Is the function one-to-one?

**75.** $f(x) = 9x - 4x^2$

**76.** $f(x) = 2\left(3x^2 - \dfrac{6}{x}\right)$

**77.** $g(t) = \dfrac{t + 3}{1 - t}$

**78.** $h(x) = 6x^3 - 12x^2 + 4$

**79.** $g(x) = x^2\sqrt{x^2 - 4}$

**80.** $g(x) = \left|\dfrac{1}{2}x^2 - 4\right|$

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**Business Capsule**



Photo courtesy of Cap Poore

**C**ardSenders is a home-based greeting card service for businesses. Cap Poore bought the company in 2003, which has licensees operating in the United Kingdom, Canada, Asia, and Mexico. Start-up costs are $6,900.00 for licensees.

**81. Research Project**   Use your school's library, the Internet, or some other reference source to find information about the start-up costs of beginning a business, such as the example above. Write a short paper about the company.

**Section 1.5**

# Limits

■ Find limits of functions graphically and numerically.
■ Use the properties of limits to evaluate limits of functions.
■ Use different analytic techniques to evaluate limits of functions.
■ Evaluate one-sided limits.
■ Recognize unbounded behavior of functions.

## The Limit of a Function

In everyday language, people refer to a speed limit, a wrestler's weight limit, the limit of one's endurance, or stretching a spring to its limit. These phrases all suggest that a limit is a bound, which on some occasions may not be reached but on other occasions may be reached or exceeded.

Consider a spring that will break only if a weight of 10 pounds or more is attached. To determine how far the spring will stretch without breaking, you could attach increasingly heavier weights and measure the spring length $s$ for each weight $w$, as shown in Figure 1.51. If the spring length approaches a value of $L$, then it is said that "the limit of $s$ as $w$ approaches 10 is $L$." A mathematical limit is much like the limit of a spring. The notation for a limit is

$$\lim_{x \to c} f(x) = L$$

which is read as "the limit of $f(x)$ as $x$ approaches $c$ is $L$."



**FIGURE 1.51**    What is the limit of $s$ as $w$ approaches 10 pounds?

$w = 0$
$w = 3$
$w = 7.5$
$w = 9.5$
$w = 9.999$
$s$

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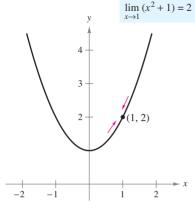
**Example 1**    **Finding a Limit**

Find the limit: $\lim_{x \to 1} (x^2 + 1)$.

**SOLUTION**    Let $f(x) = x^2 + 1$. From the graph of $f$ in Figure 1.52, it appears that $f(x)$ approaches 2 as $x$ approaches 1 from either side, and you can write

$$\lim_{x \to 1} (x^2 + 1) = 2.$$

The table yields the same conclusion. Notice that as $x$ gets closer and closer to 1, $f(x)$ gets closer and closer to 2.



$\lim_{x \to 1} (x^2 + 1) = 2$

**FIGURE 1.52**

| | *x* approaches 1. | | | | *x* approaches 1. | | |
|---|---|---|---|---|---|---|---|
| $x$ | 0.900 | 0.990 | 0.999 | 1.000 | 1.001 | 1.010 | 1.100 |
| $f(x)$ | 1.810 | 1.980 | 1.998 | 2.000 | 2.002 | 2.020 | 2.210 |
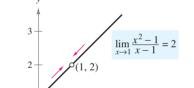
$f(x)$ approaches 2.    $f(x)$ approaches 2.
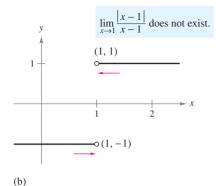
✓ **CHECKPOINT 1**
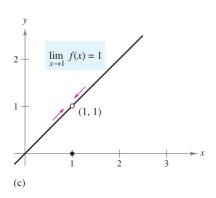
Find the limit: $\lim_{x \to 1} (2x + 4)$. ■

### Example 2  Finding Limits Graphically and Numerically

Find the limit: $\lim\limits_{x\to1} f(x)$.

**a.** $f(x) = \dfrac{x^2 - 1}{x - 1}$   **b.** $f(x) = \dfrac{|x - 1|}{x - 1}$   **c.** $f(x) = \begin{cases} x, & x \neq 1 \\ 0, & x = 1 \end{cases}$
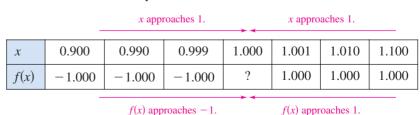
#### SOLUTION

**a.** From the graph of $f$, in Figure 1.53(a), it appears that $f(x)$ approaches 2 as $x$ approaches 1 from either side. A missing point is denoted by the open dot on the graph. This conclusion is reinforced by the table. Be sure you see that *it does not matter that $f(x)$ is undefined when $x = 1$. The limit depends only on values of $f(x)$ near 1, not at 1.*

$\lim\limits_{x\to1} \dfrac{x^2-1}{x-1} = 2$

$(1, 2)$

(a)

| | | *x* approaches 1. | | | | *x* approaches 1. | |
|------|-------|-------|-------|-------|-------|-------|-------|
| $x$ | 0.900 | 0.990 | 0.999 | 1.000 | 1.001 | 1.010 | 1.100 |
| $f(x)$ | 1.900 | 1.990 | 1.999 | ? | 2.001 | 2.010 | 2.100 |

*f(x) approaches 2.*        *f(x) approaches 2.*

**b.** From the graph of $f$, in Figure 1.53(b), you can see that $f(x) = -1$ for all values to the left of $x = 1$ and $f(x) = 1$ for all values to the right of $x = 1$. So, $f(x)$ is approaching a different value from the left of $x = 1$ than it is from the right of $x = 1$. In such situations, we say that *the limit does not exist*. This conclusion is reinforced by the table.

$\lim\limits_{x\to1} \dfrac{|x-1|}{x-1}$ does not exist.

$(1, 1)$

$(1, -1)$

(b)

| | | *x* approaches 1. | | | | *x* approaches 1. | |
|------|--------|--------|--------|-------|-------|-------|-------|
| $x$ | 0.900 | 0.990 | 0.999 | 1.000 | 1.001 | 1.010 | 1.100 |
| $f(x)$ | $-1.000$ | $-1.000$ | $-1.000$ | ? | 1.000 | 1.000 | 1.000 |

*f(x) approaches −1.*        *f(x) approaches 1.*

**c.** From the graph of $f$, in Figure 1.53(c), it appears that $f(x)$ approaches 1 as $x$ approaches 1 from either side. This conclusion is reinforced by the table. It does not matter that $f(1) = 0$. The limit depends only on values of $f(x)$ near 1, not at 1.

$\lim\limits_{x\to1} f(x) = 1$

$(1, 1)$

(c)

**FIGURE 1.53**

| | | *x* approaches 1. | | | | *x* approaches 1. | |
|------|-------|-------|-------|-------|-------|-------|-------|
| $x$ | 0.900 | 0.990 | 0.999 | 1.000 | 1.001 | 1.010 | 1.100 |
| $f(x)$ | 0.900 | 0.990 | 0.999 | ? | 1.001 | 1.010 | 1.100 |

*f(x) approaches 1.*        *f(x) approaches 1.*

✓ **CHECKPOINT 2**

Find the limit: $\lim\limits_{x\to2} f(x)$.

**a.** $f(x) = \dfrac{x^2 - 4}{x - 2}$

**b.** $f(x) = \dfrac{|x - 2|}{x - 2}$

**c.** $f(x) = \begin{cases} x^2, & x \neq 2 \\ 0, & x = 2 \end{cases}$ ■

There are three important ideas to learn from Examples 1 and 2.

**1.** Saying that the limit of $f(x)$ approaches $L$ as $x$ approaches $c$ means that the value of $f(x)$ may be made *arbitrarily close* to the number $L$ by choosing $x$ closer and closer to $c$.

**2.** For a limit to exist, you must allow $x$ to approach $c$ from *either side* of $c$. If $f(x)$ approaches a different number as $x$ approaches $c$ from the left than it does as $x$ approaches $c$ from the right, then the limit *does not exist*. [See Example 2(b).]

**3.** The value of $f(x)$ when $x = c$ has no bearing on the existence or nonexistence of the limit of $f(x)$ as $x$ approaches $c$. For instance, in Example 2(a), the limit of $f(x)$ exists as $x$ approaches 1 even though the function $f$ is not defined at $x = 1$.

---

**Definition of the Limit of a Function**

If $f(x)$ becomes arbitrarily close to a single number $L$ as $x$ approaches $c$ from either side, then

$$\lim_{x \to c} f(x) = L$$

which is read as "the **limit** of $f(x)$ as $x$ approaches $c$ is $L$."

---

## Properties of Limits

Many times the limit of $f(x)$ as $x$ approaches $c$ is simply $f(c)$, as shown in Example 1. Whenever the limit of $f(x)$ as $x$ approaches $c$ is

$$\lim_{x \to c} f(x) = f(c) \qquad \text{Substitute } c \text{ for } x.$$

the limit can be evaluated by **direct substitution.** (In the next section, you will learn that a function that has this property is *continuous at c*.) It is important that you learn to recognize the types of functions that have this property. Some basic ones are given in the following list.

---

**Properties of Limits**

Let $b$ and $c$ be real numbers, and let $n$ be a positive integer.

**1.** $\lim\limits_{x \to c} b = b$

**2.** $\lim\limits_{x \to c} x = c$

**3.** $\lim\limits_{x \to c} x^n = c^n$

**4.** $\lim\limits_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$

In Property 4, if $n$ is even, then $c$ must be positive.

By combining the properties of limits with the rules for operating with limits shown below, you can find limits for a wide variety of algebraic functions.

**Operations with Limits**

Let $b$ and $c$ be real numbers, let $n$ be a positive integer, and let $f$ and $g$ be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [bf(x)] = bL$

2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x \to c} [f(x) \cdot g(x)] = LK$

4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$

5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

6. Radical: $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$

In Property 6, if $n$ is even, then $L$ must be positive.

### DISCOVERY

Use a graphing utility to graph $y_1 = 1/x^2$. Does $y_1$ approach a limit as $x$ approaches 0? Evaluate $y_1 = 1/x^2$ at several positive and negative values of $x$ near 0 to confirm your answer. Does $\lim_{x \to 1} 1/x^2$ exist?

**Example 3** **Finding the Limit of a Polynomial Function**

Find the limit: $\lim_{x \to 2} (x^2 + 2x - 3)$.

$$\lim_{x \to 2} (x^2 + 2x - 3) = \lim_{x \to 2} x^2 + \lim_{x \to 2} 2x - \lim_{x \to 2} 3 \qquad \text{Apply Property 2.}$$
$$= 2^2 + 2(2) - 3 \qquad \text{Use direct substitution.}$$
$$= 4 + 4 - 3 \qquad \text{Simplify.}$$
$$= 5$$

✓**CHECKPOINT 3**

Find the limit: $\lim_{x \to 1} (2x^2 - x + 4)$. ■

Example 3 is an illustration of the following important result, which states that the limit of a polynomial function can be evaluated by direct substitution.

**The Limit of a Polynomial Function**

If $p$ is a polynomial function and $c$ is any real number, then

$$\lim_{x \to c} p(x) = p(c).$$

## Techniques for Evaluating Limits

Many techniques for evaluating limits are based on the following important theorem. Basically, the theorem states that if two functions agree at all but a single point $c$, then they have identical limit behavior at $x = c$.

### The Replacement Theorem

Let $c$ be a real number and let $f(x) = g(x)$ for all $x \neq c$. If the limit of $g(x)$ exists as $x \to c$, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

To apply the Replacement Theorem, you can use a result from algebra which states that for a polynomial function $p$, $p(c) = 0$ if and only if $(x - c)$ is a factor of $p(x)$. This concept is demonstrated in Example 4.

### *Example 4*  Finding the Limit of a Function

Find the limit: $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.
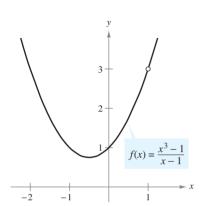
**SOLUTION**   Note that the numerator and denominator are zero when $x = 1$. This implies that $x - 1$ is a factor of both, and you can divide out this like factor.
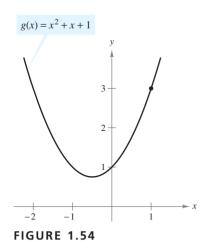
$$\frac{x^3 - 1}{x - 1} = \frac{(x - 1)(x^2 + x + 1)}{x - 1} \qquad \text{Factor numerator.}$$

$$= \frac{(x - 1)(x^2 + x + 1)}{x - 1} \qquad \text{Divide out like factor.}$$

$$= x^2 + x + 1, \quad x \neq 1 \qquad \text{Simplify.}$$

So, the rational function $(x^3 - 1)/(x - 1)$ and the polynomial function $x^2 + x + 1$ agree for all values of $x$ other than $x = 1$, and you can apply the Replacement Theorem.

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} (x^2 + x + 1) = 1^2 + 1 + 1 = 3$$

Figure 1.54 illustrates this result graphically. Note that the two graphs are identical except that the graph of $g$ contains the point $(1, 3)$, whereas this point is missing on the graph of $f$. (In the graph of $f$ in Figure 1.54, the missing point is denoted by an open dot.)



$f(x) = \dfrac{x^3 - 1}{x - 1}$

$g(x) = x^2 + x + 1$

**FIGURE 1.54**

### ✓CHECKPOINT 4

Find the limit:   $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2}$. ■
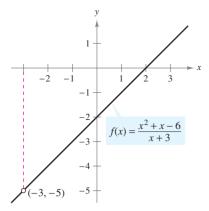
The technique used to evaluate the limit in Example 4 is called the **dividing out** technique. This technique is further demonstrated in the next example.

## DISCOVERY

Use a graphing utility to graph

$$y = \frac{x^2 + x - 6}{x + 3}.$$

Is the graph a line? Why or why not?

$f(x) = \dfrac{x^2 + x - 6}{x + 3}$

$(-3, -5)$

**FIGURE 1.55**  $f$ is undefined when $x = -3$.

### Example 5  Using the Dividing Out Technique

Find the limit:  $\lim\limits_{x \to -3} \dfrac{x^2 + x - 6}{x + 3}.$

**SOLUTION**  Direct substitution fails because both the numerator and the denominator are zero when $x = -3$.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} \qquad \begin{array}{l} \longleftarrow \lim\limits_{x \to -3}(x^2 + x - 6) = 0 \\ \longleftarrow \lim\limits_{x \to -3}(x + 3) = 0 \end{array}$$

However, because the limits of both the numerator and denominator are zero, you know that they have a *common factor* of $x + 3$. So, for all $x \neq -3$, you can divide out this factor to obtain the following.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} \qquad \text{Factor numerator.}$$

$$= \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} \qquad \text{Divide out like factor.}$$

$$= \lim_{x \to -3} (x - 2) \qquad \text{Simplify.}$$

$$= -5 \qquad \text{Direct substitution}$$

This result is shown graphically in Figure 1.55. Note that the graph of $f$ coincides with the graph of $g(x) = x - 2$, except that the graph of $f$ has a hole at $(-3, -5)$.

### ✓ CHECKPOINT 5

Find the limit:  $\lim\limits_{x \to 3} \dfrac{x^2 + x - 12}{x - 3}.$ ■

## STUDY TIP

When you try to evaluate a limit and both the numerator and denominator are zero, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to divide out like factors, as shown in Example 5. Another technique is to rationalize the numerator, as shown in Example 6.

### ✓ CHECKPOINT 6

Find the limit:  $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}.$ ■

### Example 6  Finding a Limit of a Function

Find the limit:  $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 1} - 1}{x}.$

**SOLUTION**  Direct substitution fails because both the numerator and the denominator are zero when $x = 0$. In this case, you can rewrite the fraction by rationalizing the numerator.

$$\frac{\sqrt{x + 1} - 1}{x} = \left( \frac{\sqrt{x + 1} - 1}{x} \right) \left( \frac{\sqrt{x + 1} + 1}{\sqrt{x + 1} + 1} \right)$$

$$= \frac{(x + 1) - 1}{x(\sqrt{x + 1} + 1)}$$

$$= \frac{x}{x(\sqrt{x + 1} + 1)} = \frac{1}{\sqrt{x + 1} + 1}, \quad x \neq 0$$

Now, using the Replacement Theorem, you can evaluate the limit as shown.

$$\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x + 1} + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

## One-Sided Limits

In Example 2(b), you saw that one way in which a limit can fail to exist is when a function approaches a different value from the left of $c$ than it approaches from the right of $c$. This type of behavior can be described more concisely with the concept of a **one-sided limit.**

$$\lim_{x \to c^-} f(x) = L \qquad \text{Limit from the left}$$

$$\lim_{x \to c^+} f(x) = L \qquad \text{Limit from the right}$$

The first of these two limits is read as "the limit of $f(x)$ as $x$ approaches $c$ from the left is $L$." The second is read as "the limit of $f(x)$ as $x$ approaches $c$ from the right is $L$."

### Example 7  Finding One-Sided Limits

Find the limit as $x \to 0$ from the left and the limit as $x \to 0$ from the right for the function
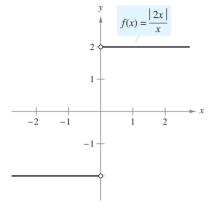
$$f(x) = \frac{|2x|}{x}.$$

**SOLUTION**  From the graph of $f$, shown in Figure 1.56, you can see that $f(x) = -2$ for all $x < 0$. So, the limit from the left is

$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2. \qquad \text{Limit from the left}$$

Because $f(x) = 2$ for all $x > 0$, the limit from the right is

$$\lim_{x \to 0^+} \frac{|2x|}{x} = 2. \qquad \text{Limit from the right}$$

$$f(x) = \frac{|2x|}{x}$$

**FIGURE 1.56**

**TECHNOLOGY**

(T) On most graphing utilities, the absolute value function is denoted by *abs*. You can verify the result in Example 7 by graphing

$$y = \frac{\text{abs}(2x)}{x}$$

in the viewing window $-3 \le x \le 3$ and $-3 \le y \le 3$.

✓ **CHECKPOINT 7**

Find each limit.    (a) $\lim_{x \to 2^-} \dfrac{|x - 2|}{x - 2}$    (b) $\lim_{x \to 2^+} \dfrac{|x - 2|}{x - 2}$ ■
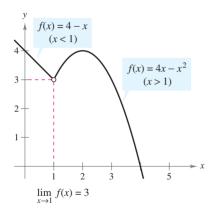
In Example 7, note that the function approaches different limits from the left and from the right. In such cases, the limit of $f(x)$ as $x \to c$ does not exist. For the limit of a function to exist as $x \to c$, *both* one-sided limits must exist and must be equal.

**Existence of a Limit**

If $f$ is a function and $c$ and $L$ are real numbers, then

$$\lim_{x \to c} f(x) = L$$

if and only if both the left and right limits are equal to $L$.

$f(x) = 4 - x$
$(x < 1)$

$f(x) = 4x - x^2$
$(x > 1)$

$\lim_{x \to 1} f(x) = 3$

**FIGURE 1.57**

✓ **CHECKPOINT 8**

Find the limit of $f(x)$ as $x$ approaches 0.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ 2x + 1, & x > 0 \end{cases} \blacksquare$$
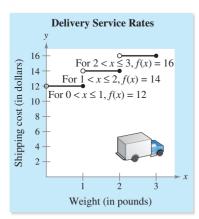
*Example 8*  **Finding One-Sided Limits**

Find the limit of $f(x)$ as $x$ approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

**SOLUTION**    Remember that you are concerned about the value of $f$ near $x = 1$ rather than at $x = 1$. So, for $x < 1, f(x)$ is given by $4 - x$, and you can use direct substitution to obtain

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4 - x)$$
$$= 4 - 1 = 3.$$

For $x > 1, f(x)$ is given by $4x - x^2$, and you can use direct substitution to obtain

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x - x^2)$$
$$= 4(1) - 1^2 = 4 - 1 = 3.$$

Because both one-sided limits exist and are equal to 3, it follows that

$$\lim_{x \to 1} f(x) = 3.$$

The graph in Figure 1.57 confirms this conclusion.

*Example 9*  **Comparing One-Sided Limits**  ®

An overnight delivery service charges $12 for the first pound and $2 for each additional pound. Let $x$ represent the weight of a parcel and let $f(x)$ represent the shipping cost.

$$f(x) = \begin{cases} 12, & 0 < x \leq 1 \\ 14, & 1 < x \leq 2 \\ 16, & 2 < x \leq 3 \end{cases}$$

Show that the limit of $f(x)$ as $x \to 2$ does not exist.



**Delivery Service Rates**

For $2 < x \leq 3, f(x) = 16$
For $1 < x \leq 2, f(x) = 14$
For $0 < x \leq 1, f(x) = 12$

Shipping cost (in dollars)

Weight (in pounds)

**FIGURE 1.58**   Demand Curve

**SOLUTION**    The graph of $f$ is shown in Figure 1.58. The limit of $f(x)$ as $x$ approaches 2 from the left is

$$\lim_{x \to 2^-} f(x) = 14$$

whereas the limit of $f(x)$ as $x$ approaches 2 from the right is

$$\lim_{x \to 2^+} f(x) = 16.$$

Because these one-sided limits are not equal, the limit of $f(x)$ as $x \to 2$ does not exist.

✓ **CHECKPOINT 9**

Show that the limit of $f(x)$ as $x \to 1$ does not exist in Example 9.  ■

## Unbounded Behavior

Example 9 shows a limit that fails to exist because the limits from the left and right differ. Another important way in which a limit can fail to exist is when $f(x)$ increases or decreases without bound as $x$ approaches $c$.

### Example 10   An Unbounded Function

Find the limit (if possible).

$$\lim_{x \to 2} \frac{3}{x - 2}$$
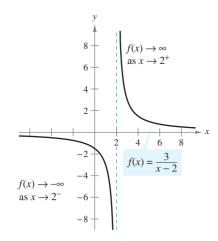
**SOLUTION**    From Figure 1.59, you can see that $f(x)$ decreases without bound as $x$ approaches 2 from the left and $f(x)$ increases without bound as $x$ approaches 2 from the right. Symbolically, you can write this as

$$\lim_{x \to 2^-} \frac{3}{x - 2} = -\infty$$

and

$$\lim_{x \to 2^+} \frac{3}{x - 2} = \infty.$$

Because $f$ is unbounded as $x$ approaches 2, the limit does not exist.



$f(x) \to \infty$ as $x \to 2^+$

$f(x) = \dfrac{3}{x - 2}$

$f(x) \to -\infty$ as $x \to 2^-$

**FIGURE 1.59**

### ✓ CHECKPOINT 10

Find the limit (if possible):    $\displaystyle\lim_{x \to -2} \frac{5}{x + 2}$.  ■

---

**STUDY TIP**

The equal sign in the statement $\displaystyle\lim_{x \to c^+} f(x) = \infty$ does not mean that the limit exists. On the contrary, it tells you how the limit *fails to exist* by denoting the unbounded behavior of $f(x)$ as $x$ approaches $c$.

---

**CONCEPT CHECK**

1. If $\displaystyle\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, what can you conclude about $\displaystyle\lim_{x \to c} f(x)$?

2. Describe how to find the limit of a polynomial function $p(x)$ as $x$ approaches $c$.

3. Is the limit of $f(x)$ as $x$ approaches $c$ always equal to $f(c)$? Why or why not?

4. If $f$ is undefined at $x = c$, can you conclude that the limit of $f(x)$ as $x$ approaches $c$ does not exist? Explain.

**Skills Review 1.5** The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 1.4.

In Exercises 1–4, evaluate the expression and simplify.

**1.** $f(x) = x^2 - 3x + 3$

(a) $f(-1)$    (b) $f(c)$    (c) $f(x + h)$

**2.** $f(x) = \begin{cases} 2x - 2, & x < 1 \\ 3x + 1, & x \geq 1 \end{cases}$

(a) $f(-1)$    (b) $f(3)$    (c) $f(t^2 + 1)$

**3.** $f(x) = x^2 - 2x + 2$    $\dfrac{f(1 + h) - f(1)}{h}$

**4.** $f(x) = 4x$    $\dfrac{f(2 + h) - f(2)}{h}$

In Exercises 5–8, find the domain and range of the function and sketch its graph.

**5.** $h(x) = -\dfrac{5}{x}$

**6.** $g(x) = \sqrt{25 - x^2}$

**7.** $f(x) = |x - 3|$

**8.** $f(x) = \dfrac{|x|}{x}$

In Exercises 9 and 10, determine whether $y$ is a function of $x$.

**9.** $9x^2 + 4y^2 = 49$

**10.** $2x^2y + 8x = 7y$

## Exercises 1.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

**1.** $\lim\limits_{x \to 2} (2x + 5)$

| $x$ | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | ? | | | |

**2.** $\lim\limits_{x \to 2} (x^2 - 3x + 1)$

| $x$ | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | ? | | | |

**3.** $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - 4}$

| $x$ | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | ? | | | |

**4.** $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - 3x + 2}$

| $x$ | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | ? | | | |

**5.** $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 1} - 1}{x}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | ? | | | |

**6.** $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 2} - \sqrt{2}}{x}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | ? | | | |

**7.** $\lim\limits_{x\to0^-} \dfrac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x}$
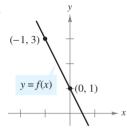
| $x$ | $-0.5$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 |
|-----|--------|--------|---------|----------|---|
| $f(x)$ | | | | | ? |

**8.** $\lim\limits_{x\to0^+} \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{2x}$
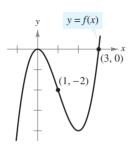
| $x$ | 0.5 | 0.1 | 0.01 | 0.001 | 0 |
|-----|-----|-----|------|-------|---|
| $f(x)$ | | | | | ? |

In Exercises 9–12, use the graph to find the limit (if it exists).

**9.**



(a) $\lim\limits_{x\to0} f(x)$

(b) $\lim\limits_{x\to-1} f(x)$

**10.**



(a) $\lim\limits_{x\to1} f(x)$

(b) $\lim\limits_{x\to3} f(x)$

**11.**



(a) $\lim\limits_{x\to0} g(x)$

(b) $\lim\limits_{x\to-1} g(x)$

**12.**
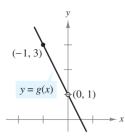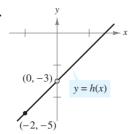


(a) $\lim\limits_{x\to-2} h(x)$

(b) $\lim\limits_{x\to0} h(x)$

In Exercises 13 and 14, find the limit of (a) $f(x) + g(x)$, (b) $f(x)g(x)$, and (c) $f(x)/g(x)$, as $x$ approaches $c$.

**13.** $\lim\limits_{x\to c} f(x) = 3$

$\lim\limits_{x\to c} g(x) = 9$

**14.** $\lim\limits_{x\to c} f(x) = \frac{3}{2}$

$\lim\limits_{x\to c} g(x) = \frac{1}{2}$

In Exercises 15 and 16, find the limit of (a) $\sqrt{f(x)}$, (b) $[3f(x)]$, and (c) $[f(x)]^2$, as $x$ approaches $c$.

**15.** $\lim\limits_{x\to c} f(x) = 16$

**16.** $\lim\limits_{x\to c} f(x) = 9$
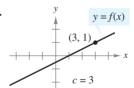
In Exercises 17–22, use the graph to find the limit (if it exists).

(a) $\lim\limits_{x\to c^+} f(x)$    (b) $\lim\limits_{x\to c^-} f(x)$    (c) $\lim\limits_{x\to c} f(x)$
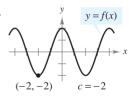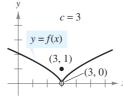
**17.**
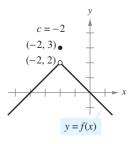


**18.**
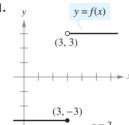


**19.**
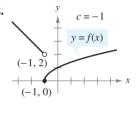


**20.**



**21.**



**22.**



In Exercises 23–40, find the limit.

**23.** $\lim\limits_{x\to2} x^2$

**24.** $\lim\limits_{x\to-2} x^3$

**25.** $\lim\limits_{x\to-3} (2x + 5)$

**26.** $\lim\limits_{x\to0} (3x - 2)$

**27.** $\lim\limits_{x\to1} (1 - x^2)$

**28.** $\lim\limits_{x\to2} (-x^2 + x - 2)$

**29.** $\lim\limits_{x\to3} \sqrt{x + 6}$

**30.** $\lim\limits_{x\to4} \sqrt[3]{x + 4}$

**31.** $\lim\limits_{x\to-3} \dfrac{2}{x + 2}$

**32.** $\lim\limits_{x\to-2} \dfrac{3x + 1}{2 - x}$

**33.** $\lim\limits_{x\to-2} \dfrac{x^2 - 1}{2x}$

**34.** $\lim\limits_{x\to-1} \dfrac{4x - 5}{3 - x}$

**35.** $\lim\limits_{x\to7} \dfrac{5x}{x + 2}$

**36.** $\lim\limits_{x\to3} \dfrac{\sqrt{x + 1}}{x - 4}$

**37.** $\lim\limits_{x\to3} \dfrac{\sqrt{x + 1} - 1}{x}$

**38.** $\lim\limits_{x\to5} \dfrac{\sqrt{x + 4} - 2}{x}$

**39.** $\lim\limits_{x\to1} \dfrac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x}$

**40.** $\lim\limits_{x\to2} \dfrac{\dfrac{1}{x+2} - \dfrac{1}{2}}{x}$

In Exercises 41–60, find the limit (if it exists).

**41.** $\displaystyle\lim_{x\to1}\frac{x^2-1}{x-1}$

**42.** $\displaystyle\lim_{x\to-1}\frac{2x^2-x-3}{x+1}$

**43.** $\displaystyle\lim_{x\to2}\frac{x-2}{x^2-4x+4}$

**44.** $\displaystyle\lim_{x\to2}\frac{2-x}{x^2-4}$

**45.** $\displaystyle\lim_{t\to4}\frac{t+4}{t^2-16}$

**46.** $\displaystyle\lim_{t\to1}\frac{t^2+t-2}{t^2-1}$

**47.** $\displaystyle\lim_{x\to-2}\frac{x^3+8}{x+2}$

**48.** $\displaystyle\lim_{x\to-1}\frac{x^3-1}{x+1}$

**49.** $\displaystyle\lim_{x\to-2}\frac{|x+2|}{x+2}$

**50.** $\displaystyle\lim_{x\to2}\frac{|x-2|}{x-2}$

**51.** $\displaystyle\lim_{x\to2}f(x)$, where $f(x)=\begin{cases}4-x, & x\neq2\\0 & x=2\end{cases}$

**52.** $\displaystyle\lim_{x\to1}f(x)$, where $f(x)=\begin{cases}x^2+2, & x\neq1\\1, & x=1\end{cases}$

**53.** $\displaystyle\lim_{x\to3}f(x)$, where $f(x)=\begin{cases}\frac{1}{3}x-2, & x\leq3\\-2x+5, & x>3\end{cases}$

**54.** $\displaystyle\lim_{s\to1}f(s)$, where $f(s)=\begin{cases}s, & s\leq1\\1-s, & s>1\end{cases}$

**55.** $\displaystyle\lim_{\Delta x\to0}\frac{2(x+\Delta x)-2x}{\Delta x}$

**56.** $\displaystyle\lim_{\Delta x\to0}\frac{4(x+\Delta x)-5-(4x-5)}{\Delta x}$

**57.** $\displaystyle\lim_{\Delta x\to0}\frac{\sqrt{x+2+\Delta x}-\sqrt{x+2}}{\Delta x}$

**58.** $\displaystyle\lim_{\Delta x\to0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$

**59.** $\displaystyle\lim_{\Delta t\to0}\frac{(t+\Delta t)^2-5(t+\Delta t)-(t^2-5t)}{\Delta t}$

**60.** $\displaystyle\lim_{\Delta t\to0}\frac{(t+\Delta t)^2-4(t+\Delta t)+2-(t^2-4t+2)}{\Delta t}$

Ⓣ **Graphical, Numerical, and Analytic Analysis** In Exercises 61–64, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.

**61.** $\displaystyle\lim_{x\to1^-}\frac{2}{x^2-1}$

**62.** $\displaystyle\lim_{x\to1^+}\frac{5}{1-x}$

**63.** $\displaystyle\lim_{x\to-2^-}\frac{1}{x+2}$

**64.** $\displaystyle\lim_{x\to0^-}\frac{x+1}{x}$

Ⓣ In Exercises 65–68, use a graphing utility to estimate the limit (if it exists).

**65.** $\displaystyle\lim_{x\to2}\frac{x^2-5x+6}{x^2-4x+4}$

**66.** $\displaystyle\lim_{x\to1}\frac{x^2+6x-7}{x^3-x^2+2x-2}$

**67.** $\displaystyle\lim_{x\to-4}\frac{x^3+4x^2+x+4}{2x^2+7x-4}$

**68.** $\displaystyle\lim_{x\to-2}\frac{4x^3+7x^2+x+6}{3x^2-x-14}$

**69. Environment** The cost (in dollars) of removing $p\%$ of the pollutants from the water in a small lake is given by

$$C=\frac{25{,}000p}{100-p}, \quad 0\leq p<100$$

where $C$ is the cost and $p$ is the percent of pollutants.

(a) Find the cost of removing 50% of the pollutants.

(b) What percent of the pollutants can be removed for $100,000?

(c) Evaluate $\displaystyle\lim_{p\to100^-}C$. Explain your results.

**70. Compound Interest** You deposit $2000 in an account that is compounded quarterly at an annual rate of $r$ (in decimal form). The balance $A$ after 10 years is

$$A=2000\left(1+\frac{r}{4}\right)^{40}.$$

Does the limit of $A$ exist as the interest rate approaches 6%? If so, what is the limit?

Ⓣ **71. Compound Interest** Consider a certificate of deposit that pays 10% (annual percentage rate) on an initial deposit of $1000. The balance $A$ after 10 years is

$$A=1000(1+0.1x)^{10/x}$$

where $x$ is the length of the compounding period (in years).

(a) Use a graphing utility to graph $A$, where $0\leq x\leq1$.

(b) Use the *zoom* and *trace* features to estimate the balance for quarterly compounding and daily compounding.

(c) Use the *zoom* and *trace* features to estimate

$$\lim_{x\to0^+}A.$$

What do you think this limit represents? Explain your reasoning.

**72. Profit** Consider the profit function $P$ for the manufacturer in Section 1.4, Exercise 71(b). Does the limit of $P$ exist as $x$ approaches 100? If so, what is the limit?

**73.** The limit of

$$f(x)=(1+x)^{1/x}$$

is a natural base for many business applications, as you will see in Section 4.2.

$$\lim_{x\to0}(1+x)^{1/x}=e\approx2.718$$

(a) Show the reasonableness of this limit by completing the table.

| $x$ | $-0.01$ | $-0.001$ | $-0.0001$ | 0 | 0.0001 | 0.001 | 0.01 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | | |

Ⓣ (b) Use a graphing utility to graph $f$ and to confirm the answer in part (a).

(c) Find the domain and range of the function.

# Continuity

■ Determine the continuity of functions.
■ Determine the continuity of functions on a closed interval.
■ Use the greatest integer function to model and solve real-life problems.
■ Use compound interest models to solve real-life problems.
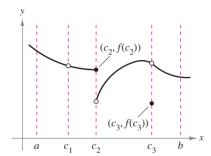
## Continuity

In mathematics, the term "continuous" has much the same meaning as it does in everyday use. To say that a function is continuous at $x = c$ means that there is no interruption in the graph of $f$ at $c$. The graph of $f$ is unbroken at $c$, and there are no holes, jumps, or gaps. As simple as this concept may seem, its precise definition eluded mathematicians for many years. In fact, it was not until the early 1800's that a precise definition was finally developed.

Before looking at this definition, consider the function whose graph is shown in Figure 1.60. This figure identifies three values of $x$ at which the function $f$ is not continuous.

**1.** At $x = c_1, f(c_1)$ is not defined.

**2.** At $x = c_2$, $\lim\limits_{x \to c_2} f(x)$ does not exist.

**3.** At $x = c_3, f(c_3) \neq \lim\limits_{x \to c_3} f(x)$.

**FIGURE 1.60**   $f$ is not continuous when $x = c_1, c_2, c_3$.

At all other points in the interval $(a, b)$, the graph of $f$ is uninterrupted, which implies that the function $f$ is continuous at all other points in the interval $(a, b)$.

---

**Definition of Continuity**

Let $c$ be a number in the interval $(a, b)$, and let $f$ be a function whose domain contains the interval $(a, b)$. The function $f$ is **continuous at the point $c$** if the following conditions are true.

**1.** $f(c)$ is defined.

**2.** $\lim\limits_{x \to c} f(x)$ exists.

**3.** $\lim\limits_{x \to c} f(x) = f(c)$.

If $f$ is continuous at every point in the interval $(a, b)$, then it is **continuous on an open interval $(a, b)$.**

---

**FIGURE 1.61**   On the interval $(a, b)$, the graph of $f$ can be traced with a pencil.

Roughly, you can say that a function is continuous on an interval if its graph on the interval can be traced using a pencil and paper without lifting the pencil from the paper, as shown in Figure 1.61.

Ⓣ Most graphing utilities can draw graphs in two different modes: *connected mode* and *dot mode*. The *connected mode* works well as long as the function is continuous on the entire interval represented by the viewing window. If, however, the function is not continuous at one or more *x*-values in the viewing window, then the *connected mode* may try to "connect" parts of the graphs that should not be connected. For instance, try graphing the function $y_1 = (x + 3)/(x - 2)$ on the viewing window $-8 \le x \le 8$ and $-6 \le y \le 6$. Do you notice any problems?

In Section 1.5, you studied several types of functions that meet the three conditions for continuity. Specifically, if *direct substitution* can be used to evaluate the limit of a function at *c*, then the function is continuous at *c*. Two types of functions that have this property are polynomial functions and rational functions.

**Continuity of Polynomial and Rational Functions**

**1.** A polynomial function is continuous at every real number.

**2.** A rational function is continuous at every number in its domain.
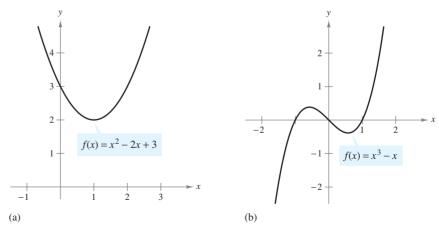
**Example 1** **Determining Continuity of a Polynomial Function**

Discuss the continuity of each function.

**a.** $f(x) = x^2 - 2x + 3$

**b.** $f(x) = x^3 - x$

**SOLUTION** Each of these functions is a *polynomial function*. So, each is continuous on the entire real line, as indicated in Figure 1.62.



$f(x) = x^2 - 2x + 3$

$f(x) = x^3 - x$

(a)                                                    (b)

**FIGURE 1.62** Both functions are continuous on $(-\infty, \infty)$.

**STUDY TIP**

A graphing utility can give misleading information about the continuity of a function. Graph the function

$$f(x) = \frac{x^3 + 8}{x + 2}$$

in the standard viewing window. Does the graph appear to be continuous? For what values of *x* is the function continuous?

**✓ CHECKPOINT 1**

Discuss the continuity of each function.

**a.** $f(x) = x^2 + x + 1$     **b.** $f(x) = x^3 + x$ ■

Polynomial functions are one of the most important types of functions used in calculus. Be sure you see from Example 1 that the graph of a polynomial function is continuous on the entire real line, and therefore has no holes, jumps, or gaps. Rational functions, on the other hand, need not be continuous on the entire real line, as shown in Example 2.

**Example 2** **Determining Continuity of a Rational Function**

Discuss the continuity of each function.
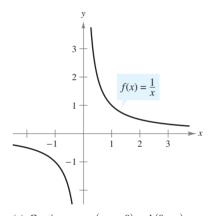
**a.** $f(x) = 1/x$    **b.** $f(x) = (x^2 - 1)/(x - 1)$    **c.** $f(x) = 1/(x^2 + 1)$
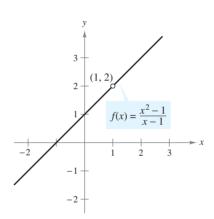
**SOLUTION**    Each of these functions is a rational function and is therefore continuous at every number in its domain.
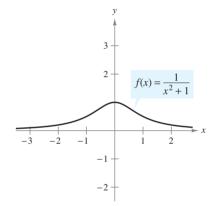
**a.** The domain of $f(x) = 1/x$ consists of all real numbers except $x = 0$. So, this function is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$. [See Figure 1.63(a).]

**b.** The domain of $f(x) = (x^2 - 1)/(x - 1)$ consists of all real numbers except $x = 1$. So, this function is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$. [See Figure 1.63(b).]

**c.** The domain of $f(x) = 1/(x^2 + 1)$ consists of all real numbers. So, this function is continuous on the entire real line. [See Figure 1.63(c).]



(a) Continuous on $(-\infty, 0)$ and $(0, \infty)$.    (b) Continuous on $(-\infty, 1)$ and $(1, \infty)$.    (c) Continuous on $(-\infty, \infty)$.

**FIGURE 1.63**

✓ **CHECKPOINT 2**

Discuss the continuity of each function.

**a.** $f(x) = \dfrac{1}{x - 1}$    **b.** $f(x) = \dfrac{x^2 - 4}{x - 2}$    **c.** $f(x) = \dfrac{1}{x^2 + 2}$ ■

Consider an open interval $I$ that contains a real number $c$. If a function $f$ is defined on $I$ (except possibly at $c$), and $f$ is not continuous at $c$, then $f$ is said to have a **discontinuity** at $c$. Discontinuities fall into two categories: **removable** and **nonremovable.** A discontinuity at $c$ is called removable if $f$ can be made continuous by appropriately defining (or redefining) $f(c)$. For instance, the function in Example 2(b) has a removable discontinuity at $(1, 2)$. To remove the discontinuity, all you need to do is redefine the function so that $f(1) = 2$.

A discontinuity at $x = c$ is nonremovable if the function cannot be made continuous at $x = c$ by defining or redefining the function at $x = c$. For instance, the function in Example 2(a) has a nonremovable discontinuity at $x = 0$.

## Continuity on a Closed Interval

The intervals discussed in Examples 1 and 2 are open. To discuss continuity on a closed interval, you can use the concept of one-sided limits, as defined in Section 1.5.

---

**Definition of Continuity on a Closed Interval**

Let $f$ be defined on a closed interval $[a, b]$. If $f$ is continuous on the open interval $(a, b)$ and

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to b^-} f(x) = f(b)$$

then $f$ is **continuous on the closed interval** $[a, b]$. Moreover, $f$ is **continuous from the right** at $a$ and **continuous from the left** at $b$.

---

Similar definitions can be made to cover continuity on intervals of the form $(a, b]$ and $[a, b)$, or on infinite intervals. For example, the function

$$f(x) = \sqrt{x}$$

is continuous on the infinite interval $[0, \infty)$.

**Example 3**  **Examining Continuity at an Endpoint**
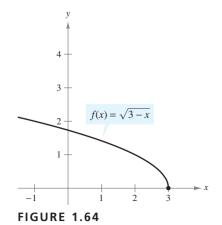
Discuss the continuity of

$$f(x) = \sqrt{3 - x}.$$

**SOLUTION**   Notice that the domain of $f$ is the set $(-\infty, 3]$. Moreover, $f$ is continuous from the left at $x = 3$ because

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{3 - x}$$
$$= 0$$
$$= f(3).$$

For all $x < 3$, the function $f$ satisfies the three conditions for continuity. So, you can conclude that $f$ is continuous on the interval $(-\infty, 3]$, as shown in Figure 1.64.



$f(x) = \sqrt{3 - x}$

**FIGURE 1.64**

✓ **CHECKPOINT 3**
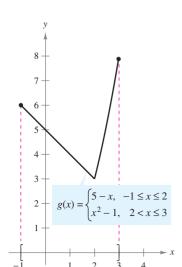
Discuss the continuity of $f(x) = \sqrt{x - 2}$.  ■

---

**STUDY TIP**
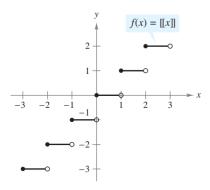
When working with radical functions of the form

$$f(x) = \sqrt{g(x)}$$

remember that the domain of $f$ coincides with the solution of $g(x) \geq 0$.

**FIGURE 1.65**

| Example 4 | **Examining Continuity on a Closed Interval**

Discuss the continuity of $g(x) = \begin{cases} 5 - x, & -1 \le x \le 2 \\ x^2 - 1, & 2 < x \le 3 \end{cases}$.

**SOLUTION**    The polynomial functions $5 - x$ and $x^2 - 1$ are continuous on the intervals $[-1, 2]$ and $(2, 3]$, respectively. So, to conclude that $g$ is continuous on the entire interval $[-1, 3]$, you only need to check the behavior of $g$ when $x = 2$. You can do this by taking the one-sided limits when $x = 2$.

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (5 - x) = 3 \qquad \text{Limit from the left}$$

and

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (x^2 - 1) = 3 \qquad \text{Limit from the right}$$

Because these two limits are equal,

$$\lim_{x \to 2} g(x) = g(2) = 3.$$

So, $g$ is continuous at $x = 2$ and, consequently, it is continuous on the entire interval $[-1, 3]$. The graph of $g$ is shown in Figure 1.65.

✓ **CHECKPOINT 4**

Discuss the continuity of $f(x) = \begin{cases} x + 2, & -1 \le x < 3 \\ 14 - x^2, & 3 \le x \le 5 \end{cases}$. ■

## The Greatest Integer Function



**FIGURE 1.66**    Greatest Integer Function

Many functions that are used in business applications are **step functions.** For instance, the function in Example 9 in Section 1.5 is a step function. The **greatest integer function** is another example of a step function. This function is denoted by

$$[\![x]\!] = \text{greatest integer less than or equal to } x.$$

For example,

$$[\![-2.1]\!] = \text{greatest integer less than or equal to } -2.1 = -3$$
$$[\![-2]\!] = \text{greatest integer less than or equal to } -2 = -2$$
$$[\![1.5]\!] = \text{greatest integer less than or equal to } 1.5 = 1.$$

Note that the graph of the greatest integer function (Figure 1.66) jumps up one unit at each integer. This implies that the function is not continuous at each integer.

In real-life applications, the domain of the greatest integer function is often restricted to nonnegative values of $x$. In such cases this function serves the purpose of **truncating** the decimal portion of $x$. For example, 1.345 is truncated to 1 and 3.57 is truncated to 3. That is,

$$[\![1.345]\!] = 1 \qquad \text{and} \qquad [\![3.57]\!] = 3.$$

**TECHNOLOGY**

(T) Use a graphing utility to calculate the following.

**a.** $[\![3.5]\!]$    **b.** $[\![-3.5]\!]$    **c.** $[\![0]\!]$

## Example 5  Modeling a Cost Function  ®

A bookbinding company produces 10,000 books in an eight-hour shift. The fixed cost *per shift* amounts to $5000, and the unit cost per book is $3. Using the greatest integer function, you can write the cost of producing $x$ books as

$$C = 5000\left(1 + \left[\!\left[\frac{x-1}{10{,}000}\right]\!\right]\right) + 3x.$$

Sketch the graph of this cost function.

**SOLUTION**   Note that during the first eight-hour shift

$$\left[\!\left[\frac{x-1}{10{,}000}\right]\!\right] = 0, \quad 1 \le x \le 10{,}000$$

which implies

$$C = 5000\left(1 + \left[\!\left[\frac{x-1}{10{,}000}\right]\!\right]\right) + 3x = 5000 + 3x.$$
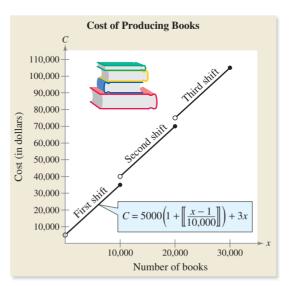
During the second eight-hour shift

$$\left[\!\left[\frac{x-1}{10{,}000}\right]\!\right] = 1, \quad 10{,}001 \le x \le 20{,}000$$

which implies

$$C = 5000\left(1 + \left[\!\left[\frac{x-1}{10{,}000}\right]\!\right]\right) + 3x$$
$$= 10{,}000 + 3x.$$

The graph of $C$ is shown in Figure 1.67. Note the graph's discontinuities.



**FIGURE 1.67**

R. R. Donnelley & Sons Company is one of the world's largest commercial printers. It prints and binds a major share of the national publications in the United States, including *Time*, *Newsweek*, and *TV Guide*.

AP/Wide World Photos

✓ **CHECKPOINT 5**

Use a graphing utility to graph the cost function in Example 5. ∎
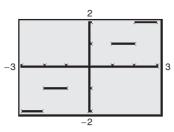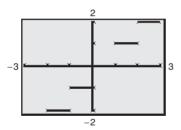
**TECHNOLOGY**

**Step Functions and Compound Functions**

(T) To graph a step function or compound function with a graphing utility, you must be familiar with the utility's programming language. For instance, different graphing utilities have different "integer truncation" functions. One is IPart($x$), and it yields the truncated integer part of $x$. For example, IPart($-1.2$) $= -1$ and IPart($3.4$) $= 3$. The other function is Int($x$), which is the greatest integer function. The graphs of these two functions are shown below. When graphing a step function, you should set your graphing utility to *dot mode*.



Graph of $f(x) = $ IPart($x$)
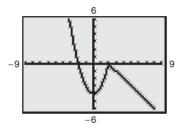


Graph of $f(x) = $ Int($x$)

On some graphing utilities, you can graph a piecewise-defined function such as

$$f(x) = \begin{cases} x^2 - 4, & x \le 2 \\ -x + 2, & 2 < x \end{cases}.$$
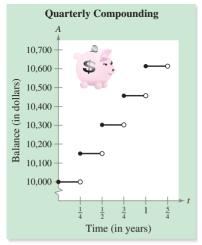
The graph of this function is shown below.



Consult the user's guide for your graphing utility for specific keystrokes you can use to graph these functions.

(T) You can use a spreadsheet or the *table* feature of a graphing utility to create a table. Try doing this for the data shown at the right. (Consult the user's manual of a spreadsheet software program for specific instructions on how to create a table.)

**Quarterly Compounding**



**FIGURE 1.68**

## Extended Application: Compound Interest

Banks and other financial institutions differ on how interest is paid to an account. If the interest is added to the account so that future interest is paid on previously earned interest, then the interest is said to be compounded. Suppose, for example, that you deposited $10,000 in an account that pays 6% interest, compounded quarterly. Because the 6% is the annual interest rate, the quarterly rate is $\frac{1}{4}(0.06) = 0.015$ or 1.5%. The balances during the first five quarters are shown below.

| Quarter | Balance |
|---------|---------|
| 1st | $10,000.00 |
| 2nd | $10,000.00 + (0.015)(10,000.00) = \$10,150.00$ |
| 3rd | $10,150.00 + (0.015)(10,150.00) = \$10,302.25$ |
| 4th | $10,302.25 + (0.015)(10,302.25) = \$10,456.78$ |
| 5th | $10,456.78 + (0.015)(10,456.78) = \$10,613.63$ |

**Example 6**  Graphing Compound Interest  (R)

Sketch the graph of the balance in the account described above.

**SOLUTION**  Let $A$ represent the balance in the account and let $t$ represent the time, in years. You can use the greatest integer function to represent the balance, as shown.

$$A = 10,000(1 + 0.015)^{[\![4t]\!]}$$

From the graph shown in Figure 1.68, notice that the function has a discontinuity at each quarter.

✓**CHECKPOINT 6**

Write an equation that gives the balance of the account in Example 6 if the annual interest rate is 8%. ■

**CONCEPT CHECK**

1. Describe the continuity of a polynomial function.
2. Describe the continuity of a rational function.
3. If a function $f$ is continuous at every point in the interval $(a, b)$, then what can you say about $f$ on an open interval $(a, b)$?
4. Describe in your own words what it means to say that a function $f$ is continuous at $x = c$.

## Skills Review 1.6

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4, 0.5, and 1.5.

In Exercises 1–4, simplify the expression.

**1.** $\dfrac{x^2 + 6x + 8}{x^2 - 6x - 16}$

**2.** $\dfrac{x^2 - 5x - 6}{x^2 - 9x + 18}$

**3.** $\dfrac{2x^2 - 2x - 12}{4x^2 - 24x + 36}$

**4.** $\dfrac{x^3 - 16x}{x^3 + 2x^2 - 8x}$

In Exercises 5–8, solve for $x$.

**5.** $x^2 + 7x = 0$

**6.** $x^2 + 4x - 5 = 0$

**7.** $3x^2 + 8x + 4 = 0$

**8.** $x^3 + 5x^2 - 24x = 0$

In Exercises 9 and 10, find the limit.

**9.** $\lim\limits_{x \to 3} (2x^2 - 3x + 4)$

**10.** $\lim\limits_{x \to -2} (3x^3 - 8x + 7)$

## Exercises 1.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, determine whether the function is continuous on the entire real line. Explain your reasoning.

**1.** $f(x) = 5x^3 - x^2 + 2$

**2.** $f(x) = (x^2 - 1)^3$

**3.** $f(x) = \dfrac{1}{x^2 - 4}$

**4.** $f(x) = \dfrac{1}{9 - x^2}$

**5.** $f(x) = \dfrac{1}{4 + x^2}$

**6.** $f(x) = \dfrac{3x}{x^2 + 1}$

**7.** $f(x) = \dfrac{2x - 1}{x^2 - 8x + 15}$

**8.** $f(x) = \dfrac{x + 4}{x^2 - 6x + 5}$
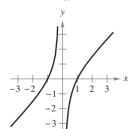
**9.** $g(x) = \dfrac{x^2 - 4x + 4}{x^2 - 4}$
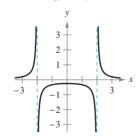
**10.** $g(x) = \dfrac{x^2 - 9x + 20}{x^2 - 16}$

In Exercises 11–34, describe the interval(s) on which the function is continuous. Explain why the function is continuous on the interval(s). If the function has a discontinuity, identify the conditions of continuity that are not satisfied.
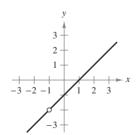
**11.** $f(x) = \dfrac{x^2 - 1}{x}$
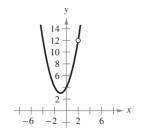


**12.** $f(x) = \dfrac{1}{x^2 - 4}$



**13.** $f(x) = \dfrac{x^2 - 1}{x + 1}$



**14.** $f(x) = \dfrac{x^3 - 8}{x - 2}$



**15.** $f(x) = x^2 - 2x + 1$

**16.** $f(x) = 3 - 2x - x^2$

**17.** $f(x) = \dfrac{x}{x^2 - 1}$

**18.** $f(x) = \dfrac{x - 3}{x^2 - 9}$

**19.** $f(x) = \dfrac{x}{x^2 + 1}$
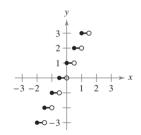
**20.** $f(x) = \dfrac{1}{x^2 + 1}$
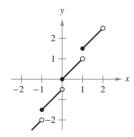
**21.** $f(x) = \dfrac{x - 5}{x^2 - 9x + 20}$

**22.** $f(x) = \dfrac{x - 1}{x^2 + x - 2}$

**23.** $f(x) = [\![2x]\!] + 1$

**24.** $f(x) = \dfrac{[\![x]\!]}{2} + x$



**25.** $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \geq 1 \end{cases}$

**26.** $f(x) = \begin{cases} 3 + x, & x \leq 2 \\ x^2 + 1, & x > 2 \end{cases}$

**27.** $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases}$

**28.** $f(x) = \begin{cases} x^2 - 4, & x \leq 0 \\ 3x + 1, & x > 0 \end{cases}$

**29.** $f(x) = \dfrac{|x + 1|}{x + 1}$

**30.** $f(x) = \dfrac{|4 - x|}{4 - x}$

**31.** $f(x) = [\![x - 1]\!]$

**32.** $f(x) = x - [\![x]\!]$

**33.** $h(x) = f(g(x)), \quad f(x) = \dfrac{1}{\sqrt{x}}, \quad g(x) = x - 1, x > 1$

**34.** $h(x) = f(g(x)), \quad f(x) = \dfrac{1}{x - 1}, \quad g(x) = x^2 + 5$

In Exercises 35–38, discuss the continuity of the function on the closed interval. If there are any discontinuities, determine whether they are removable.

| Function | Interval |
|---|---|
| **35.** $f(x) = x^2 - 4x - 5$ | $[-1, 5]$ |
| **36.** $f(x) = \dfrac{5}{x^2 + 1}$ | $[-2, 2]$ |
| **37.** $f(x) = \dfrac{1}{x - 2}$ | $[1, 4]$ |
| **38.** $f(x) = \dfrac{x}{x^2 - 4x + 3}$ | $[0, 4]$ |

In Exercises 39–44, sketch the graph of the function and describe the interval(s) on which the function is continuous.

**39.** $f(x) = \dfrac{x^2 - 16}{x - 4}$

**40.** $f(x) = \dfrac{2x^2 + x}{x}$

**41.** $f(x) = \dfrac{x^3 + x}{x}$

**42.** $f(x) = \dfrac{x - 3}{4x^2 - 12x}$

**43.** $f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$

**44.** $f(x) = \begin{cases} x^2 - 4, & x \leq 0 \\ 2x + 4, & x > 0 \end{cases}$

In Exercises 45 and 46, find the constant $a$ (Exercise 45) and the constants $a$ and $b$ (Exercise 46) such that the function is continuous on the entire real line.

**45.** $f(x) = \begin{cases} x^3, & x \leq 2 \\ ax^2, & x > 2 \end{cases}$

**46.** $f(x) = \begin{cases} 2, & x \leq -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases}$

(T) In Exercises 47–52, use a graphing utility to graph the function. Use the graph to determine any $x$-value(s) at which the function is not continuous. Explain why the function is not continuous at the $x$-value(s).

**47.** $h(x) = \dfrac{1}{x^2 - x - 2}$

**48.** $k(x) = \dfrac{x - 4}{x^2 - 5x + 4}$

**49.** $f(x) = \begin{cases} 2x - 4, & x \leq 3 \\ x^2 - 2x, & x > 3 \end{cases}$

**50.** $f(x) = \begin{cases} 3x - 1, & x \leq 1 \\ x + 1, & x > 1 \end{cases}$
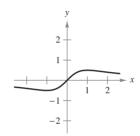
**51.** $f(x) = x - 2[\![x]\!]$
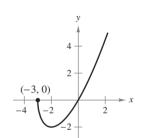
**52.** $f(x) = [\![2x - 1]\!]$

In Exercises 53–56, describe the interval(s) on which the function is continuous.
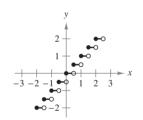
**53.** $f(x) = \dfrac{x}{x^2 + 1}$

**54.** $f(x) = x\sqrt{x + 3}$

**55.** $f(x) = \dfrac{1}{2}[\![2x]\!]$



**56.** $f(x) = \dfrac{x+1}{\sqrt{x}}$
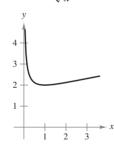


(T) **Writing**    In Exercises 57 and 58, use a graphing utility to graph the function on the interval $[-4, 4]$. Does the graph of the function appear to be continuous on this interval? Is the function in fact continuous on $[-4, 4]$? Write a short paragraph about the importance of examining a function analytically as well as graphically.

**57.** $f(x) = \dfrac{x^2 + x}{x}$

**58.** $f(x) = \dfrac{x^3 - 8}{x - 2}$

**59. Compound Interest**    A deposit of $7500 is made in an account that pays 6% compounded quarterly. The amount $A$ in the account after $t$ years is

$$A = 7500(1.015)^{[\![4t]\!]}, \quad t \ge 0.$$

(a) Sketch the graph of $A$. Is the graph continuous? Explain your reasoning.

(b) What is the balance after 7 years?

**60. Environmental Cost**    The cost $C$ (in millions of dollars) of removing $x$ percent of the pollutants emitted from the smokestack of a factory can be modeled by

$$C = \dfrac{2x}{100 - x}.$$

(a) What is the implied domain of $C$? Explain your reasoning.

(T) (b) Use a graphing utility to graph the cost function. Is the function continuous on its domain? Explain your reasoning.

(c) Find the cost of removing 75% of the pollutants from the smokestack.

(T) **61. Consumer Awareness**    A shipping company's charge for sending an overnight package from New York to Atlanta is $12.80 for the first pound and $2.50 for each additional pound or fraction thereof. Use the greatest integer function to create a model for the charge $C$ for overnight delivery of a package weighing $x$ pounds. Use a graphing utility to graph the function, and discuss its continuity.

**62. Consumer Awareness**    The United States Postal Service first class mail rates are $0.41 for the first ounce and $0.17 for each additional ounce or fraction thereof up to 3.5 ounces. A model for the cost $C$ (in dollars) of a first class mailing that weighs 3.5 ounces or less is given below. *(Source: United States Postal Service)*

$$C(x) = \begin{cases} 0.41, & 0 \le x \le 1 \\ 0.58, & 1 < x \le 2 \\ 0.75, & 2 < x \le 3 \\ 0.92, & 3 < x \le 3.5 \end{cases}$$

(T) (a) Use a graphing utility to graph the function and discuss its continuity. At what values is the function not continuous? Explain your reasoning.

(b) Find the cost of mailing a 2.5-ounce letter.

**63. Salary Contract**    A union contract guarantees a 9% yearly increase for 5 years. For a current salary of $28,500, the salaries for the next 5 years are given by

$$S = 28,500(1.09)^{[\![t]\!]}$$

where $t = 0$ represents the present year.

(T) (a) Use the greatest integer function of a graphing utility to graph the salary function, and discuss its continuity.

(b) Find the salary during the fifth year (when $t = 5$).

**64. Inventory Management**    The number of units in inventory in a small company is

$$N = 25\left(2\left[\!\left[\dfrac{t+2}{2}\right]\!\right] - t\right), \quad 0 \le t \le 12$$

where the real number $t$ is the time in months.

(T) (a) Use the greatest integer function of a graphing utility to graph this function, and discuss its continuity.

(b) How often must the company replenish its inventory?

**65. Owning a Franchise**    You have purchased a franchise. You have determined a linear model for your revenue as a function of time. Is the model a continuous function? Would your actual revenue be a continuous function of time? Explain your reasoning.

**66. Biology**    The gestation period of rabbits is about 29 to 35 days. Therefore, the population of a form (rabbits' home) can increase dramatically in a short period of time. The table gives the population of a form, where $t$ is the time in months and $N$ is the rabbit population.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|----|----|----|----|----|
| $N$ | 2 | 8 | 10 | 14 | 10 | 15 | 12 |

Graph the population as a function of time. Find any points of discontinuity in the function. Explain your reasoning.

**67. Profit**    Consider the profit function $P$ for the manufacturer in Section 1.4, Exercise 71(b). Is the function continuous at $x = 100$? Explain.

# Algebra Review

## Order of Operations

Much of the algebra in this chapter involves evaluation of algebraic expressions. When you evaluate an algebraic expression, you need to know the priorities assigned to different operations. These priorities are called the *order of operations*.

1. Perform operations inside *symbols of grouping or absolute value symbols*, starting with the innermost symbol.

2. Evaluate all *exponential* expressions.

3. Perform all *multiplications* and *divisions* from left to right.

4. Perform all *additions* and *subtractions* from left to right.

### Example 1    Using Order of Operations

Evaluate each expression.

**a.** $7 - [(5 \cdot 3) + 2^3]$

**b.** $[36 \div (3^2 \cdot 2)] + 6$

**c.** $36 - [3^2 \cdot (2 \div 6)]$

**d.** $10 - 2(8 + |5 - 7|)$

**SOLUTION**

**a.** $\begin{aligned} 7 - [(5 \cdot 3) + 2^3] &= 7 - [15 + 2^3] \\ &= 7 - [15 + 8] \\ &= 7 - 23 \\ &= -16 \end{aligned}$

Multiply inside parentheses.
Evaluate exponential expression.
Add inside brackets.
Subtract.

**b.** $\begin{aligned} [36 \div (3^2 \cdot 2)] + 6 &= [36 \div (9 \cdot 2)] + 6 \\ &= [36 \div 18] + 6 \\ &= 2 + 6 \\ &= 8 \end{aligned}$

Evaluate exponential expression inside parentheses.
Multiply inside parentheses.
Divide inside brackets.
Add.

**c.** $\begin{aligned} 36 - [3^2 \cdot (2 \div 6)] &= 36 - \left[3^2 \cdot \tfrac{1}{3}\right] \\ &= 36 - \left[9 \cdot \tfrac{1}{3}\right] \\ &= 36 - 3 \\ &= 33 \end{aligned}$

Divide inside parentheses.
Evaluate exponential expression.
Multiply inside brackets.
Subtract.

**d.** $\begin{aligned} 10 - 2(8 + |5 - 7|) &= 10 - 2(8 + |-2|) \\ &= 10 - 2(8 + 2) \\ &= 10 - 2(10) \\ &= 10 - 20 \\ &= -10 \end{aligned}$

Subtract inside absolute value symbols.
Evaluate absolute value.
Add inside parentheses.
Multiply.
Subtract.

**TECHNOLOGY**

Ⓣ Most scientific and graphing calculators use the same order of operations listed above. Try entering the expressions in Example 1 into your calculator. Do you get the same results?

## Solving Equations

A second algebraic skill in this chapter is solving an equation in one variable.

1. To solve a *linear equation*, you can add or subtract the same quantity from each side of the equation. You can also multiply or divide each side of the equation by the same *nonzero* quantity.

2. To solve a *quadratic equation*, you can take the square root of each side, use factoring, or use the Quadratic Formula.

3. To solve a *radical equation*, isolate the radical on one side of the equation and square each side of the equation.

4. To solve an *absolute value equation*, use the definition of absolute value to rewrite the equation as two equations.

**Example 2**  **Solving Equations**

Solve each equation.

**a.** $3x - 3 = 5x - 7$

**b.** $2x^2 = 10$

**c.** $2x^2 + 5x - 6 = 6$

**d.** $\sqrt{2x - 7} = 5$

**SOLUTION**

**STUDY TIP**

You should be aware that solving radical equations can sometimes lead to *extraneous solutions* (those that do not satisfy the original equation). For example, squaring both sides of the following equation yields two possible solutions, one of which is extraneous.

$$\sqrt{x} = x - 2$$
$$x = x^2 - 4x + 4$$
$$0 = x^2 - 5x + 4$$
$$= (x - 4)(x - 1)$$
$$x - 4 = 0 \implies x = 4$$
$$\text{(solution)}$$
$$x - 1 = 0 \implies x = 1$$
$$\text{(extraneous)}$$

| | |
|---|---|
| **a.** $3x - 3 = 5x - 7$ | Write original (linear) equation. |
| $-3 = 2x - 7$ | Subtract $3x$ from each side. |
| $4 = 2x$ | Add 7 to each side. |
| $2 = x$ | Divide each side by 2. |
| **b.** $2x^2 = 10$ | Write original (quadratic) equation. |
| $x^2 = 5$ | Divide each side by 2. |
| $x = \pm\sqrt{5}$ | Take the square root of each side. |
| **c.** $2x^2 + 5x - 6 = 6$ | Write original (quadratic) equation. |
| $2x^2 + 5x - 12 = 0$ | Write in general form. |
| $(2x - 3)(x + 4) = 0$ | Factor. |
| $2x - 3 = 0 \implies x = \frac{3}{2}$ | Set first factor equal to zero. |
| $x + 4 = 0 \implies x = -4$ | Set second factor equal to zero. |
| **d.** $\sqrt{2x - 7} = 5$ | Write original (radical) equation. |
| $2x - 7 = 25$ | Square each side. |
| $2x = 32$ | Add 7 to each side. |
| $x = 16$ | Divide each side by 2. |

# Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 109.
Answers to odd-numbered Review Exercises are given in the back of the text.*

## Section 1.1                                                     Review Exercises

- Plot points in a coordinate plane.                                   *1–4*
- Read data presented graphically.                                     *5–8*
- Find the distance between two points in a coordinate plane.          *9–12*

  $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Find the midpoints of line segments connecting two points.          *13–16*

  $$\text{Midpoint} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

- Interpret real-life data that is presented graphically.             *17, 18*
- Translate points in a coordinate plane.                             *19, 20*
- Construct a bar graph from real-life data.                            *21*

## Section 1.2

- Sketch graphs of equations by hand.                                  *22–31*
- Find the $x$- and $y$-intercepts of graphs of equations algebraically and   *32, 33*
  graphically using a graphing utility.
- Write the standard forms of equations of circles, given the center and a    *34, 35*
  point on the circle.

  $$(x - h)^2 + (y - k)^2 = r^2$$

- Convert equations of circles from general form to standard form by completing   *36, 37*
  the square, and sketch the circles.
- Find the points of intersection of two graphs algebraically and graphically   *38–41*
  using a graphing utility.
- Find the break-even point for a business.                            *42, 43*

  The break-even point occurs when the revenue $R$ is equal to the cost $C$.

- Find the equilibrium points of supply equations and demand equations.   *44*

  The equilibrium point is the point of intersection of the graphs
  of the supply and demand equations.

## Section 1.3

- Use the slope-intercept form of a linear equation to sketch graphs of lines.   *45–50*

  $$y = mx + b$$

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* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student
Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review
and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a
*Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands
and instructions for a wide variety of graphing calculators, including the most recent models.

### Section 1.3   (continued)                                   Review Exercises

- Find slopes of lines passing through two points.                        *51–54*

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Use the point-slope form to write equations of lines and graph equations using a    *55–58*
  graphing utility.

$$y - y_1 = m(x - x_1)$$

- Find equations of parallel and perpendicular lines.                     *59, 60*

  Parallel lines: $m_1 = m_2$    Perpendicular lines: $m_1 = -\dfrac{1}{m_2}$

- Use linear equations to solve real-life problems such as predicting future    *61, 62*
  sales or creating a linear depreciation schedule.

### Section 1.4

- Use the Vertical Line Test to decide whether equations define functions.    *63–66*
- Use function notation to evaluate functions.                            *67, 68*
- Use a graphing utility to graph functions and find the domains and ranges    *69–74*
  of functions.
- Combine functions to create other functions.                            *75, 76*
- Use the Horizontal Line Test to determine whether functions have inverse functions.    *77–80*
  If they do, find the inverse functions.

### Section 1.5

- Determine whether limits exist. If they do, find the limits.            *81–98*
- Use a table to estimate one-sided limits.                               *99, 100*
- Determine whether statements about limits are true or false.            *101–106*

### Section 1.6

- Determine whether functions are continuous at a point, on an open interval, and    *107–114*
  on a closed interval.
- Determine the constant such that $f$ is continuous.                     *115, 116*
- Use analytic and graphical models of real-life data to solve real-life problems.    *117–121*

## Study Strategies
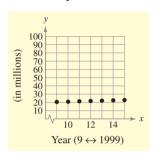
- **Use a Graphing Utility**   A graphing calculator or graphing software for a computer can help you in this course in two
  important ways. As an *exploratory device*, a graphing utility allows you to learn concepts by allowing you to compare
  graphs of equations. For instance, sketching the graphs of $y = x^2$, $y = x^2 + 1$, and $y = x^2 - 1$ helps confirm that adding
  (or subtracting) a constant to (or from) a function shifts the graph of the function vertically. As a *problem-solving tool*, a
  graphing utility frees you of some of the drudgery of sketching complicated graphs by hand. The time that you save can be
  spent using mathematics to solve real-life problems.

- **Use the Skills Review Exercises**   Each exercise set in this text begins with a set of skills review exercises. We urge you
  to begin each homework session by quickly working all of these exercises (all are answered in the back of the text). The "old"
  skills covered in these exercises are needed to master the "new" skills in the section exercise set. The skills review exercises
  remind you that mathematics is cumulative—to be successful in this course, you must retain "old" skills.

- **Use the Additional Study Aids**   The additional study aids were prepared specifically to help you master the concepts
  discussed in the text. They are the *Student Solutions Guide*, student website, and the *Graphing Technology Guide*.

# Review Exercises
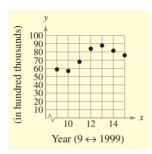
In Exercises 1–4, plot the points

**1.** $(2, 3), (0, 6)$   **2.** $(-5, 1), (4, -3)$

**3.** $(0.5, -4), (-1, -2)$   **4.** $(-1.5, 0), (6, -5)$
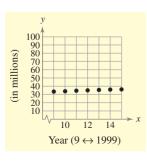
In Exercises 5–8, match the data with the real-life situation that it represents. [The graphs are labeled (a)–(d).]

**5.** Population of Texas

**6.** Population of California

**7.** Number of unemployed workers in the United States

**8.** Best Buy sales

(a)

(b)

(c)

(d)

In Exercises 9–12, find the distance between the two points.
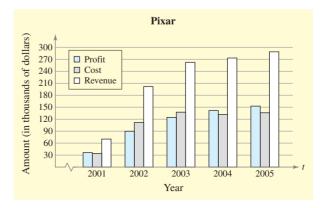
**9.** $(0, 0), (5, 2)$

**10.** $(1, 2), (4, 3)$

**11.** $(-1, 3), (-4, 6)$

**12.** $(6, 8), (-3, 7)$

In Exercises 13–16, find the midpoint of the line segment connecting the two points.

**13.** $(5, 6), (9, 2)$   **14.** $(0, 0), (-4, 8)$

**15.** $(-10, 4), (-6, 8)$   **16.** $(7, -9), (-3, 5)$

In Exercises 17 and 18, use the graph below, which gives the revenues, costs, and profits for Pixar from 2001 through 2005. (Pixar develops and produces animated feature films.)  *(Source: Pixar)*

**17.** Write an equation that relates the revenue $R$, cost $C$, and profit $P$. Explain the relationship between the heights of the bars and the equation.

**18.** Estimate the revenue, cost, and profit for Pixar for each year.

**19.** Translate the triangle whose vertices are $(1, 3), (2, 4)$, and $(5, 6)$ three units to the right and four units up. Find the coordinates of the translated vertices.

**20.** Translate the rectangle whose vertices are $(-2, 1), (-1, 2), (1, 0)$, and $(0, -1)$ four units to the right and one unit down.

Ⓣ **21.** **Biology**   The following data represent six intertidal
Ⓑ invertebrate species collected from four stations along the Maine coast.

| | | | |
|---|---|---|---|
| *Mytilus* | 105 | *Gammarus* | 75 |
| *Littorina* | 66 | *Arbacia* | 7 |
| *Nassarius* | 113 | *Mya* | 19 |

Use a graphing utility to construct a bar graph that represents the data.   *(Source: Adapted from Haefner, Exploring Marine Biology: Laboratory and Field Exercises)*

In Exercises 22–31, sketch the graph of the equation.

**22.** $y = 4x - 12$   **23.** $y = 4 - 3x$

**24.** $y = x^2 + 5$   **25.** $y = 1 - x^2$

**26.** $y = |4 - x|$   **27.** $y = |2x - 3|$

**28.** $y = x^3 + 4$   **29.** $y = 2x^3 - 1$

**30.** $y = \sqrt{4x + 1}$   **31.** $y = \sqrt{2x}$

In Exercises 32 and 33, find the $x$- and $y$-intercepts of the graph of the equation algebraically. Use a graphing utility to verify your results.

**32.** $4x + y + 3 = 0$

**33.** $y = (x - 1)^3 + 2(x - 1)^2$

In Exercises 34 and 35, write the standard form of the equation of the circle.

**34.** Center: $(0, 0)$

Solution point: $(2, \sqrt{5})$

**35.** Center: $(2, -1)$

Solution point: $(-1, 7)$

In Exercises 36 and 37, complete the square to write the equation of the circle in standard form. Determine the radius and center of the circle. Then sketch the circle.

**36.** $x^2 + y^2 - 6x + 8y = 0$

**37.** $x^2 + y^2 + 10x + 4y - 7 = 0$

In Exercises 38–41, find the point(s) of intersection of the graphs algebraically. Then use a graphing utility to verify your results.

**38.** $2x - 3y = 13, \quad 5x + 3y = 1$

**39.** $x^2 + y^2 = 5, \quad x - y = 1$    **40.** $y = x^3, \quad y = x$

**41.** $x^2 + y = 4, \quad 2x - y = 1$

**42. Break-Even Analysis**    A student organization wants to raise money by having a T-shirt sale. Each shirt costs $8. The silk screening costs $200 for the design, plus $2 per shirt. Each shirt will sell for $14.

(a) Find equations for the total cost $C$ and the total revenue $R$ for selling $x$ shirts.

(b) Find the break-even point.

**43. Break-Even Analysis**    You are starting a part-time business. You make an initial investment of $6000. The unit cost of the product is $6.50, and the selling price is $13.90.

(a) Find equations for the total cost $C$ and the total revenue $R$ for selling $x$ units of the product.

(b) Find the break-even point.

**44. Supply and Demand**    The demand and supply equations for a cordless screwdriver are given by

$p = 91.4 - 0.009x$        Demand equation

$p = 6.4 + 0.008x$        Supply equation

where $p$ is the price in dollars and $x$ represents the number of units. Find the equilibrium point for this market.

In Exercises 45–50, find the slope and $y$-intercept (if possible) of the linear equation. Then sketch the graph of the equation.

**45.** $3x + y = -2$

**46.** $-\frac{1}{3}x + \frac{5}{6}y = 1$

**47.** $y = -\frac{5}{3}$

**48.** $x = -3$

**49.** $-2x - 5y - 5 = 0$    **50.** $3.2x - 0.8y + 5.6 = 0$

In Exercises 51–54, find the slope of the line passing through the two points.

**51.** $(0, 0), (7, 6)$        **52.** $(-1, 5), (-5, 7)$

**53.** $(10, 17), (-11, -3)$        **54.** $(-11, -3), (-1, -3)$

(T) In Exercises 55–58, find an equation of the line that passes through the point and has the given slope. Then use a graphing utility to graph the line.

| Point | Slope | Point | Slope |
|---|---|---|---|
| **55.** $(3, -1)$ | $m = -2$ | **56.** $(-3, -3)$ | $m = \frac{1}{2}$ |
| **57.** $(1.5, -4)$ | $m = 0$ | **58.** $(8, 2)$ | $m$ is undefined |
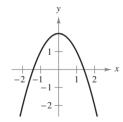
In Exercises 59 and 60, find the general form of the equation of the line passing through the point and satisfying the given condition.
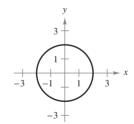
**59.** Point: $(-3, 6)$

(a) Slope is $\frac{7}{8}$

(b) Parallel to the line $4x + 2y = 7$

(c) Passes through the origin

(d) Perpendicular to the line $3x - 2y = 2$

**60.** Point: $(1, -3)$

(a) Parallel to the $x$-axis

(b) Perpendicular to the $x$-axis

(c) Parallel to the line $-4x + 5y = -3$

(d) Perpendicular to the line $5x - 2y = 3$

**61. Demand**    When a wholesaler sold a product at $32 per unit, sales were 750 units per week. After a price increase of $5 per unit, however, the sales dropped to 700 units per week.

(a) Write the quantity demanded $x$ as a linear function of the price $p$.

(b) *Linear Interpolation*    Predict the number of units sold at a price of $34.50 per unit.

(c) *Linear Extrapolation*    Predict the number of units sold at a price of $42.00 per unit.

(T) **62. Linear Depreciation**    A printing company purchases an advanced color copier/printer for $117,000. After 9 years, the equipment will be obsolete and have no value.

(a) Write a linear equation giving the value $v$ of the equipment in terms of the time $t$.

(b) Use a graphing utility to graph the function.

(c) Use a graphing utility to estimate the value of the equipment after 4 years.

(d) Use a graphing utility to estimate the time when the equipment's value will be $84,000.

In Exercises 63–66, use the Vertical Line Test to deter-
mine whether $y$ is a function of $x$.

**63.** $y = -x^2 + 2$
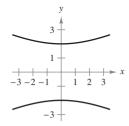


**64.** $x^2 + y^2 = 4$
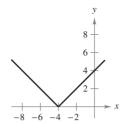


**65.** $y^2 - \frac{1}{4}x^2 = 4$



**66.** $y = |x + 4|$



In Exercises 67 and 68, evaluate the function at the
specified values of the independent variable. Simplify
the result.

**67.** $f(x) = 3x + 4$

   (a) $f(1)$      (b) $f(x + 1)$      (c) $f(2 + \Delta x)$

**68.** $f(x) = x^2 + 4x + 3$

   (a) $f(0)$      (b) $f(x - 1)$      (c) $f(x + \Delta x) - f(x)$

(T) In Exercises 69–74, use a graphing utility to graph the
function. Then find the domain and range of the
function.

**69.** $f(x) = x^3 + 2x^2 - x + 2$

**70.** $f(x) = 2$

**71.** $f(x) = \sqrt{x + 1}$

**72.** $f(x) = \dfrac{x - 3}{x^2 + x - 12}$

**73.** $f(x) = -|x| + 3$

**74.** $f(x) = -\frac{12}{13}x - \frac{7}{8}$

In Exercises 75 and 76, use $f$ and $g$ to find the combi-
nations of the functions.

(a) $f(x) + g(x)$      (b) $f(x) - g(x)$      (c) $f(x)g(x)$

(d) $\dfrac{f(x)}{g(x)}$      (e) $f(g(x))$      (f) $g(f(x))$
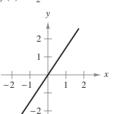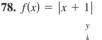
**75.** $f(x) = 1 + x^2,\quad g(x) = 2x - 1$
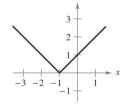
**76.** $f(x) = 2x - 3,\quad g(x) = \sqrt{x + 1}$

In Exercises 77–80, find the inverse function of $f$ (if it
exists).

**77.** $f(x) = \frac{3}{2}x$



**78.** $f(x) = |x + 1|$
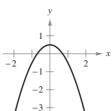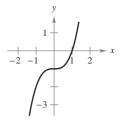


**79.** $f(x) = -x^2 + \frac{1}{2}$



**80.** $f(x) = x^3 - 1$



In Exercises 81–98, find the limit (if it exists).

**81.** $\lim\limits_{x \to 2} (5x - 3)$

**82.** $\lim\limits_{x \to 2} (2x + 9)$

**83.** $\lim\limits_{x \to 2} (5x - 3)(2x + 3)$

**84.** $\lim\limits_{x \to 2} \dfrac{5x - 3}{2x + 9}$

**85.** $\lim\limits_{t \to 3} \dfrac{t^2 + 1}{t}$

**86.** $\lim\limits_{t \to 0} \dfrac{t^2 + 1}{t}$

**87.** $\lim\limits_{t \to 1} \dfrac{t + 1}{t - 2}$

**88.** $\lim\limits_{t \to 2} \dfrac{t + 1}{t - 2}$

**89.** $\lim\limits_{x \to -2} \dfrac{x + 2}{x^2 - 4}$

**90.** $\lim\limits_{x \to 3^-} \dfrac{x^2 - 9}{x - 3}$

**91.** $\lim\limits_{x \to 0^+} \left( x - \dfrac{1}{x} \right)$

**92.** $\lim\limits_{x \to 1/2} \dfrac{2x - 1}{6x - 3}$

**93.** $\lim\limits_{x \to 0} \dfrac{[1/(x - 2)] - 1}{x}$

**94.** $\lim\limits_{x \to 0} \dfrac{[1/(x - 4)] - (1/4)}{x}$

**95.** $\lim\limits_{t \to 0} \dfrac{(1/\sqrt{t + 4}) - (1/2)}{t}$

**96.** $\lim\limits_{s \to 0} \dfrac{(1/\sqrt{1 + s}) - 1}{s}$

**97.** $\lim\limits_{\Delta x \to 0} \dfrac{(x + \Delta x)^3 - (x + \Delta x) - (x^3 - x)}{\Delta x}$

**98.** $\lim\limits_{\Delta x \to 0} \dfrac{1 - (x + \Delta x)^2 - (1 - x^2)}{\Delta x}$

In Exercises 99 and 100, use a table to estimate the
limit.

**99.** $\lim\limits_{x \to 1^+} \dfrac{\sqrt{2x + 1} - \sqrt{3}}{x - 1}$

**100.** $\lim\limits_{x \to 1^+} \dfrac{1 - \sqrt[3]{x}}{x - 1}$

**True or False?**  In Exercises 101–106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**101.** $\lim\limits_{x \to 0} \dfrac{|x|}{x} = 1$

**102.** $\lim\limits_{x \to 0} x^3 = 0$

**103.** $\lim\limits_{x \to 0} \sqrt{x} = 0$

**104.** $\lim\limits_{x \to 0} \sqrt[3]{x} = 0$

**105.** $\lim\limits_{x \to 2} f(x) = 3, \quad f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

**106.** $\lim\limits_{x \to 3} f(x) = 1, \quad f(x) = \begin{cases} x - 2, & x \le 3 \\ -x^2 + 8x - 14, & x > 3 \end{cases}$

In Exercises 107–114, describe the interval(s) on which the function is continuous. Explain why the function is continuous on the interval(s). If the function has a discontinuity, identify the conditions of continuity that are not satisfied.

**107.** $f(x) = \dfrac{1}{(x + 4)^2}$

**108.** $f(x) = \dfrac{x + 2}{x}$

**109.** $f(x) = \dfrac{3}{x + 1}$

**110.** $f(x) = \dfrac{x + 1}{2x + 2}$

**111.** $f(x) = [\![x + 3]\!]$

**112.** $f(x) = [\![x]\!] - 2$

**113.** $f(x) = \begin{cases} x, & x \le 0 \\ x + 1, & x > 0 \end{cases}$

**114.** $f(x) = \begin{cases} x, & x \le 0 \\ x^2, & x > 0 \end{cases}$

In Exercises 115 and 116, find the constant $a$ such that $f$ is continuous on the entire real line.

**115.** $f(x) = \begin{cases} -x + 1, & x \le 3 \\ ax - 8, & x > 3 \end{cases}$

**116.** $f(x) = \begin{cases} x + 1, & x < 1 \\ 2x + a, & x \ge 1 \end{cases}$

**117. Consumer Awareness**  The cost $C$ (in dollars) of making $x$ photocopies at a copy shop is given below.

$$C(x) = \begin{cases} 0.15x, & 0 < x \le 25 \\ 0.10x, & 25 < x \le 100 \\ 0.07x, & 100 < x \le 500 \\ 0.05x, & x > 500 \end{cases}$$

(T) (a) Use a graphing utility to graph the function and discuss its continuity. At what values is the function not continuous? Explain your reasoning.

(b) Find the cost of making 100 copies.

**118. Salary Contract**  A union contract guarantees a 10% salary increase yearly for 3 years. For a current salary of $28,000, the salary $S$ (in thousands of dollars) for the next 3 years is given by

$$S(t) = \begin{cases} 28.00, & 0 < t \le 1 \\ 30.80, & 1 < t \le 2 \\ 33.88, & 2 < t \le 3 \end{cases}$$

where $t = 0$ represents the present year. Does the limit of $S$ exist as $t$ approaches 2? Explain your reasoning.

(T) **119. Consumer Awareness**  A pay-as-you-go cellular phone charges $1 for the first time you access the phone and $0.10 for each additional minute or fraction thereof. Use the greatest integer function to create a model for the cost $C$ of a phone call lasting $t$ minutes. Use a graphing utility to graph the function, and discuss its continuity.

**120. Recycling**  A recycling center pays $0.50 for each pound of aluminum cans. Twenty-four aluminum cans weigh one pound. A mathematical model for the amount $A$ paid by the recycling center is

$$A = \frac{1}{2}\left[\!\!\left[\frac{x}{24}\right]\!\!\right]$$

where $x$ is the number of cans.

(T) (a) Use a graphing utility to graph the function and then discuss its continuity.

(b) How much does the recycling center pay out for 1500 cans?

**121. National Debt**  The table lists the national debt $D$ (in billions of dollars) for selected years. A mathematical model for the national debt is

$$D = 4.2845t^3 - 97.655t^2 + 861.14t + 2571.1,$$
$$2 \le t \le 15$$

where $t = 2$ represents 1992.  *(Source: U.S. Department of the Treasury)*

| $t$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $D$ | 4001.8 | 4351.0 | 4643.3 | 4920.6 | 5181.5 |

| $t$ | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|
| $D$ | 5369.2 | 5478.2 | 5605.5 | 5628.7 | 5769.9 |

| $t$ | 12 | 13 | 14 | 15 |
|---|---|---|---|---|
| $D$ | 6198.4 | 6760.0 | 7354.7 | 7905.3 |

(T) (a) Use a graphing utility to graph the model.

(b) Create a table that compares the values given by the model with the actual data.

(c) Use the model to estimate the national debt in 2010.

## Chapter Test

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, (a) find the distance between the points, (b) find the midpoint of the line segment joining the points, and (c) find the slope of the line passing through the points.

**1.** $(1, -1), (-4, 4)$      **2.** $\left(\frac{5}{2}, 2\right), (0, 2)$      **3.** $\left(3\sqrt{2}, 2\right), \left(\sqrt{2}, 1\right)$

**4.** Sketch the graph of the circle whose general equation is

$$x^2 + y^2 - 4x - 2y - 4 = 0.$$

**5.** The demand and supply equations for a product are $p = 65 - 2.1x$ and $p = 43 + 1.9x$, respectively, where $p$ is the price in dollars and $x$ represents the number of units in thousands. Find the equilibrium point for this market.

In Exercises 6–8, find the slope and $y$-intercept (if possible) of the linear equation. Then sketch the graph of the equation.

**6.** $y = \frac{1}{5}x - 2$      **7.** $x - \frac{7}{4} = 0$      **8.** $-x - 0.4y + 2.5 = 0$

In Exercises 9–11, (a) graph the function and label the intercepts, (b) determine the domain and range of the function, (c) find the value of the function when $x$ is $-3$, $-2$, and 3, and (d) determine whether the function is one-to-one.

**9.** $f(x) = 2x + 5$      **10.** $f(x) = x^2 - x - 2$      **11.** $f(x) = |x| - 4$

In Exercises 12 and 13, find the inverse function of $f$. Then check your results algebraically by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

**12.** $f(x) = 4x + 6$          **13.** $f(x) = \sqrt[3]{8 - 3x}$

In Exercises 14–17, find the limit (if it exists).

**14.** $\lim\limits_{x \to 0} \dfrac{x + 5}{x - 5}$    **15.** $\lim\limits_{x \to 5} \dfrac{x + 5}{x - 5}$    **16.** $\lim\limits_{x \to -3} \dfrac{x^2 + 2x - 3}{x^2 + 4x + 3}$    **17.** $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 9} - 3}{x}$

In Exercises 18–20, describe the interval(s) on which the function is continuous. Explain why the function is continuous on the interval(s). If the function has a discontinuity at a point, identify all conditions of continuity that are not satisfied.

**18.** $f(x) = \dfrac{x^2 - 16}{x - 4}$    **19.** $f(x) = \sqrt{5 - x}$    **20.** $f(x) = \begin{cases} 1 - x, & x < 1 \\ x - x^2, & x \geq 1 \end{cases}$

| $t$ | 0 | 1 | 2 |
|-----|------|------|------|
| $y$ | 2167 | 2149 | 2135 |

| $t$ | 3 | 4 | 5 |
|-----|------|------|------|
| $y$ | 2127 | 2113 | 2101 |

Table for 21

**21.** The table lists the numbers of farms $y$ (in thousands) in the United States for selected years. A mathematical model for the data is given by $y = 0.54t^2 - 15.4t + 2166$, where $t$ represents the year, with $t = 0$ corresponding to 2000. *(Source: U.S. Department of Agriculture)*

(a) Compare the values given by the model with the actual data. How well does the model fit the data? Explain your reasoning.

(b) Use the model to predict the number of farms in 2009.

# 2 Differentiation

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Higher-order derivatives are used to determine the acceleration function of a sports car. The acceleration function shows the changes in the car's velocity. As the car reaches its "cruising" speed, is the acceleration increasing or decreasing? (See Section 2.6, Exercise 45.)

## Applications

Differentiation has many real-life applications. The applications listed below represent a sample of the applications in this chapter.

# The Derivative and the Slope of a Graph

■ Identify tangent lines to a graph at a point.
■ Approximate the slopes of tangent lines to graphs at points.
■ Use the limit definition to find the slopes of graphs at points.
■ Use the limit definition to find the derivatives of functions.
■ Describe the relationship between differentiability and continuity.

## Tangent Line to a Graph



**FIGURE 2.1**   The slope of a nonlinear graph changes from one point to another.

Calculus is a branch of mathematics that studies rates of change of functions. In this course, you will learn that rates of change have many applications in real life. In Section 1.3, you learned how the slope of a line indicates the rate at which the line rises or falls. For a line, this rate (or slope) is the same at every point on the line. For graphs other than lines, the rate at which the graph rises or falls changes from point to point. For instance, in Figure 2.1, the parabola is rising more quickly at the point $(x_1, y_1)$ than it is at the point $(x_2, y_2)$. At the vertex $(x_3, y_3)$, the graph levels off, and at the point $(x_4, y_4)$, the graph is falling.

To determine the rate at which a graph rises or falls at a *single point*, you can find the slope of the **tangent line** at the point. In simple terms, the tangent line to the graph of a function $f$ at a point $P(x_1, y_1)$ is the line that best approximates the graph at that point, as shown in Figure 2.1. Figure 2.2 shows other examples of tangent lines.



**FIGURE 2.2**   Tangent Line to a Graph at a Point



**FIGURE 2.3**   Tangent Line to a Circle
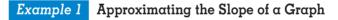
When Isaac Newton (1642–1727) was working on the "tangent line problem," he realized that it is difficult to define precisely what is meant by a tangent to a general curve. From geometry, you know that a line is tangent to a circle if the line intersects the circle at only one point, as shown in Figure 2.3. Tangent lines to a noncircular graph, however, can intersect the graph at more than one point. For instance, in the second graph in Figure 2.2, if the tangent line were extended, it would intersect the graph at a point other than the point of tangency. In this section, you will see how the notion of a limit can be used to define a general tangent line.

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### DISCOVERY

Use a graphing utility to graph $f(x) = 2x^3 - 4x^2 + 3x - 5$. On the same screen, sketch the graphs of $y = x - 5$, $y = 2x - 5$, and $y = 3x - 5$. Which of these lines, if any, appears to be tangent to the graph of $f$ at the point $(0, -5)$? Explain your reasoning.

## Slope of a Graph

Because a tangent line approximates the graph at a point, the problem of finding the slope of a graph at a point becomes one of finding the slope of the tangent line at the point.



$f(x) = x^2$

**FIGURE 2.4**

### *Example 1* Approximating the Slope of a Graph

Use the graph in Figure 2.4 to approximate the slope of the graph of $f(x) = x^2$ at the point $(1, 1)$.

**SOLUTION**  From the graph of $f(x) = x^2$, you can see that the tangent line at $(1, 1)$ rises approximately two units for each unit change in $x$. So, the slope of the tangent line at $(1, 1)$ is given by

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} \approx \frac{2}{1} = 2.$$

Because the tangent line at the point $(1, 1)$ has a slope of about 2, you can conclude that the graph has a slope of about 2 at the point $(1, 1)$.
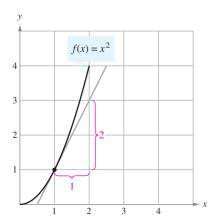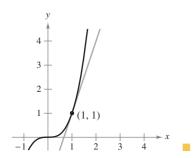
✓**CHECKPOINT 1**

Use the graph to approximate the slope of the graph of $f(x) = x^3$ at the point $(1, 1)$.



$(1, 1)$
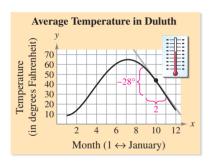
**STUDY TIP**

When visually approximating the slope of a graph, note that the scales on the horizontal and vertical axes may differ. When this happens (as it frequently does in applications), the slope of the tangent line is distorted, and you must be careful to account for the difference in scales.

### *Example 2* Interpreting Slope ®

Figure 2.5 graphically depicts the average monthly temperature (in degrees Fahrenheit) in Duluth, Minnesota. Estimate the slope of this graph at the indicated point and give a physical interpretation of the result. *(Source: National Oceanic and Atmospheric Administration)*
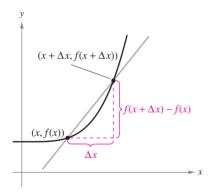
**SOLUTION**  From the graph, you can see that the tangent line at the given point falls approximately 28 units for each two-unit change in $x$. So, you can estimate the slope at the given point to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} \approx \frac{-28}{2}$$
$$= -14 \text{ degrees per month.}$$

This means that you can expect the average daily temperatures in November to be about 14 degrees *lower* than the corresponding temperatures in October.



**Average Temperature in Duluth**

$-28°$

Month (1 ↔ January)

**FIGURE 2.5**

✓**CHECKPOINT 2**

For which months do the slopes of the tangent lines appear to be positive? Negative? Interpret these slopes in the context of the problem. ■

## Slope and the Limit Process
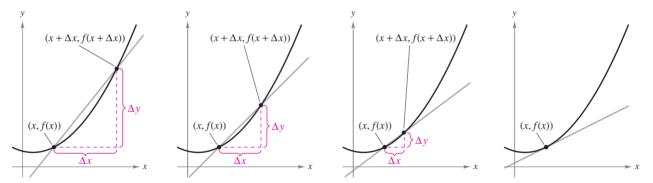


**FIGURE 2.6** The Secant Line Through the Two Points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$

In Examples 1 and 2, you approximated the slope of a graph at a point by making a careful graph and then "eyeballing" the tangent line at the point of tangency. A more precise method of approximating the slope of a tangent line makes use of a **secant line** through the point of tangency and a second point on the graph, as shown in Figure 2.6. If $(x, f(x))$ is the point of tangency and $(x + \Delta x, f(x + \Delta x))$ is a second point on the graph of $f$, then the slope of the secant line through the two points is

$$m_{\text{sec}} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$    Slope of secant line

The right side of this equation is called the **difference quotient.** The denominator $\Delta x$ is the **change in $x$,** and the numerator is the **change in $y$.** The beauty of this procedure is that you obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 2.7. Using the limit process, you can find the *exact* slope of the tangent line at $(x, f(x))$, which is also the slope of the graph of $f$ at $(x, f(x))$.



**FIGURE 2.7** As $\Delta x$ approaches 0, the secant lines approach the tangent line.

---

**Definition of the Slope of a Graph**

The **slope** $m$ of the graph of $f$ at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by

$$m = \lim_{\Delta x \to 0} m_{\text{sec}} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided this limit exists.

---

**STUDY TIP**

$\Delta x$ is used as a variable to represent the change in $x$ in the definition of the slope of a graph. Other variables may also be used. For instance, this definition is sometimes written as

$$m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

**FIGURE 2.8**
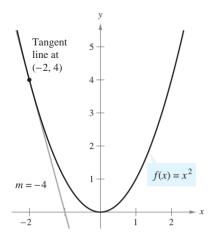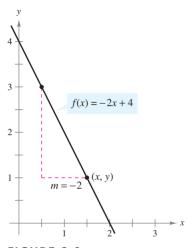
### Example 3    Finding Slope by the Limit Process

Find the slope of the graph of $f(x) = x^2$ at the point $(-2, 4)$.

**SOLUTION**    Begin by finding an expression that represents the slope of a secant line at the point $(-2, 4)$.

$$m_{\text{sec}} = \frac{f(-2 + \Delta x) - f(-2)}{\Delta x} \qquad \text{Set up difference quotient.}$$

$$= \frac{(-2 + \Delta x)^2 - (-2)^2}{\Delta x} \qquad \text{Use } f(x) = x^2.$$

$$= \frac{4 - 4\,\Delta x + (\Delta x)^2 - 4}{\Delta x} \qquad \text{Expand terms.}$$

$$= \frac{-4\,\Delta x + (\Delta x)^2}{\Delta x} \qquad \text{Simplify.}$$

$$= \frac{\Delta x(-4 + \Delta x)}{\Delta x} \qquad \text{Factor and divide out.}$$

$$= -4 + \Delta x, \quad \Delta x \neq 0 \qquad \text{Simplify.}$$

Next, take the limit of $m_{\text{sec}}$ as $\Delta x \to 0$.

$$m = \lim_{\Delta x \to 0} m_{\text{sec}} = \lim_{\Delta x \to 0} (-4 + \Delta x) = -4$$

So, the graph of $f$ has a slope of $-4$ at the point $(-2, 4)$, as shown in Figure 2.8.

### ✓ CHECKPOINT 3

Find the slope of the graph of $f(x) = x^2$ at the point $(2, 4)$. ■

### Example 4    Finding the Slope of a Graph

Find the slope of the graph of $f(x) = -2x + 4$.

**SOLUTION**    You know from your study of linear functions that the line given by $f(x) = -2x + 4$ has a slope of $-2$, as shown in Figure 2.9. This conclusion is consistent with the limit definition of slope.

$$m = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[-2(x + \Delta x) + 4] - [-2x + 4]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - 2\,\Delta x + 4 + 2x - 4}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\,\Delta x}{\Delta x} = -2$$

### ✓ CHECKPOINT 4

Find the slope of the graph of $f(x) = 2x + 5$. ■



**FIGURE 2.9**

It is important that you see the distinction between the ways the difference quotients were set up in Examples 3 and 4. In Example 3, you were finding the slope of a graph at a specific point $(c, f(c))$. To find the slope, you can use the following form of a difference quotient.

$$m = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} \qquad \text{Slope at specific point}$$

In Example 4, however, you were finding a formula for the slope at *any* point on the graph. In such cases, you should use $x$, rather than $c$, in the difference quotient.

$$m = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad \text{Formula for slope}$$

Except for linear functions, this form will always produce a function of $x$, which can then be evaluated to find the slope at any desired point.

### Example 5   Finding a Formula for the Slope of a Graph

Find a formula for the slope of the graph of $f(x) = x^2 + 1$. What are the slopes at the points $(-1, 2)$ and $(2, 5)$?

**SOLUTION**

$$\begin{aligned}
m_{\text{sec}} &= \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Set up difference quotient.} \\
&= \frac{[(x + \Delta x)^2 + 1] - (x^2 + 1)}{\Delta x} && \text{Use } f(x) = x^2 + 1. \\
&= \frac{x^2 + 2x\,\Delta x + (\Delta x)^2 + 1 - x^2 - 1}{\Delta x} && \text{Expand terms.} \\
&= \frac{2x\,\Delta x + (\Delta x)^2}{\Delta x} && \text{Simplify.} \\
&= \frac{\Delta x(2x + \Delta x)}{\Delta x} && \text{Factor and divide out.} \\
&= 2x + \Delta x, \quad \Delta x \neq 0 && \text{Simplify.}
\end{aligned}$$

Next, take the limit of $m_{\text{sec}}$ as $\Delta x \to 0$.

$$\begin{aligned}
m &= \lim_{\Delta x \to 0} m_{\text{sec}} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x) \\
&= 2x
\end{aligned}$$

Using the formula $m = 2x$, you can find the slopes at the specified points. At $(-1, 2)$ the slope is $m = 2(-1) = -2$, and at $(2, 5)$ the slope is $m = 2(2) = 4$. The graph of $f$ is shown in Figure 2.10.



$f(x) = x^2 + 1$

Tangent line at $(2, 5)$

Tangent line at $(-1, 2)$

**FIGURE 2.10**

### ✓ CHECKPOINT 5

Find a formula for the slope of the graph of $f(x) = 4x^2 + 1$. What are the slopes at the points $(0, 1)$ and $(1, 5)$?



$(1, 5)$

$(0, 1)$

### STUDY TIP

The slope of the graph of $f(x) = x^2 + 1$ varies for different values of $x$. For what value of $x$ is the slope equal to 0?

## The Derivative of a Function

In Example 5, you started with the function $f(x) = x^2 + 1$ and used the limit process to derive another function, $m = 2x$, that represents the slope of the graph of $f$ at the point $(x, f(x))$. This derived function is called the **derivative** of $f$ at $x$. It is denoted by $f'(x)$, which is read as "$f$ prime of $x$."

**Definition of the Derivative**

The **derivative of $f$ at $x$** is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided this limit exists. A function is **differentiable** at $x$ if its derivative exists at $x$. The process of finding derivatives is called **differentiation.**

In addition to $f'(x)$, other notations can be used to denote the derivative of $y = f(x)$. The most common are

$$\frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad \text{and} \quad D_x[y].$$

*Example 6*  **Finding a Derivative**

Find the derivative of $f(x) = 3x^2 - 2x$.

**SOLUTION**

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[3(x + \Delta x)^2 - 2(x + \Delta x)] - (3x^2 - 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2 + 6x\,\Delta x + 3(\Delta x)^2 - 2x - 2\,\Delta x - 3x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\,\Delta x + 3(\Delta x)^2 - 2\,\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(6x + 3\,\Delta x - 2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (6x + 3\,\Delta x - 2)$$

$$= 6x - 2$$

So, the derivative of $f(x) = 3x^2 - 2x$ is $f'(x) = 6x - 2$.

✓ **CHECKPOINT 6**

Find the derivative of $f(x) = x^2 - 5x$.  ■

In many applications, it is convenient to use a variable other than $x$ as the independent variable. Example 7 shows a function that uses $t$ as the independent variable.

## Example 7  Finding a Derivative

Find the derivative of $y$ with respect to $t$ for the function

$$y = \frac{2}{t}.$$

**SOLUTION**  Consider $y = f(t)$, and use the limit process as shown.

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \qquad \text{Set up difference quotient.}$$

$$= \lim_{\Delta t \to 0} \frac{\dfrac{2}{t + \Delta t} - \dfrac{2}{t}}{\Delta t} \qquad \text{Use } f(t) = 2/t.$$

$$= \lim_{\Delta t \to 0} \frac{\dfrac{2t - 2t - 2\,\Delta t}{t(t + \Delta t)}}{\Delta t} \qquad \text{Expand terms.}$$

$$= \lim_{\Delta t \to 0} \frac{-2\,\Delta t}{t(\Delta t)(t + \Delta t)} \qquad \text{Factor and divide out.}$$

$$= \lim_{\Delta t \to 0} \frac{-2}{t(t + \Delta t)} \qquad \text{Simplify.}$$

$$= -\frac{2}{t^2} \qquad \text{Evaluate the limit.}$$

So, the derivative of $y$ with respect to $t$ is

$$\frac{dy}{dt} = -\frac{2}{t^2}.$$

Remember that the derivative of a function gives you a formula for finding the slope of the tangent line at any point on the graph of the function. For example, the slope of the tangent line to the graph of $f$ at the point $(1, 2)$ is given by

$$f'(1) = -\frac{2}{1^2} = -2.$$

To find the slopes of the graph at other points, substitute the $t$-coordinate of the point into the derivative, as shown below.

| Point | $t$-Coordinate | Slope |
|-------|----------------|-------|
| $(2, 1)$ | $t = 2$ | $m = f'(2) = -\dfrac{2}{2^2} = -\dfrac{1}{2}$ |
| $(-2, -1)$ | $t = -2$ | $m = f'(-2) = -\dfrac{2}{(-2)^2} = -\dfrac{1}{2}$ |

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*Specific calculator keystroke instructions for operations in this and other technology boxes can be found at *college.hmco.com/info/larsonapplied*.

## Differentiability and Continuity

Not every function is differentiable. Figure 2.11 shows some common situations in which a function will not be differentiable at a point—vertical tangent lines, discontinuities, and sharp turns in the graph. Each of the functions shown in Figure 2.11 is differentiable at every value of *x except x = 0.*



**FIGURE 2.11**     Functions That Are Not Differentiable at $x = 0$

In Figure 2.11, you can see that all but one of the functions are continuous at $x = 0$ but none are differentiable there. This shows that continuity is not a strong enough condition to guarantee differentiability. On the other hand, if a function is differentiable at a point, then it must be continuous at that point. This important result is stated in the following theorem.

---

**Differentiability Implies Continuity**

If a function *f* is differentiable at $x = c$, then *f* is continuous at $x = c$.

---

**CONCEPT CHECK**

1. What is the name of the line that best approximates the slope of a graph at a point?

2. What is the name of a line through the point of tangency and a second point on the graph?

3. Sketch a graph of a function whose derivative is always negative.

4. Sketch a graph of a function whose derivative is always positive.

In Exercises 1–3, find an equation of the line containing $P$ and $Q$.

**1.** $P(2, 1)$, $Q(2, 4)$

**2.** $P(2, 2)$, $Q(-5, 2)$

**3.** $P(2, 0)$, $Q(3, -1)$

In Exercises 4–7, find the limit.

**4.** $\lim\limits_{\Delta x \to 0} \dfrac{2x\Delta x + (\Delta x)^2}{\Delta x}$

**5.** $\lim\limits_{\Delta x \to 0} \dfrac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$

**6.** $\lim\limits_{\Delta x \to 0} \dfrac{1}{x(x + \Delta x)}$

**7.** $\lim\limits_{\Delta x \to 0} \dfrac{(x + \Delta x)^2 - x^2}{\Delta x}$

In Exercises 8–10, find the domain of the function.

**8.** $f(x) = \dfrac{1}{x - 1}$

**9.** $f(x) = \dfrac{1}{5}x^3 - 2x^2 + \dfrac{1}{3}x - 1$
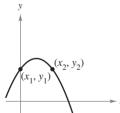
**10.** $f(x) = \dfrac{6x}{x^3 + x}$

## Exercises 2.1
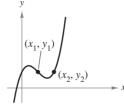
In Exercises 1–4, trace the graph and sketch the tangent lines at $(x_1, y_1)$ and $(x_2, y_2)$.

**1.**



**2.**
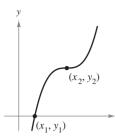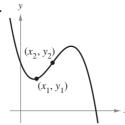


**3.**
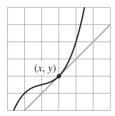


**4.**
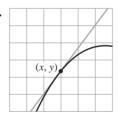


In Exercises 5–10, estimate the slope of the graph at the point $(x, y)$. (Each square on the grid is 1 unit by 1 unit.)

**5.**



**6.**



**7.**



**8.**
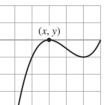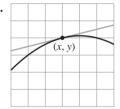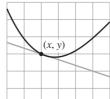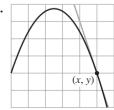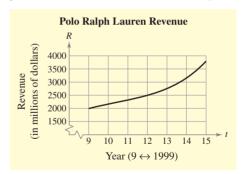


**9.**
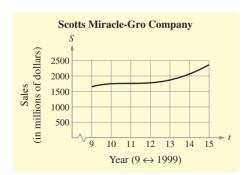


**10.**



**11. Revenue** The graph represents the revenue $R$ (in millions of dollars per year) for Polo Ralph Lauren from 1999 through 2005, where $t$ represents the year, with $t = 9$ corresponding to 1999. Estimate the slopes of the graph for the years 2002 and 2004. *(Source: Polo Ralph Lauren Corp.)*



**Polo Ralph Lauren Revenue**
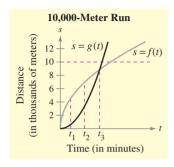
**12. Sales** The graph represents the sales $S$ (in millions of dollars per year) for Scotts Miracle-Gro Company from 1999 through 2005, where $t$ represents the year, with $t = 9$ corresponding to 1999. Estimate the slopes of the graph for the years 2001 and 2004. *(Source: Scotts Miracle-Gro Company)*



**Scotts Miracle-Gro Company**

Year (9 ↔ 1999)

**13. Consumer Trends** The graph shows the number of visitors $V$ to a national park in hundreds of thousands during a one-year period, where $t = 1$ corresponds to January. Estimate the slopes of the graph at $t = 1$, 8, and 12.



**Visitors to a National Park**

Month (1 ↔ January)

**14. Athletics** Two long distance runners starting out side by side begin a 10,000-meter run. Their distances are given by $s = f(t)$ and $s = g(t)$, where $s$ is measured in thousands of meters and $t$ is measured in minutes.



**10,000-Meter Run**

Time (in minutes)

(a) Which runner is running faster at $t_1$?

(b) What conclusion can you make regarding their rates at $t_2$?

(c) What conclusion can you make regarding their rates at $t_3$?

(d) Which runner finishes the race first? Explain.

In Exercises 15–24, use the limit definition to find the slope of the tangent line to the graph of $f$ at the given point.

**15.** $f(x) = 6 - 2x$; $(2, 2)$     **16.** $f(x) = 2x + 4$; $(1, 6)$

**17.** $f(x) = -1$; $(0, -1)$     **18.** $f(x) = 6$; $(-2, 6)$

**19.** $f(x) = x^2 - 1$; $(2, 3)$     **20.** $f(x) = 4 - x^2$; $(2, 0)$

**21.** $f(x) = x^3 - x$; $(2, 6)$

**22.** $f(x) = x^3 + 2x$; $(1, 3)$

**23.** $f(x) = 2\sqrt{x}$; $(4, 4)$

**24.** $f(x) = \sqrt{x + 1}$; $(8, 3)$

In Exercises 25–38, use the limit definition to find the derivative of the function.

**25.** $f(x) = 3$     **26.** $f(x) = -2$

**27.** $f(x) = -5x$     **28.** $f(x) = 4x + 1$

**29.** $g(s) = \frac{1}{3}s + 2$     **30.** $h(t) = 6 - \frac{1}{2}t$

**31.** $f(x) = x^2 - 4$     **32.** $f(x) = 1 - x^2$

**33.** $h(t) = \sqrt{t - 1}$     **34.** $f(x) = \sqrt{x + 2}$

**35.** $f(t) = t^3 - 12t$     **36.** $f(t) = t^3 + t^2$

**37.** $f(x) = \dfrac{1}{x + 2}$     **38.** $g(s) = \dfrac{1}{s - 1}$

In Exercises 39–46, use the limit definition to find an equation of the tangent line to the graph of $f$ at the given point. Then verify your results by using a graphing utility to graph the function and its tangent line at the point.

**39.** $f(x) = \frac{1}{2}x^2$; $(2, 2)$     **40.** $f(x) = -x^2$; $(-1, -1)$

**41.** $f(x) = (x - 1)^2$; $(-2, 9)$     **42.** $f(x) = 2x^2 - 1$; $(0, -1)$

**43.** $f(x) = \sqrt{x} + 1$; $(4, 3)$     **44.** $f(x) = \sqrt{x + 2}$; $(7, 3)$

**45.** $f(x) = \dfrac{1}{x}$; $(1, 1)$     **46.** $f(x) = \dfrac{1}{x - 1}$; $(2, 1)$

In Exercises 47–50, find an equation of the line that is tangent to the graph of $f$ and parallel to the given line.

| *Function* | *Line* |
|---|---|
| **47.** $f(x) = -\frac{1}{4}x^2$ | $x + y = 0$ |
| **48.** $f(x) = x^2 + 1$ | $2x + y = 0$ |
| **49.** $f(x) = -\frac{1}{2}x^3$ | $6x + y + 4 = 0$ |
| **50.** $f(x) = x^2 - x$ | $x + 2y - 6 = 0$ |
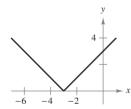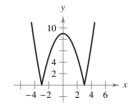
In Exercises 51–58, describe the x-values at which the function is differentiable. Explain your reasoning.

**51.** $y = |x + 3|$
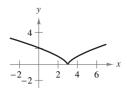


**52.** $y = |x^2 - 9|$
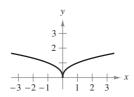


**53.** $y = (x - 3)^{2/3}$
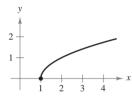


**54.** $y = x^{2/5}$
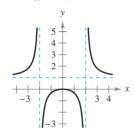


**55.** $y = \sqrt{x - 1}$
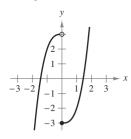


**56.** $y = \dfrac{x^2}{x^2 - 4}$
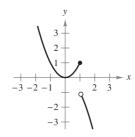


**57.** $y = \begin{cases} x^3 + 3, & x < 0 \\ x^3 - 3, & x \geq 0 \end{cases}$



**58.** $y = \begin{cases} x^2, & x \leq 1 \\ -x^2, & x > 1 \end{cases}$
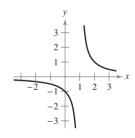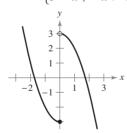


In Exercises 59 and 60, describe the x-values at which f is differentiable.

**59.** $f(x) = \dfrac{1}{x - 1}$



**60.** $f(x) = \begin{cases} x^2 - 3, & x \leq 0 \\ 3 - x^2, & x > 0 \end{cases}$



In Exercises 61 and 62, identify a function f that has the given characteristics. Then sketch the function.

**61.** $f(0) = 2; f'(x) = -3, -\infty < x < \infty$

**62.** $f(-2) = f(4) = 0; f'(1) = 0, f'(x) < 0$
   for $x < 1; f'(x) > 0$ for $x > 1$

(T) **Graphical, Numerical, and Analytic Analysis**    In Exercises 63–66, use a graphing utility to graph f on the interval $[-2, 2]$. Complete the table by graphically estimating the slopes of the graph at the given points. Then evaluate the slopes analytically and compare your results with those obtained graphically.

| $x$ | $-2$ | $-\frac{3}{2}$ | $-1$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ | $\frac{3}{2}$ | $2$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | | | | |
| $f'(x)$ | | | | | | | | | |

**63.** $f(x) = \frac{1}{4}x^3$            **64.** $f(x) = \frac{1}{2}x^2$

**65.** $f(x) = -\frac{1}{2}x^3$          **66.** $f(x) = -\frac{3}{2}x^2$

(T) In Exercises 67–70, find the derivative of the given function f. Then use a graphing utility to graph f and its derivative in the same viewing window. What does the x-intercept of the derivative indicate about the graph of f?

**67.** $f(x) = x^2 - 4x$
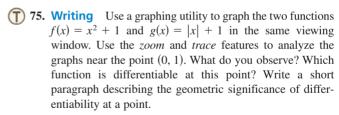
**68.** $f(x) = 2 + 6x - x^2$

**69.** $f(x) = x^3 - 3x$

**70.** $f(x) = x^3 - 6x^2$

**True or False?**    In Exercises 71–74, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**71.** The slope of the graph of $y = x^2$ is different at every point on the graph of f.

**72.** If a function is continuous at a point, then it is differentiable at that point.

**73.** If a function is differentiable at a point, then it is continuous at that point.

**74.** A tangent line to a graph can intersect the graph at more than one point.

(T) **75.** **Writing**    Use a graphing utility to graph the two functions $f(x) = x^2 + 1$ and $g(x) = |x| + 1$ in the same viewing window. Use the *zoom* and *trace* features to analyze the graphs near the point $(0, 1)$. What do you observe? Which function is differentiable at this point? Write a short paragraph describing the geometric significance of differentiability at a point.

## Section 2.2

## Some Rules for Differentiation

- ■ Find the derivatives of functions using the Constant Rule.
- ■ Find the derivatives of functions using the Power Rule.
- ■ Find the derivatives of functions using the Constant Multiple Rule.
- ■ Find the derivatives of functions using the Sum and Difference Rules.
- ■ Use derivatives to answer questions about real-life situations.

### The Constant Rule

In Section 2.1, you found derivatives by the limit process. This process is tedious, even for simple functions, but fortunately there are rules that greatly simplify differentiation. These rules allow you to calculate derivatives without the *direct* use of limits.

> **The Constant Rule**
>
> The derivative of a constant function is zero. That is,
>
> $$\frac{d}{dx}[c] = 0, \qquad c \text{ is a constant.}$$

**PROOF**   Let $f(x) = c$. Then, by the limit definition of the derivative, you can write

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0.$$

So, $\dfrac{d}{dx}[c] = 0$.

The slope of a horizontal line is zero.

$f(x) = c$

The derivative of a constant function is zero.
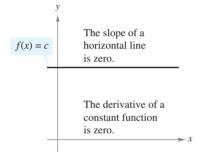
**FIGURE 2.12**

> **STUDY TIP**
>
> Note in Figure 2.12 that the Constant Rule is equivalent to saying that the slope of a horizontal line is zero.

**STUDY TIP**

An interpretation of the Constant Rule says that the tangent line to a constant function is the function itself. Find an equation of the tangent line to $f(x) = -4$ at $x = 3$.

### *Example 1*   Finding Derivatives of Constant Functions

**a.** $\dfrac{d}{dx}[7] = 0$

**b.** If $f(x) = 0$, then $f'(x) = 0$.

**c.** If $y = 2$, then $\dfrac{dy}{dx} = 0$.

**d.** If $g(t) = -\dfrac{3}{2}$, then $g'(t) = 0$.

**✓ CHECKPOINT 1**

Find the derivative of each function.

**a.** $f(x) = -2$     **b.** $y = \pi$     **c.** $g(w) = \sqrt{5}$     **d.** $s(t) = 320.5$ ■

## The Power Rule

The binomial expansion process is used to prove the Power Rule.

$$(x + \Delta x)^2 = x^2 + 2x\,\Delta x + (\Delta x)^2$$

$$(x + \Delta x)^3 = x^3 + 3x^2\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$(x + \Delta x)^n = x^n + nx^{n-1}\,\Delta x + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n$$

$(\Delta x)^2$ is a factor of these terms.

---

**The (Simple) Power Rule**

$$\frac{d}{dx}[x^n] = nx^{n-1}, \qquad n \text{ is any real number.}$$

---

**PROOF**    We prove only the case in which $n$ is a positive integer. Let $f(x) = x^n$. Using the binomial expansion, you can write

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\,\Delta x + \dfrac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n - x^n}{\Delta x}$$
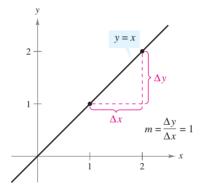
$$= \lim_{\Delta x \to 0} \left[ nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdots + (\Delta x)^{n-1} \right]$$

$$= nx^{n-1} + 0 + \cdots + 0 = nx^{n-1}.$$

For the Power Rule, the case in which $n = 1$ is worth remembering as a separate differentiation rule. That is,

$$\frac{d}{dx}[x] = 1. \qquad \text{The derivative of } x \text{ is 1.}$$

This rule is consistent with the fact that the slope of the line given by $y = x$ is 1. (See Figure 2.13.)



**FIGURE 2.13**    The slope of the line $y = x$ is 1.

**Example 2**   **Applying the Power Rule**

Find the derivative of each function.

| *Function* | *Derivative* |
|---|---|
| **a.** $f(x) = x^3$ | $f'(x) = 3x^2$ |
| **b.** $y = \dfrac{1}{x^2} = x^{-2}$ | $\dfrac{dy}{dx} = (-2)x^{-3} = -\dfrac{2}{x^3}$ |
| **c.** $g(t) = t$ | $g'(t) = 1$ |
| **d.** $R = x^4$ | $\dfrac{dR}{dx} = 4x^3$ |

In Example 2(b), note that *before* differentiating, you should rewrite $1/x^2$ as $x^{-2}$. Rewriting is the first step in *many* differentiation problems.

| Original Function: $y = \dfrac{1}{x^2}$ | ⇨ | Rewrite: $y = x^{-2}$ | ⇨ | Differentiate: $\dfrac{dy}{dx} = (-2)x^{-3}$ | ⇨ | Simplify: $\dfrac{dy}{dx} = -\dfrac{2}{x^3}$ |
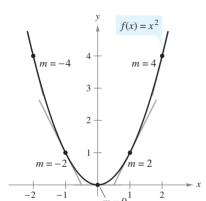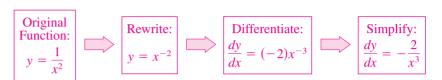
Remember that the derivative of a function $f$ is another function that gives the slope of the graph of $f$ at any point at which $f$ is differentiable. So, you can use the derivative to find slopes, as shown in Example 3.

**Example 3**   **Finding the Slope of a Graph**

Find the slopes of the graph of

$$f(x) = x^2 \qquad \text{Original function}$$

when $x = -2, -1, 0, 1,$ and 2.

**SOLUTION**   Begin by using the Power Rule to find the derivative of $f$.

$$f'(x) = 2x \qquad \text{Derivative}$$

You can use the derivative to find the slopes of the graph of $f$, as shown.

| *x-Value* | *Slope of Graph of f* |
|---|---|
| $x = -2$ | $m = f'(-2) = 2(-2) = -4$ |
| $x = -1$ | $m = f'(-1) = 2(-1) = -2$ |
| $x = 0$ | $m = f'(0) = 2(0) = 0$ |
| $x = 1$ | $m = f'(1) = 2(1) = 2$ |
| $x = 2$ | $m = f'(2) = 2(2) = 4$ |
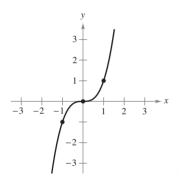
The graph of $f$ is shown in Figure 2.14.  ■



**FIGURE 2.14**

## The Constant Multiple Rule

To prove the Constant Multiple Rule, the following property of limits is used.

$$\lim_{x \to a} cg(x) = c\left[\lim_{x \to a} g(x)\right]$$

---

**The Constant Multiple Rule**

If $f$ is a differentiable function of $x$, and $c$ is a real number, then

$$\frac{d}{dx}[cf(x)] = cf'(x), \qquad c \text{ is a constant.}$$

---

**PROOF**   Apply the definition of the derivative to produce

$$\frac{d}{dx}[cf(x)] = \lim_{\Delta x \to 0} \frac{cf(x + \Delta x) - cf(x)}{\Delta x} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta x \to 0} c\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$$

$$= c\left[\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\right] = cf'(x).$$

Informally, the Constant Multiple Rule states that constants can be factored out of the differentiation process.

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[(\phantom{)}f(x)] = cf'(x)$$

The usefulness of this rule is often overlooked, especially when the constant appears in the denominator, as shown below.

$$\frac{d}{dx}\left[\frac{f(x)}{c}\right] = \frac{d}{dx}\left[\frac{1}{c}f(x)\right] = \frac{1}{c}\left(\frac{d}{dx}[(\phantom{)}f(x)]\right) = \frac{1}{c}f'(x)$$

To use the Constant Multiple Rule efficiently, look for constants that can be factored out *before* differentiating. For example,

$$\frac{d}{dx}[5x^2] = 5\frac{d}{dx}[x^2] \qquad \text{Factor out 5.}$$

$$= 5(2x) \qquad \text{Differentiate.}$$

$$= 10x \qquad \text{Simplify.}$$

and

$$\frac{d}{dx}\left[\frac{x^2}{5}\right] = \frac{1}{5}\left(\frac{d}{dx}[x^2]\right) \qquad \text{Factor out } \tfrac{1}{5}.$$

$$= \frac{1}{5}(2x) \qquad \text{Differentiate.}$$

$$= \frac{2}{5}x. \qquad \text{Simplify.}$$

**Example 4**   **Using the Power and Constant Multiple Rules**

Differentiate each function.

**a.** $y = 2x^{1/2}$    **b.** $f(t) = \dfrac{4t^2}{5}$

**SOLUTION**

**a.** Using the Constant Multiple Rule and the Power Rule, you can write

$$\frac{dy}{dx} = \frac{d}{dx}[2x^{1/2}] = 2\underbrace{\frac{d}{dx}[x^{1/2}]}_{\text{Constant Multiple Rule}} = 2\underbrace{\left(\frac{1}{2}x^{-1/2}\right)}_{\text{Power Rule}} = x^{-1/2} = \frac{1}{\sqrt{x}}.$$

**b.** Begin by rewriting $f(t)$ as

$$f(t) = \frac{4t^2}{5} = \frac{4}{5}t^2.$$

Then, use the Constant Multiple Rule and the Power Rule to obtain

$$f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\left[\frac{d}{dt}(t^2)\right] = \frac{4}{5}(2t) = \frac{8}{5}t.$$

You may find it helpful to combine the Constant Multiple Rule and the Power Rule into one combined rule.

$$\frac{d}{dx}[cx^n] = cnx^{n-1}, \quad n \text{ is a real number, } c \text{ is a constant.}$$

For instance, in Example 4(b), you can apply this combined rule to obtain

$$\frac{d}{dt}\left[\frac{4}{5}t^2\right] = \left(\frac{4}{5}\right)(2)(t) = \frac{8}{5}t.$$

The three functions in the next example are simple, yet errors are frequently made in differentiating functions involving constant multiples of the first power of $x$. Keep in mind that

$$\frac{d}{dx}[cx] = c, \quad c \text{ is a constant.}$$

**Example 5**   **Applying the Constant Multiple Rule**

Find the derivative of each function.

| *Original Function* | *Derivative* |
|---|---|
| **a.** $y = -\dfrac{3x}{2}$ | $y' = -\dfrac{3}{2}$ |
| **b.** $y = 3\pi x$ | $y' = 3\pi$ |
| **c.** $y = -\dfrac{x}{2}$ | $y' = -\dfrac{1}{2}$ |

✓**CHECKPOINT 4**

Differentiate each function.

**a.** $y = 4x^2$

**b.** $f(x) = 16x^{1/2}$ ◾

✓**CHECKPOINT 5**

Find the derivative of each function.

**a.** $y = \dfrac{t}{4}$

**b.** $y = -\dfrac{2x}{5}$ ◾

Parentheses can play an important role in the use of the Constant Multiple Rule and the Power Rule. In Example 6, be sure you understand the mathematical conventions involving the use of parentheses.

### Example 6   Using Parentheses When Differentiating

Find the derivative of each function.

**a.** $y = \dfrac{5}{2x^3}$    **b.** $y = \dfrac{5}{(2x)^3}$    **c.** $y = \dfrac{7}{3x^{-2}}$    **d.** $y = \dfrac{7}{(3x)^{-2}}$

**SOLUTION**

| Function | Rewrite | Differentiate | Simplify |
|---|---|---|---|
| **a.** $y = \dfrac{5}{2x^3}$ | $y = \dfrac{5}{2}(x^{-3})$ | $y' = \dfrac{5}{2}(-3x^{-4})$ | $y' = -\dfrac{15}{2x^4}$ |
| **b.** $y = \dfrac{5}{(2x)^3}$ | $y = \dfrac{5}{8}(x^{-3})$ | $y' = \dfrac{5}{8}(-3x^{-4})$ | $y' = -\dfrac{15}{8x^4}$ |
| **c.** $y = \dfrac{7}{3x^{-2}}$ | $y = \dfrac{7}{3}(x^2)$ | $y' = \dfrac{7}{3}(2x)$ | $y' = \dfrac{14x}{3}$ |
| **d.** $y = \dfrac{7}{(3x)^{-2}}$ | $y = 63(x^2)$ | $y' = 63(2x)$ | $y' = 126x$ |

**✓ CHECKPOINT 6**

Find the derivative of each function.

**a.** $y = \dfrac{9}{4x^2}$    **b.** $y = \dfrac{9}{(4x)^2}$ ■

**STUDY TIP**

When differentiating functions involving radicals, you should rewrite the function with rational exponents. For instance, you should rewrite $y = \sqrt[3]{x}$ as $y = x^{1/3}$, and you should rewrite

$$y = \dfrac{1}{\sqrt[3]{x^4}} \ \text{ as } \ y = x^{-4/3}.$$

### Example 7   Differentiating Radical Functions

Find the derivative of each function.

**a.** $y = \sqrt{x}$    **b.** $y = \dfrac{1}{2\sqrt[3]{x^2}}$    **c.** $y = \sqrt{2x}$

**SOLUTION**

| Function | Rewrite | Differentiate | Simplify |
|---|---|---|---|
| **a.** $y = \sqrt{x}$ | $y = x^{1/2}$ | $y' = \left(\dfrac{1}{2}\right)x^{-1/2}$ | $y' = \dfrac{1}{2\sqrt{x}}$ |
| **b.** $y = \dfrac{1}{2\sqrt[3]{x^2}}$ | $y = \dfrac{1}{2}x^{-2/3}$ | $y' = \dfrac{1}{2}\left(-\dfrac{2}{3}\right)x^{-5/3}$ | $y' = -\dfrac{1}{3x^{5/3}}$ |
| **c.** $y = \sqrt{2x}$ | $y = \sqrt{2}(x^{1/2})$ | $y' = \sqrt{2}\left(\dfrac{1}{2}\right)x^{-1/2}$ | $y' = \dfrac{1}{\sqrt{2x}}$ |

**✓ CHECKPOINT 7**

Find the derivative of each function.

**a.** $y = \sqrt{5x}$

**b.** $y = \sqrt[3]{x}$ ■

## The Sum and Difference Rules

The next two rules are ones that you might expect to be true, and you may have used them without thinking about it. For instance, if you were asked to differentiate $y = 3x + 2x^3$, you would probably write

$$y' = 3 + 6x^2$$

without questioning your answer. The validity of differentiating a sum or difference of functions term by term is given by the Sum and Difference Rules.

---

**The Sum and Difference Rules**

The derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

---

**PROOF**    Let $h(x) = f(x) + g(x)$. Then, you can prove the Sum Rule as shown.

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x) + g(x + \Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[ \frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$= f'(x) + g'(x)$$

So,

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).$$

The Difference Rule can be proved in a similar manner.

The Sum and Difference Rules can be extended to the sum or difference of any finite number of functions. For instance, if $y = f(x) + g(x) + h(x)$, then $y' = f'(x) + g'(x) + h'(x)$.

---

**STUDY TIP**

Look back at Example 6 on page 120. Notice that the example asks for the derivative of the difference of two functions. Verify this result by using the Difference Rule.

With the four differentiation rules listed in this section, you can differentiate *any* polynomial function.

$f(x) = x^3 - 4x + 2$

## *Example 8*   Using the Sum and Difference Rules

Find the slope of the graph of $f(x) = x^3 - 4x + 2$ at the point $(1, -1)$.

**SOLUTION**   The derivative of $f(x)$ is

$$f'(x) = 3x^2 - 4.$$

So, the slope of the graph of $f$ at $(1, -1)$ is

$$\text{Slope} = f'(1) = 3(1)^2 - 4 = -1$$

as shown in Figure 2.15.

$(1, -1)$

Slope = −1

**FIGURE 2.15**

### ✓ CHECKPOINT 8

Find the slope of the graph of $f(x) = x^2 - 5x + 1$ at the point $(2, -5)$. ■

Example 8 illustrates the use of the derivative for determining the shape of a graph. A rough sketch of the graph of $f(x) = x^3 - 4x + 2$ might lead you to think that the point $(1, -1)$ is a minimum point of the graph. After finding the slope at this point to be $-1$, however, you can conclude that the minimum point (where the slope is 0) is farther to the right. (You will study techniques for finding minimum and maximum points in Section 3.2.)

$g(x) = -\frac{1}{2}x^4 + 3x^3 - 2x$

## *Example 9*   Using the Sum and Difference Rules

Find an equation of the tangent line to the graph of

$$g(x) = -\frac{1}{2}x^4 + 3x^3 - 2x$$

at the point $\left(-1, -\frac{3}{2}\right)$.

**SOLUTION**   The derivative of $g(x)$ is $g'(x) = -2x^3 + 9x^2 - 2$, which implies that the slope of the graph at the point $\left(-1, -\frac{3}{2}\right)$ is

$$\begin{aligned}\text{Slope} = g'(-1) &= -2(-1)^3 + 9(-1)^2 - 2 \\ &= 2 + 9 - 2 \\ &= 9\end{aligned}$$

Slope = 9

$\left(-1, -\frac{3}{2}\right)$

**FIGURE 2.16**

as shown in Figure 2.16. Using the point-slope form, you can write the equation of the tangent line at $\left(-1, -\frac{3}{2}\right)$ as shown.

$$y - \left(-\frac{3}{2}\right) = 9[x - (-1)] \qquad \text{Point-slope form}$$

$$y = 9x + \frac{15}{2} \qquad \text{Equation of tangent line}$$

### ✓ CHECKPOINT 9

Find an equation of the tangent line to the graph of $f(x) = -x^2 + 3x - 2$ at the point $(2, 0)$. ■

## Application

### *Example 10*  Modeling Revenue  (R)

From 2000 through 2005, the revenue $R$ (in millions of dollars per year) for Microsoft Corporation can be modeled by

$$R = -110.194t^3 + 993.98t^2 + 1155.6t + 23,036, \quad 0 \le t \le 5$$

where $t$ represents the year, with $t = 0$ corresponding to 2000. At what rate was Microsoft's revenue changing in 2001?  *(Source: Microsoft Corporation)*

**SOLUTION**   One way to answer this question is to find the derivative of the revenue model with respect to time.

$$\frac{dR}{dt} = -330.582t^2 + 1987.96t + 1155.6, \quad 0 \le t \le 5$$

In 2001 (when $t = 1$), the rate of change of the revenue with respect to time is given by

$$-330.582(1)^2 + 1987.96(1) + 1155.6 \approx 2813.$$

Because $R$ is measured in millions of dollars and $t$ is measured in years, it follows that the derivative $dR/dt$ is measured in millions of dollars per year. So, at the end of 2001, Microsoft's revenue was increasing at a rate of about \$2813 million per year, as shown in Figure 2.17.



**Microsoft Revenue**

Slope ≈ 2813

Year (0 ↔ 2000)

**FIGURE 2.17**

### ✓ CHECKPOINT 10

From 1998 through 2005, the revenue per share $R$ (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44, \quad 8 \le t \le 15$$

where $t$ represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?  *(Source: McDonald's Corporation)* ∎

---

#### CONCEPT CHECK

1. What is the derivative of any constant function?
2. Write a verbal description of the Power Rule.
3. According to the Sum Rule, the derivative of the sum of two differentiable functions is equal to what?
4. According to the Difference Rule, the derivative of the difference of two differentiable functions is equal to what?

| **Skills Review 2.2** | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 0.4. |
| --- | --- |

In Exercises 1 and 2, evaluate each expression when $x = 2$.

**1.** (a) $2x^2$   (b) $(2x)^2$   (c) $2x^{-2}$

**2.** (a) $\dfrac{1}{(3x)^2}$   (b) $\dfrac{1}{4x^3}$   (c) $\dfrac{(2x)^{-3}}{4x^{-2}}$

In Exercises 3–6, simplify the expression.

**3.** $4(3)x^3 + 2(2)x$

**4.** $\frac{1}{2}(3)x^2 - \frac{3}{2}x^{1/2}$

**5.** $\left(\frac{1}{4}\right)x^{-3/4}$

**6.** $\frac{1}{3}(3)x^2 - 2\left(\frac{1}{2}\right)x^{-1/2} + \frac{1}{3}x^{-2/3}$

In Exercises 7–10, solve the equation.

**7.** $3x^2 + 2x = 0$

**8.** $x^3 - x = 0$

**9.** $x^2 + 8x - 20 = 0$
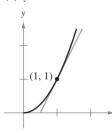
**10.** $x^2 - 10x - 24 = 0$

## Exercises 2.2
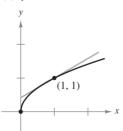
In Exercises 1–4, find the slope of the tangent line to $y = x^n$ at the point $(1, 1)$.
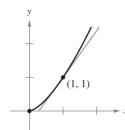
**1.** (a) $y = x^2$
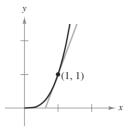
(b) $y = x^{1/2}$
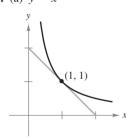
**2.** (a) $y = x^{3/2}$
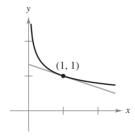
(b) $y = x^3$
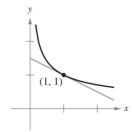
**3.** (a) $y = x^{-1}$

(b) $y = x^{-1/3}$

**4.** (a) $y = x^{-1/2}$

(b) $y = x^{-2}$
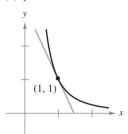
In Exercises 5–22, find the derivative of the function.

**5.** $y = 3$

**6.** $f(x) = -2$

**7.** $y = x^4$

**8.** $h(x) = 2x^5$

**9.** $f(x) = 4x + 1$

**10.** $g(x) = 3x - 1$

**11.** $g(x) = x^2 + 5x$

**12.** $y = t^2 - 6$

**13.** $f(t) = -3t^2 + 2t - 4$

**14.** $y = x^3 - 9x^2 + 2$

**15.** $s(t) = t^3 - 2t + 4$

**16.** $y = 2x^3 - x^2 + 3x - 1$

**17.** $y = 4t^{4/3}$

**18.** $h(x) = x^{5/2}$

**19.** $f(x) = 4\sqrt{x}$

**20.** $g(x) = 4\sqrt[3]{x} + 2$

**21.** $y = 4x^{-2} + 2x^2$

**22.** $s(t) = 4t^{-1} + 1$

In Exercises 23–28, use Example 6 as a model to find the derivative.

| Function | Rewrite | Differentiate | Simplify |
|---|---|---|---|
| **23.** $y = \dfrac{1}{x^3}$ | ▮ | ▮ | ▮ |
| **24.** $y = \dfrac{2}{3x^2}$ | ▮ | ▮ | ▮ |
| **25.** $y = \dfrac{1}{(4x)^3}$ | ▮ | ▮ | ▮ |
| **26.** $y = \dfrac{\pi}{(3x)^2}$ | ▮ | ▮ | ▮ |
| **27.** $y = \dfrac{\sqrt{x}}{x}$ | ▮ | ▮ | ▮ |
| **28.** $y = \dfrac{4x}{x^{-3}}$ | ▮ | ▮ | ▮ |

In Exercises 29–34, find the value of the derivative of the function at the given point.

| Function | Point |
|---|---|
| **29.** $f(x) = \dfrac{1}{x}$ | $(1, 1)$ |
| **30.** $f(t) = 4 - \dfrac{4}{3t}$ | $\left(\dfrac{1}{2}, \dfrac{4}{3}\right)$ |
| **31.** $f(x) = -\frac{1}{2}x(1 + x^2)$ | $(1, -1)$ |
| **32.** $y = 3x\left(x^2 - \dfrac{2}{x}\right)$ | $(2, 18)$ |
| **33.** $y = (2x + 1)^2$ | $(0, 1)$ |
| **34.** $f(x) = 3(5 - x)^2$ | $(5, 0)$ |

In Exercises 35–48, find $f'(x)$.

**35.** $f(x) = x^2 - \dfrac{4}{x} - 3x^{-2}$

**36.** $f(x) = x^2 - 3x - 3x^{-2} + 5x^{-3}$

**37.** $f(x) = x^2 - 2x - \dfrac{2}{x^4}$     **38.** $f(x) = x^2 + 4x + \dfrac{1}{x}$

**39.** $f(x) = x(x^2 + 1)$     **40.** $f(x) = (x^2 + 2x)(x + 1)$

**41.** $f(x) = (x + 4)(2x^2 - 1)$

**42.** $f(x) = (3x^2 - 5x)(x^2 + 2)$

**43.** $f(x) = \dfrac{2x^3 - 4x^2 + 3}{x^2}$     **44.** $f(x) = \dfrac{2x^2 - 3x + 1}{x}$

**45.** $f(x) = \dfrac{4x^3 - 3x^2 + 2x + 5}{x^2}$

**46.** $f(x) = \dfrac{-6x^3 + 3x^2 - 2x + 1}{x}$

**47.** $f(x) = x^{4/5} + x$     **48.** $f(x) = x^{1/3} - 1$

Ⓣ In Exercises 49–52, (a) find an equation of the tangent line to the graph of the function at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

| Function | Point |
|---|---|
| **49.** $y = -2x^4 + 5x^2 - 3$ | $(1, 0)$ |
| **50.** $y = x^3 + x$ | $(-1, -2)$ |
| **51.** $f(x) = \sqrt[3]{x} + \sqrt[5]{x}$ | $(1, 2)$ |
| **52.** $f(x) = \dfrac{1}{\sqrt[3]{x^2}} - x$ | $(-1, 2)$ |

In Exercises 53–56, determine the point(s), if any, at which the graph of the function has a horizontal tangent line.

**53.** $y = -x^4 + 3x^2 - 1$     **54.** $y = x^3 + 3x^2$

**55.** $y = \frac{1}{2}x^2 + 5x$     **56.** $y = x^2 + 2x$

In Exercises 57 and 58, (a) sketch the graphs of $f$ and $g$, (b) find $f'(1)$ and $g'(1)$, (c) sketch the tangent line to each graph when $x = 1$, and (d) explain the relationship between $f'$ and $g'$.

**57.** $f(x) = x^3$     **58.** $f(x) = x^2$
   $g(x) = x^3 + 3$     $g(x) = 3x^2$
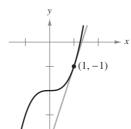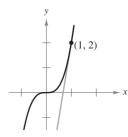
**59.** Use the Constant Rule, the Constant Multiple Rule, and the Sum Rule to find $h'(1)$ given that $f'(1) = 3$.
   (a) $h(x) = f(x) - 2$     (b) $h(x) = 2f(x)$
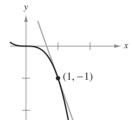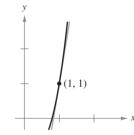


(c) $h(x) = -f(x)$     (d) $h(x) = -1 + 2f(x)$
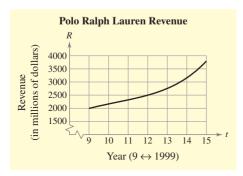
**60. Revenue**   The revenue $R$ (in millions of dollars per year) for Polo Ralph Lauren from 1999 through 2005 can be modeled by

$$R = 0.59221t^4 - 18.0042t^3 + 175.293t^2 - 316.42t$$
$$- 116.5$$
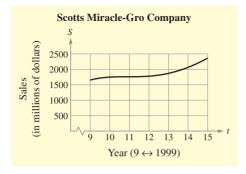
where $t$ is the year, with $t = 9$ corresponding to 1999. *(Source: Polo Ralph Lauren Corp.)*

**Polo Ralph Lauren Revenue**

Revenue (in millions of dollars) vs. Year (9 ↔ 1999)

(a) Find the slopes of the graph for the years 2002 and 2004.

(b) Compare your results with those obtained in Exercise 11 in Section 2.1.

(c) What are the units for the slope of the graph? Interpret the slope of the graph in the context of the problem.

**61. Sales**   The sales $S$ (in millions of dollars per year) for Scotts Miracle-Gro Company from 1999 through 2005 can be modeled by
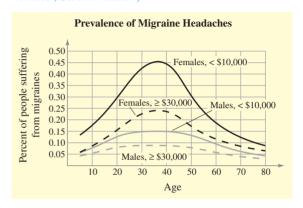
$$S = -1.29242t^4 + 69.9530t^3 - 1364.615t^2$$
$$+ 11,511.47t - 33,932.9$$

where $t$ is the year, with $t = 9$ corresponding to 1999. *(Source: Scotts Miracle-Gro Company)*

**Scotts Miracle-Gro Company**

Sales (in millions of dollars) vs. Year (9 ↔ 1999)

(a) Find the slopes of the graph for the years 2001 and 2004.

(b) Compare your results with those obtained in Exercise 12 in Section 2.1.

(c) What are the units for the slope of the graph? Interpret the slope of the graph in the context of the problem.

**62. Cost**   The variable cost for manufacturing an electrical component is \$7.75 per unit, and the fixed cost is \$500. Write the cost $C$ as a function of $x$, the number of units produced. Show that the derivative of this cost function is a constant and is equal to the variable cost.

**63. Political Fundraiser**   A politician raises funds by selling tickets to a dinner for \$500. The politician pays \$150 for each dinner and has fixed costs of \$7000 to rent a dining hall and wait staff. Write the profit $P$ as a function of $x$, the number of dinners sold. Show that the derivative of the profit function is a constant and is equal to the increase in profit from each dinner sold.

Ⓑ **64. Psychology: Migraine Prevalence**   The graph illustrates the prevalence of migraine headaches in males and females in selected income groups. *(Source: Adapted from Sue/Sue/Sue, Understanding Abnormal Behavior, Seventh Edition)*

**Prevalence of Migraine Headaches**

Percent of people suffering from migraines vs. Age

(a) Write a short paragraph describing your general observations about the prevalence of migraines in females and males with respect to age group and income bracket.

(b) Describe the graph of the derivative of each curve, and explain the significance of each derivative. Include an explanation of the units of the derivatives, and indicate the time intervals in which the derivatives would be positive and negative.

Ⓣ In Exercises 65 and 66, use a graphing utility to graph $f$ and $f'$ over the given interval. Determine any points at which the graph of $f$ has horizontal tangents.

| Function | Interval |
|---|---|
| **65.** $f(x) = 4.1x^3 - 12x^2 + 2.5x$ | $[0, 3]$ |
| **66.** $f(x) = x^3 - 1.4x^2 - 0.96x + 1.44$ | $[-2, 2]$ |

**True or False?**   In Exercises 67 and 68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**67.** If $f'(x) = g'(x)$, then $f(x) = g(x)$.

**68.** If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.

# Rates of Change: Velocity and Marginals

- Find the average rates of change of functions over intervals.
- Find the instantaneous rates of change of functions at points.
- Find the marginal revenues, marginal costs, and marginal profits for products.

## Average Rate of Change

In Sections 2.1 and 2.2, you studied the two primary applications of derivatives.

1. **Slope**  The derivative of $f$ is a function that gives the slope of the graph of $f$ at a point $(x, f(x))$.

2. **Rate of Change**  The derivative of $f$ is a function that gives the rate of change of $f(x)$ with respect to $x$ at the point $(x, f(x))$.

In this section, you will see that there are many real-life applications of rates of change. A few are velocity, acceleration, population growth rates, unemployment rates, production rates, and water flow rates. Although rates of change often involve change with respect to time, you can investigate the rate of change of one variable with respect to any other related variable.

When determining the rate of change of one variable with respect to another, you must be careful to distinguish between *average* and *instantaneous* rates of change. The distinction between these two rates of change is comparable to the distinction between the slope of the secant line through two points on a graph and the slope of the tangent line at one point on the graph.
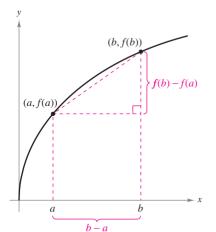


**FIGURE 2.18**

**Definition of Average Rate of Change**

If $y = f(x)$, then the **average rate of change** of $y$ with respect to $x$ on the interval $[a, b]$ is

$$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\Delta y}{\Delta x}.$$

Note that $f(a)$ is the value of the function at the *left* endpoint of the interval, $f(b)$ is the value of the function at the *right* endpoint of the interval, and $b - a$ is the width of the interval, as shown in Figure 2.18.

**STUDY TIP**

In real-life problems, it is important to list the units of measure for a rate of change. The units for $\Delta y/\Delta x$ are "$y$-units" per "$x$-units." For example, if $y$ is measured in miles and $x$ is measured in hours, then $\Delta y/\Delta x$ is measured in *miles per hour*.

**Example 1**  Medicine  (R)

The concentration $C$ (in milligrams per milliliter) of a drug in a patient's blood-stream is monitored over 10-minute intervals for 2 hours, where $t$ is measured in minutes, as shown in the table. Find the average rate of change over each interval.

**a.** $[0, 10]$    **b.** $[0, 20]$    **c.** $[100, 110]$

| $t$ | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C$ | 0 | 2 | 17 | 37 | 55 | 73 | 89 | 103 | 111 | 113 | 113 | 103 | 68 |

**SOLUTION**

**a.** For the interval $[0, 10]$, the average rate of change is

Value of $C$ at right endpoint

Value of $C$ at left endpoint

$$\frac{\Delta C}{\Delta t} = \frac{2 - 0}{10 - 0} = \frac{2}{10} = 0.2 \text{ milligram per milliliter per minute.}$$

Width of interval

**b.** For the interval $[0, 20]$, the average rate of change is

$$\frac{\Delta C}{\Delta t} = \frac{17 - 0}{20 - 0} = \frac{17}{20} = 0.85 \text{ milligram per milliliter per minute.}$$
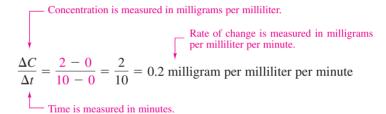
**c.** For the interval $[100, 110]$, the average rate of change is

$$\frac{\Delta C}{\Delta t} = \frac{103 - 113}{110 - 100} = \frac{-10}{10} = -1 \text{ milligram per milliliter per minute.}$$

✓**CHECKPOINT 1**

Use the table in Example 1 to find the average rate of change over each interval.

**a.** $[0, 120]$    **b.** $[90, 100]$    **c.** $[90, 120]$ ■

The rates of change in Example 1 are in milligrams per milliliter per minute because the concentration is measured in milligrams per milliliter and the time is measured in minutes.

Concentration is measured in milligrams per milliliter.

Rate of change is measured in milligrams per milliliter per minute.

$$\frac{\Delta C}{\Delta t} = \frac{2 - 0}{10 - 0} = \frac{2}{10} = 0.2 \text{ milligram per milliliter per minute}$$

Time is measured in minutes.

**STUDY TIP**

In Example 1, the average rate of change is positive when the concentration increases and negative when the concentration decreases, as shown in Figure 2.19.



**Drug Concentration in Bloodstream**

**FIGURE 2.19**

A common application of an average rate of change is to find the **average velocity** of an object that is moving in a straight line. That is,

$$\text{Average velocity} = \frac{\text{change in distance}}{\text{change in time}}.$$

This formula is demonstrated in Example 2.

**Example 2** **Finding an Average Velocity** Ⓡ

If a free-falling object is dropped from a height of 100 feet, and *air resistance is neglected*, the height $h$ (in feet) of the object at time $t$ (in seconds) is given by

$$h = -16t^2 + 100. \qquad \text{(See Figure 2.20.)}$$

Find the average velocity of the object over each interval.

**a.** $[1, 2]$ **b.** $[1, 1.5]$ **c.** $[1, 1.1]$

**SOLUTION** You can use the position equation $h = -16t^2 + 100$ to determine the heights at $t = 1$, $t = 1.1$, $t = 1.5$, and $t = 2$, as shown in the table.

| $t$ (in seconds) | 0 | 1 | 1.1 | 1.5 | 2 |
|---|---|---|---|---|---|
| $h$ (in feet) | 100 | 84 | 80.64 | 64 | 36 |

**a.** For the interval $[1, 2]$, the object falls from a height of 84 feet to a height of 36 feet. So, the average velocity is

$$\frac{\Delta h}{\Delta t} = \frac{36 - 84}{2 - 1} = \frac{-48}{1} = -48 \text{ feet per second.}$$

**b.** For the interval $[1, 1.5]$, the average velocity is

$$\frac{\Delta h}{\Delta t} = \frac{64 - 84}{1.5 - 1} = \frac{-20}{0.5} = -40 \text{ feet per second.}$$

**c.** For the interval $[1, 1.1]$, the average velocity is

$$\frac{\Delta h}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6 \text{ feet per second.}$$



**FIGURE 2.20** Some falling objects have considerable air resistance. Other falling objects have negligible air resistance. When modeling a falling-body problem, you must decide whether to account for air resistance or neglect it.

✓**CHECKPOINT 2**

The height $h$ (in feet) of a free-falling object at time $t$ (in seconds) is given by $h = -16t^2 + 180$. Find the average velocity of the object over each interval.

**a.** $[0, 1]$ **b.** $[1, 2]$ **c.** $[2, 3]$ ■

**STUDY TIP**

In Example 2, the average velocities are negative because the object is moving downward.

## Instantaneous Rate of Change and Velocity

Suppose in Example 2 you wanted to find the rate of change of $h$ at the instant $t = 1$ second. Such a rate is called an **instantaneous rate of change.** You can approximate the instantaneous rate of change at $t = 1$ by calculating the average rate of change over smaller and smaller intervals of the form $[1, 1 + \Delta t]$, as shown in the table. From the table, it seems reasonable to conclude that the instantaneous rate of change of the height when $t = 1$ is $-32$ feet per second.

<center>$\Delta t$ approaches 0.</center>

| $\Delta t$ | 1 | 0.5 | 0.1 | 0.01 | 0.001 | 0.0001 | 0 |
|---|---|---|---|---|---|---|---|
| $\dfrac{\Delta h}{\Delta t}$ | $-48$ | $-40$ | $-33.6$ | $-32.16$ | $-32.016$ | $-32.0016$ | $-32$ |

<center>$\dfrac{\Delta h}{\Delta t}$ approaches $-32$.</center>

**Definition of Instantaneous Rate of Change**

The **instantaneous rate of change** (or simply **rate of change**) of $y = f(x)$ at $x$ is the limit of the average rate of change on the interval $[x, x + \Delta x]$, as $\Delta x$ approaches 0.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If $y$ is a distance and $x$ is time, then the rate of change is a **velocity.**

*Example 3*   **Finding an Instantaneous Rate of Change**    Ⓡ

Find the velocity of the object in Example 2 when $t = 1$.

**SOLUTION**    From Example 2, you know that the height of the falling object is given by

$h = -16t^2 + 100.$        Position function

By taking the derivative of this position function, you obtain the velocity function.

$h'(t) = -32t$        Velocity function

The velocity function gives the velocity at *any* time. So, when $t = 1$, the velocity is

$h'(1) = -32(1)$
$\quad\quad = -32$ feet per second.

✓**CHECKPOINT 3**

Find the velocities of the object in Checkpoint 2 when $t = 1.75$ and $t = 2$. ■
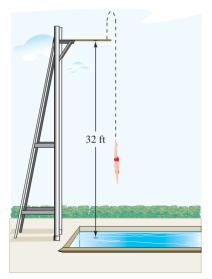
The general **position function** for a free-falling object, neglecting air resistance, is

$$h = -16t^2 + v_0t + h_0 \qquad \text{Position function}$$

where $h$ is the height (in feet), $t$ is the time (in seconds), $v_0$ is the initial velocity (in feet per second), and $h_0$ is the initial height (in feet). Remember that the model assumes that positive velocities indicate upward motion and negative velocities indicate downward motion. The derivative $h' = -32t + v_0$ is the **velocity function.** The absolute value of the velocity is the **speed** of the object.

### *Example 4*  Finding the Velocity of a Diver  ®

At time $t = 0$, a diver jumps from a diving board that is 32 feet high, as shown in Figure 2.21. Because the diver's initial velocity is 16 feet per second, his position function is

$$h = -16t^2 + 16t + 32. \qquad \text{Position function}$$

**a.** When does the diver hit the water?

**b.** What is the diver's velocity at impact?

**SOLUTION**

**a.** To find the time at which the diver hits the water, let $h = 0$ and solve for $t$.

$$-16t^2 + 16t + 32 = 0 \qquad \text{Set } h \text{ equal to 0.}$$
$$-16(t^2 - t - 2) = 0 \qquad \text{Factor out common factor.}$$
$$-16(t + 1)(t - 2) = 0 \qquad \text{Factor.}$$
$$t = -1 \text{ or } t = 2 \qquad \text{Solve for } t.$$

The solution $t = -1$ does not make sense in the problem because it would mean the diver hits the water 1 second before he jumps. So, you can conclude that the diver hits the water when $t = 2$ seconds.

**b.** The velocity at time $t$ is given by the derivative

$$h' = -32t + 16. \qquad \text{Velocity function}$$

The velocity at time $t = 2$ is $-32(2) + 16 = -48$ feet per second.



**FIGURE 2.21**

**✓ CHECKPOINT 4**

Give the position function of a diver who jumps from a board 12 feet high with initial velocity 16 feet per second. Then find the diver's velocity function.

■

In Example 4, note that the diver's initial velocity is $v_0 = 16$ feet per second (upward) and his initial height is $h_0 = 32$ feet.

Initial velocity is 16 feet per second.

Initial height is 32 feet.

$$h = -16t^2 + 16t + 32$$

## Rates of Change in Economics:  Marginals

Another important use of rates of change is in the field of economics. Economists refer to *marginal profit*, *marginal revenue*, and *marginal cost* as the rates of change of the profit, revenue, and cost with respect to the number $x$ of units produced or sold. An equation that relates these three quantities is

$$P = R - C$$

where $P$, $R$, and $C$ represent the following quantities.

$P =$ total profit

$R =$ total revenue

and

$C =$ total cost
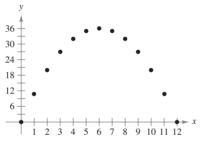
The derivatives of these quantities are called the **marginal profit, marginal revenue,** and **marginal cost,** respectively.

$$\frac{dP}{dx} = \text{marginal profit}$$
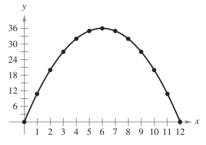
$$\frac{dR}{dx} = \text{marginal revenue}$$

$$\frac{dC}{dx} = \text{marginal cost}$$

In many business and economics problems, the number of units produced or sold is restricted to positive integer values, as indicated in Figure 2.22(a). (Of course, it could happen that a sale involves half or quarter units, but it is hard to conceive of a sale involving $\sqrt{2}$ units.) The variable that denotes such units is called a **discrete variable.** To analyze a function of a discrete variable $x$, you can temporarily assume that $x$ is a **continuous variable** and is able to take on any real value in a given interval, as indicated in Figure 2.22(b). Then, you can use the methods of calculus to find the $x$-value that corresponds to the marginal revenue, maximum profit, minimum cost, or whatever is called for. Finally, you should round the solution to the nearest sensible $x$-value—cents, dollars, units, or days, depending on the context of the problem.

(a)  Function of a Discrete Variable

(b)  Function of a Continuous Variable

**FIGURE 2.22**

*Example 5*  **Finding the Marginal Profit**  Ⓡ

The profit derived from selling $x$ units of an alarm clock is given by

$$P = 0.0002x^3 + 10x.$$

**a.** Find the marginal profit for a production level of 50 units.

**b.** Compare this with the actual gain in profit obtained by increasing the production level from 50 to 51 units.

**SOLUTION**

**a.** Because the profit is $P = 0.0002x^3 + 10x$, the marginal profit is given by the derivative

$$dP/dx = 0.0006x^2 + 10.$$

When $x = 50$, the marginal profit is

$$0.0006(50)^2 + 10 = 1.5 + 10$$
$$= \$11.50 \text{ per unit.} \qquad \textcolor{magenta}{\text{Marginal profit for } x = 50}$$

**b.** For $x = 50$, the actual profit is

$$P = (0.0002)(50)^3 + 10(50) \qquad \textcolor{magenta}{\text{Substitute 50 for } x.}$$
$$= 25 + 500$$
$$= \$525.00 \qquad \textcolor{magenta}{\text{Actual profit for } x = 50}$$

and for $x = 51$, the actual profit is

$$P = (0.0002)(51)^3 + 10(51) \qquad \textcolor{magenta}{\text{Substitute 51 for } x.}$$
$$\approx 26.53 + 510$$
$$= \$536.53. \qquad \textcolor{magenta}{\text{Actual profit for } x = 51}$$

So, the additional profit obtained by increasing the production level from 50 to 51 units is

$$536.53 - 525.00 = \$11.53. \qquad \textcolor{magenta}{\text{Extra profit for one unit}}$$

Note that the actual profit increase of $11.53 (when $x$ increases from 50 to 51 units) can be approximated by the marginal profit of $11.50 per unit (when $x = 50$), as shown in Figure 2.23.



**Marginal Profit**
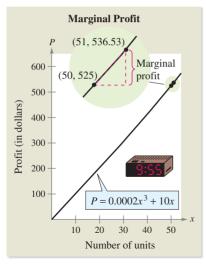
**FIGURE 2.23**

✓**CHECKPOINT 5**

Use the profit function in Example 5 to find the marginal profit for a production level of 100 units. Compare this with the actual gain in profit by increasing production from 100 to 101 units. ■

---

**STUDY TIP**

The reason the marginal profit gives a good approximation of the actual change in profit is that the graph of $P$ is nearly straight over the interval $50 \le x \le 51$. You will study more about the use of marginals to approximate actual changes in Section 3.8.

The profit function in Example 5 is unusual in that the profit continues to increase as long as the number of units sold increases. In practice, it is more common to encounter situations in which sales can be increased only by lowering the price per item. Such reductions in price will ultimately cause the profit to decline.

The number of units $x$ that consumers are willing to purchase at a given price per unit $p$ is given by the **demand function**

$$p = f(x).$$    Demand function

The total revenue $R$ is then related to the price per unit and the quantity demanded (or sold) by the equation

$$R = xp.$$    Revenue function

---

**Example 6**  **Finding a Demand Function**  Ⓡ

A business sells 2000 items per month at a price of \$10 each. It is estimated that monthly sales will increase 250 units for each \$0.25 reduction in price. Use this information to find the demand function and total revenue function.

**SOLUTION**  From the given estimate, $x$ increases 250 units each time $p$ drops \$0.25 from the original cost of \$10. This is described by the equation

$$x = 2000 + 250\left(\frac{10 - p}{0.25}\right)$$

$$= 2000 + 10{,}000 - 1000p$$

$$= 12{,}000 - 1000p.$$

Solving for $p$ in terms of $x$ produces

$$p = 12 - \frac{x}{1000}.$$    Demand function

This, in turn, implies that the revenue function is

$$R = xp$$    Formula for revenue

$$= x\left(12 - \frac{x}{1000}\right)$$

$$= 12x - \frac{x^2}{1000}.$$    Revenue function

The graph of the demand function is shown in Figure 2.24. Notice that as the price decreases, the quantity demanded increases.



**Demand Function**

Price (in dollars)

REGULAR \$10.00
REDUCED
\$8.75

$p = 12 - \dfrac{x}{1000}$

Number of units
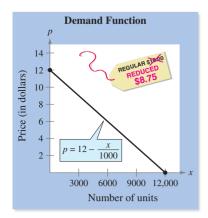
**FIGURE 2.24**

---

✓ **CHECKPOINT 6**
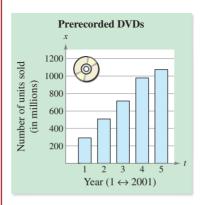
Find the demand function in Example 6 if monthly sales increase 200 units for each \$0.10 reduction in price. ■

## TECHNOLOGY

### Modeling a Demand Function

Ⓣ To model a demand function, you need data that indicate how many units of a product will sell at a given price. As you might imagine, such data are not easy to obtain for a new product. After a product has been on the market awhile, however, its sales history can provide the necessary data.

As an example, consider the two bar graphs shown below. From these graphs, you can see that from 2001 through 2005, the number of prerecorded DVDs sold increased from about 300 million to about 1100 million. During that time, the price per unit dropped from an average price of about $18 to an average price of about $15.   *(Source: Kagan Research, LLC)*



The information in the two bar graphs is combined in the table, where $x$ represents the units sold (in millions) and $p$ represents the price (in dollars).

| $t$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $x$ | 291.5 | 507.5 | 713.0 | 976.6 | 1072.4 |
| $p$ | 18.40 | 17.11 | 15.83 | 15.51 | 14.94 |

By entering the ordered pairs $(x, p)$ into a graphing utility, you can find that the power model for the demand for prerecorded DVDs is:
$p = 44.55x^{-0.155}$, $291.5 \leq x \leq 1072.4$. A graph of this demand function and its data points is shown below

### Example 7   Finding the Marginal Revenue ®

A fast-food restaurant has determined that the monthly demand for its hamburgers is given by

$$p = \frac{60,000 - x}{20,000}.$$

Figure 2.25 shows that as the price decreases, the quantity demanded increases. The table shows the demands for hamburgers at various prices.

| $x$ | 60,000 | 50,000 | 40,000 | 30,000 | 20,000 | 10,000 | 0 |
|---|---|---|---|---|---|---|---|
| $p$ | $0.00 | $0.50 | $1.00 | $1.50 | $2.00 | $2.50 | $3.00 |

**Demand Function**
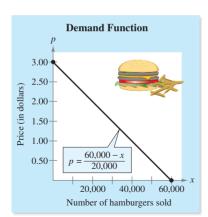


**FIGURE 2.25**   As the price decreases, more hamburgers are sold.

Find the increase in revenue per hamburger for monthly sales of 20,000 hamburgers. In other words, find the marginal revenue when $x = 20,000$.

**SOLUTION**   Because the demand is given by

$$p = \frac{60,000 - x}{20,000}$$

and the revenue is given by $R = xp$, you have

$$R = xp = x\left(\frac{60,000 - x}{20,000}\right)$$

$$= \frac{1}{20,000}(60,000x - x^2).$$

By differentiating, you can find the marginal revenue to be

$$\frac{dR}{dx} = \frac{1}{20,000}(60,000 - 2x).$$

So, when $x = 20,000$, the marginal revenue is

$$\frac{1}{20,000}[60,000 - 2(20,000)] = \frac{20,000}{20,000} = \$1 \text{ per unit.}$$

✓**CHECKPOINT 7**

Find the revenue function and marginal revenue for a demand function of $p = 2000 - 4x$.  ∎

---

**STUDY TIP**

Writing a demand function in the form $p = f(x)$ is a convention used in economics. From a consumer's point of view, it might seem more reasonable to think that the quantity demanded is a function of the price. Mathematically, however, the two points of view are equivalent because a typical demand function is one-to-one and so has an inverse function. For instance, in Example 7, you could write the demand function as $x = 60,000 - 20,000p$.

**Example 8** **Finding the Marginal Profit** Ⓡ

Suppose in Example 7 that the cost of producing $x$ hamburgers is

$$C = 5000 + 0.56x, \quad 0 \le x \le 50{,}000.$$

Find the profit and the marginal profit for each production level.

**a.** $x = 20{,}000$ **b.** $x = 24{,}400$ **c.** $x = 30{,}000$

**SOLUTION** From Example 7, you know that the total revenue from selling $x$ hamburgers is

$$R = \frac{1}{20{,}000}(60{,}000x - x^2).$$

Because the total profit is given by $P = R - C$, you have

$$P = \frac{1}{20{,}000}(60{,}000x - x^2) - (5000 + 0.56x)$$

$$= 3x - \frac{x^2}{20{,}000} - 5000 - 0.56x$$

$$= 2.44x - \frac{x^2}{20{,}000} - 5000. \qquad \text{See Figure 2.26.}$$

So, the marginal profit is

$$\frac{dP}{dx} = 2.44 - \frac{x}{10{,}000}.$$

Using these formulas, you can compute the profit and marginal profit.

| Production | Profit | Marginal Profit |
|---|---|---|
| **a.** $x = 20{,}000$ | $P = \$23{,}800.00$ | $2.44 - \dfrac{20{,}000}{10{,}000} = \$0.44$ per unit |
| **b.** $x = 24{,}400$ | $P = \$24{,}768.00$ | $2.44 - \dfrac{24{,}400}{10{,}000} = \$0.00$ per unit |
| **c.** $x = 30{,}000$ | $P = \$23{,}200.00$ | $2.44 - \dfrac{30{,}000}{10{,}000} = -\$0.56$ per unit |



**Profit Function**

$$P = 2.44x - \frac{x^2}{20{,}000} - 5000$$
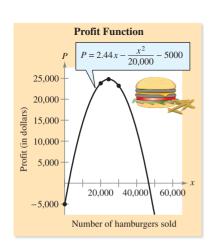
**FIGURE 2.26** Demand Curve

✓**CHECKPOINT 8**

From Example 8, compare the marginal profit when 10,000 units are produced with the actual increase in profit from 10,000 units to 10,001 units. ∎

**CONCEPT CHECK**

1. You are asked to find the rate of change of a function over a certain interval. Should you find the average rate of change or the instantaneous rate of change?

2. You are asked to find the rate of change of a function at a certain instant. Should you find the average rate of change or the instantaneous rate of change?

3. If a variable can take on any real value in a given interval, is the variable discrete or continuous?

4. What does a demand function represent?

**Skills Review 2.3**    The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.1 and 2.2.

In Exercises 1 and 2, evaluate the expression.

**1.** $\dfrac{-63 - (-105)}{21 - 7}$

**2.** $\dfrac{-37 - 54}{16 - 3}$

In Exercises 3–10, find the derivative of the function.

**3.** $y = 4x^2 - 2x + 7$

**4.** $y = -3t^3 + 2t^2 - 8$

**5.** $s = -16t^2 + 24t + 30$

**6.** $y = -16x^2 + 54x + 70$

**7.** $A = \frac{1}{10}(-2r^3 + 3r^2 + 5r)$

**8.** $y = \frac{1}{9}(6x^3 - 18x^2 + 63x - 15)$

**9.** $y = 12x - \dfrac{x^2}{5000}$

**10.** $y = 138 + 74x - \dfrac{x^3}{10,000}$

## Exercises 2.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**1. Research and Development**  The table shows the amounts $A$ (in billions of dollars per year) spent on R&D in the United States from 1980 through 2004, where $t$ is the year, with $t = 0$ corresponding to 1980. Approximate the average rate of change of $A$ during each period. *(Source: U.S. National Science Foundation)*

(a) 1980–1985    (b) 1985–1990    (c) 1990–1995

(d) 1995–2000    (e) 1980–2004    (f) 1990–2004

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|----|----|----|----|-----|-----|-----|
| $A$ | 63 | 72 | 81 | 90 | 102 | 115 | 120 |

| $t$ | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|-----|-----|-----|-----|-----|-----|
| $A$ | 126 | 134 | 142 | 152 | 161 | 165 |

| $t$ | 13 | 14 | 15 | 16 | 17 | 18 |
|-----|-----|-----|-----|-----|-----|-----|
| $A$ | 166 | 169 | 184 | 197 | 212 | 228 |

| $t$ | 19 | 20 | 21 | 22 | 23 | 24 |
|-----|-----|-----|-----|-----|-----|-----|
| $A$ | 245 | 267 | 277 | 276 | 292 | 312 |

**2. Trade Deficit**  The graph shows the values $I$ (in billions of dollars per year) of goods imported to the United States and the values $E$ (in billions of dollars per year) of goods exported from the United States from 1980 through 2005. Approximate each indicated average rate of change. *(Source: U.S. International Trade Administration)*

(a) Imports: 1980–1990    (b) Exports: 1980–1990

(c) Imports: 1990–2000    (d) Exports: 1990–2000
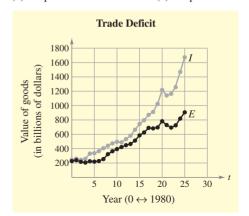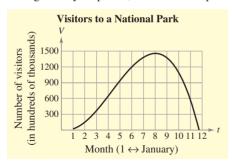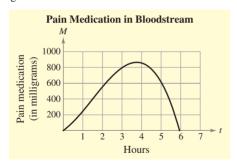
(e) Imports: 1980–2005    (f) Exports: 1980–2005



Figure for 2

(T) In Exercises 3–12, use a graphing utility to graph the function and find its average rate of change on the interval. Compare this rate with the instantaneous rates of change at the endpoints of the interval.

**3.** $f(t) = 3t + 5; [1, 2]$

**4.** $h(x) = 2 - x; [0, 2]$

**5.** $h(x) = x^2 - 4x + 2; [-2, 2]$

**6.** $f(x) = x^2 - 6x - 1; [-1, 3]$

**7.** $f(x) = 3x^{4/3}; [1, 8]$

**8.** $f(x) = x^{3/2}; [1, 4]$

**9.** $f(x) = \dfrac{1}{x}; [1, 4]$

**10.** $f(x) = \dfrac{1}{\sqrt{x}}; [1, 4]$

**11.** $g(x) = x^4 - x^2 + 2; [1, 3]$

**12.** $g(x) = x^3 - 1; [-1, 1]$

**13. Consumer Trends** The graph shows the number of visitors $V$ to a national park in hundreds of thousands during a one-year period, where $t = 1$ represents January.

**Visitors to a National Park**



(a) Estimate the rate of change of $V$ over the interval $[9, 12]$ and explain your results.

(b) Over what interval is the average rate of change approximately equal to the rate of change at $t = 8$? Explain your reasoning.

**14. Medicine** The graph shows the estimated number of milligrams of a pain medication $M$ in the bloodstream $t$ hours after a 1000-milligram dose of the drug has been given.

**Pain Medication in Bloodstream**



(a) Estimate the one-hour interval over which the average rate of change is the greatest.

(b) Over what interval is the average rate of change approximately equal to the rate of change at $t = 4$? Explain your reasoning.

**15. Medicine** The effectiveness $E$ (on a scale from 0 to 1) of a pain-killing drug $t$ hours after entering the bloodstream is given by

$$E = \frac{1}{27}(9t + 3t^2 - t^3), \quad 0 \le t \le 4.5.$$

Find the average rate of change of $E$ on each indicated interval and compare this rate with the instantaneous rates of change at the endpoints of the interval.

(a) $[0, 1]$     (b) $[1, 2]$     (c) $[2, 3]$     (d) $[3, 4]$

**16. Chemistry: Wind Chill** At 0° Celsius, the heat loss $H$ (in kilocalories per square meter per hour) from a person's body can be modeled by

$$H = 33(10\sqrt{v} - v + 10.45)$$

where $v$ is the wind speed (in meters per second).

(a) Find $\dfrac{dH}{dv}$ and interpret its meaning in this situation.

(b) Find the rates of change of $H$ when $v = 2$ and when $v = 5$.

**17. Velocity** The height $s$ (in feet) at time $t$ (in seconds) of a silver dollar dropped from the top of the Washington Monument is given by

$$s = -16t^2 + 555.$$

(a) Find the average velocity on the interval $[2, 3]$.

(b) Find the instantaneous velocities when $t = 2$ and when $t = 3$.

(c) How long will it take the dollar to hit the ground?

(d) Find the velocity of the dollar when it hits the ground.

Ⓑ **18. Physics: Velocity** A racecar travels northward on a straight, level track at a constant speed, traveling 0.750 kilometer in 20.0 seconds. The return trip over the same track is made in 25.0 seconds.

(a) What is the average velocity of the car in meters per second for the first leg of the run?

(b) What is the average velocity for the total trip?

*(Source: Shipman/Wilson/Todd, An Introduction to Physical Science, Eleventh Edition)*

**Marginal Cost** In Exercises 19–22, find the marginal cost for producing $x$ units. (The cost is measured in dollars.)

**19.** $C = 4500 + 1.47x$         **20.** $C = 205,000 + 9800x$

**21.** $C = 55,000 + 470x - 0.25x^2, \quad 0 \le x \le 940$

**22.** $C = 100(9 + 3\sqrt{x})$

**Marginal Revenue** In Exercises 23–26, find the marginal revenue for producing $x$ units. (The revenue is measured in dollars.)

**23.** $R = 50x - 0.5x^2$         **24.** $R = 30x - x^2$

**25.** $R = -6x^3 + 8x^2 + 200x$    **26.** $R = 50(20x - x^{3/2})$

**Marginal Profit** In Exercises 27–30, find the marginal profit for producing $x$ units. (The profit is measured in dollars.)

**27.** $P = -2x^2 + 72x - 145$

**28.** $P = -0.25x^2 + 2000x - 1,250,000$

**29.** $P = -0.00025x^2 + 12.2x - 25,000$

**30.** $P = -0.5x^3 + 30x^2 - 164.25x - 1000$

**31. Marginal Cost**   The cost $C$ (in dollars) of producing $x$ units of a product is given by

$C = 3.6\sqrt{x} + 500.$

(a) Find the additional cost when the production increases from 9 to 10 units.

(b) Find the marginal cost when $x = 9$.

(c) Compare the results of parts (a) and (b).

**32. Marginal Revenue**   The revenue $R$ (in dollars) from renting $x$ apartments can be modeled by

$R = 2x(900 + 32x - x^2).$

(a) Find the additional revenue when the number of rentals is increased from 14 to 15.

(b) Find the marginal revenue when $x = 14$.

(c) Compare the results of parts (a) and (b).

**33. Marginal Profit**   The profit $P$ (in dollars) from selling $x$ units of calculus textbooks is given by

$P = -0.05x^2 + 20x - 1000.$

(a) Find the additional profit when the sales increase from 150 to 151 units.

(b) Find the marginal profit when $x = 150$.

(c) Compare the results of parts (a) and (b).

**34. Population Growth**   The population $P$ (in thousands) of Japan can be modeled by

$P = -14.71t^2 + 785.5t + 117{,}216$

where $t$ is time in years, with $t = 0$ corresponding to 1980. *(Source: U.S. Census Bureau)*

(a) Evaluate $P$ for $t = 0$, 10, 15, 20, and 25. Explain these values.

(b) Determine the population growth rate, $dP/dt$.

(c) Evaluate $dP/dt$ for the same values as in part (a). Explain your results.

**35. Health**   The temperature $T$ (in degrees Fahrenheit) of a person during an illness can be modeled by the equation $T = -0.0375t^2 + 0.3t + 100.4$, where $t$ is time in hours since the person started to show signs of a fever.

(T) (a) Use a graphing utility to graph the function. Be sure to choose an appropriate window.

(b) Do the slopes of the tangent lines appear to be positive or negative? What does this tell you?

(c) Evaluate the function for $t = 0$, 4, 8, and 12.

(d) Find $dT/dt$ and explain its meaning in this situation.

(e) Evaluate $dT/dt$ for $t = 0$, 4, 8, and 12.

**36. Marginal Profit**   The profit $P$ (in dollars) from selling $x$ units of a product is given by

$P = 36{,}000 + 2048\sqrt{x} - \dfrac{1}{8x^2}, \quad 150 \le x \le 275.$

Find the marginal profit for each of the following sales.

(a) $x = 150$      (b) $x = 175$      (c) $x = 200$

(d) $x = 225$      (e) $x = 250$      (f) $x = 275$

**37. Profit**   The monthly demand function and cost function for $x$ newspapers at a newsstand are given by $p = 5 - 0.001x$ and $C = 35 + 1.5x$.

(a) Find the monthly revenue $R$ as a function of $x$.

(b) Find the monthly profit $P$ as a function of $x$.

(c) Complete the table.

| $x$ | 600 | 1200 | 1800 | 2400 | 3000 |
|-----|-----|------|------|------|------|
| $dR/dx$ | | | | | |
| $dP/dx$ | | | | | |
| $P$ | | | | | |

(B) **38. Economics**   Use the table to answer the questions below.

| Quantity produced and sold $(Q)$ | Price $(p)$ | Total revenue $(TR)$ | Marginal revenue $(MR)$ |
|------|------|------|------|
| 0 | 160 | 0 | — |
| 2 | 140 | 280 | 130 |
| 4 | 120 | 480 | 90 |
| 6 | 100 | 600 | 50 |
| 8 | 80 | 640 | 10 |
| 10 | 60 | 600 | -30 |

(T) (a) Use the *regression* feature of a graphing utility to find a quadratic model that relates the total revenue $(TR)$ to the quantity produced and sold $(Q)$.

(b) Using derivatives, find a model for marginal revenue from the model you found in part (a).

(c) Calculate the marginal revenue for all values of $Q$ using your model in part (b), and compare these values with the actual values given. How good is your model?

*(Source: Adapted from Taylor, Economics, Fifth Edition)*

**39. Marginal Profit**   When the price of a glass of lemonade at a lemonade stand was $1.75, 400 glasses were sold. When the price was lowered to $1.50, 500 glasses were sold. Assume that the demand function is linear and that the variable and fixed costs are $0.10 and $25, respectively.

(a) Find the profit $P$ as a function of $x$, the number of glasses of lemonade sold.

(T) (b) Use a graphing utility to graph $P$, and comment about the slopes of $P$ when $x = 300$ and when $x = 700$.

(c) Find the marginal profits when 300 glasses of lemonade are sold and when 700 glasses of lemonade are sold.

**40. Marginal Cost** The cost $C$ of producing $x$ units is modeled by $C = v(x) + k$, where $v$ represents the variable cost and $k$ represents the fixed cost. Show that the marginal cost is independent of the fixed cost.

**41. Marginal Profit** When the admission price for a baseball game was $6 per ticket, 36,000 tickets were sold. When the price was raised to $7, only 33,000 tickets were sold. Assume that the demand function is linear and that the variable and fixed costs for the ballpark owners are $0.20 and $85,000, respectively.

(a) Find the profit $P$ as a function of $x$, the number of tickets sold.

(T) (b) Use a graphing utility to graph $P$, and comment about the slopes of $P$ when $x = 18,000$ and when $x = 36,000$.

(c) Find the marginal profits when 18,000 tickets are sold and when 36,000 tickets are sold.

**42. Marginal Profit** In Exercise 41, suppose ticket sales decreased to 30,000 when the price increased to $7. How would this change the answers?

**43. Profit** The demand function for a product is given by $p = 50/\sqrt{x}$ for $1 \le x \le 8000$, and the cost function is given by $C = 0.5x + 500$ for $0 \le x \le 8000$.

Find the marginal profits for (a) $x = 900$, (b) $x = 1600$, (c) $x = 2500$, and (d) $x = 3600$.

If you were in charge of setting the price for this product, what price would you set? Explain your reasoning.

**44. Inventory Management** The annual inventory cost for a manufacturer is given by

$C = 1{,}008{,}000/Q + 6.3Q$

where $Q$ is the order size when the inventory is replenished. Find the change in annual cost when $Q$ is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.

**45. *MAKE A DECISION: FUEL COST*** A car is driven 15,000 miles a year and gets $x$ miles per gallon. Assume that the average fuel cost is $2.95 per gallon. Find the annual cost of fuel $C$ as a function of $x$ and use this function to complete the table.

| $x$ | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|-----|----|----|----|----|----|----|----|
| $C$ | | | | | | | |
| $dC/dx$ | | | | | | | |

Who would benefit more from a 1 mile per gallon increase in fuel efficiency—the driver who gets 15 miles per gallon or the driver who gets 35 miles per gallon? Explain.

**46. Gasoline Sales** The number $N$ of gallons of regular unleaded gasoline sold by a gasoline station at a price of $p$ dollars per gallon is given by $N = f(p)$.

(a) Describe the meaning of $f'(2.959)$

(b) Is $f'(2.959)$ usually positive or negative? Explain.

**47. Dow Jones Industrial Average** The table shows the year-end closing prices $p$ of the Dow Jones Industrial Average (DJIA) from 1992 through 2006, where $t$ is the year, and $t = 2$ corresponds to 1992. *(Source: Dow Jones Industrial Average)*

| $t$ | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|
| $p$ | 3301.11 | 3754.09 | 3834.44 | 5117.12 | 6448.26 |

| $t$ | 7 | 8 | 9 | 10 | 11 |
|-----|---|---|---|----|----|
| $p$ | 7908.24 | 9181.43 | 11,497.12 | 10,786.85 | 10,021.50 |

| $t$ | 12 | 13 | 14 | 15 | 16 |
|-----|----|----|----|----|----|
| $p$ | 8341.63 | 10,453.92 | 10,783.01 | 10,717.50 | 12,463.15 |

(a) Determine the average rate of change in the value of the DJIA from 1992 to 2006.

(b) Estimate the instantaneous rate of change in 1998 by finding the average rate of change from 1996 to 2000.

(c) Estimate the instantaneous rate of change in 1998 by finding the average rate of change from 1997 to 1999.

(d) Compare your answers for parts (b) and (c). Which interval do you think produced the best estimate for the instantaneous rate of change in 1998?

(B) **48. Biology** Many populations in nature exhibit logistic growth, which consists of four phases, as shown in the figure. Describe the rate of growth of the population in each phase, and give possible reasons as to why the rates might be changing from phase to phase. *(Source: Adapted from Levine/Miller, Biology: Discovering Life, Second Edition)*

## Section 2.4

# The Product and Quotient Rules

- Find the derivatives of functions using the Product Rule.
- Find the derivatives of functions using the Quotient Rule.
- Simplify derivatives.
- Use derivatives to answer questions about real-life situations.

## The Product Rule

In Section 2.2, you saw that the derivative of a sum or difference of two functions is simply the sum or difference of their derivatives. The rules for the derivative of a product or quotient of two functions are not as simple.

**STUDY TIP**

Rather than trying to remember the formula for the Product Rule, it can be more helpful to remember its verbal statement: *the first function times the derivative of the second plus the second function times the derivative of the first.*

**The Product Rule**

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

**PROOF**  Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve clever steps that may not appear to follow clearly from a prior step. The proof below involves such a step—adding and subtracting the same quantity. (This step is shown in color.) Let $F(x) = f(x)g(x)$.

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x + \Delta x)g(x) + f(x + \Delta x)g(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[ f(x + \Delta x)\frac{g(x + \Delta x) - g(x)}{\Delta x} + g(x)\frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} f(x + \Delta x)\frac{g(x + \Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x)\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \left[ \lim_{\Delta x \to 0} f(x + \Delta x) \right] \left[ \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \right]$$

$$\quad + \left[ \lim_{\Delta x \to 0} g(x) \right] \left[ \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

$$= f(x)g'(x) + g(x)f'(x)$$

*Example 1*  **Finding the Derivative of a Product**

Find the derivative of $y = (3x - 2x^2)(5 + 4x)$.

**SOLUTION**   Using the Product Rule, you can write

$$\frac{dy}{dx} = \overbrace{(3x - 2x^2)}^{\text{First}} \overbrace{\frac{d}{dx}[5 + 4x]}^{\substack{\text{Derivative} \\ \text{of second}}} + \overbrace{(5 + 4x)}^{\text{Second}} \overbrace{\frac{d}{dx}[3x - 2x^2]}^{\substack{\text{Derivative} \\ \text{of first}}}$$

$$= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$$

$$= (12x - 8x^2) + (15 - 8x - 16x^2)$$

$$= 15 + 4x - 24x^2.$$

✓**CHECKPOINT 1**

Find the derivative of $y = (4x + 3x^2)(6 - 3x)$.  ■

> **STUDY TIP**
>
> In general, the derivative of the product of two functions is not equal to the product of the derivatives of the two functions. To see this, compare the product of the derivatives of $f(x) = 3x - 2x^2$ and $g(x) = 5 + 4x$ with the derivative found in Example 1.

In the next example, notice that the first step in differentiating is *rewriting the original function*.

*Example 2*  **Finding the Derivative of a Product**

Find the derivative of

$$f(x) = \left(\frac{1}{x} + 1\right)(x - 1). \qquad \text{Original function}$$

**SOLUTION**   Rewrite the function. Then use the Product Rule to find the derivative.

$$f(x) = (x^{-1} + 1)(x - 1) \qquad \text{Rewrite function.}$$

$$f'(x) = (x^{-1} + 1)\frac{d}{dx}[x - 1] + (x - 1)\frac{d}{dx}[x^{-1} + 1] \qquad \text{Product Rule}$$

$$= (x^{-1} + 1)(1) + (x - 1)(-x^{-2})$$

$$= \frac{1}{x} + 1 - \frac{x - 1}{x^2}$$

$$= \frac{x + x^2 - x + 1}{x^2} \qquad \text{Write with common denominator.}$$

$$= \frac{x^2 + 1}{x^2} \qquad \text{Simplify.}$$

✓**CHECKPOINT 2**
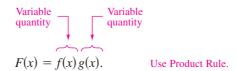
Find the derivative of

$$f(x) = \left(\frac{1}{x} + 1\right)(2x + 1). \quad ■$$

You now have two differentiation rules that deal with products—the Constant Multiple Rule and the Product Rule. The difference between these two rules is that the Constant Multiple Rule deals with the product of a constant and a variable quantity:

Constant ⌐   Variable quantity

$$F(x) = c\,f(x)$$   Use Constant Multiple Rule.

whereas the Product Rule deals with the product of two variable quantities:

Variable quantity   Variable quantity

$$F(x) = f(x)\,g(x).$$   Use Product Rule.

The next example compares these two rules.

### Example 3   Comparing Differentiation Rules

Find the derivative of each function.

**a.**  $y = 2x(x^2 + 3x)$

**b.**  $y = 2(x^2 + 3x)$

**SOLUTION**

**a.** By the Product Rule,

$$\frac{dy}{dx} = (2x)\frac{d}{dx}[x^2 + 3x] + (x^2 + 3x)\frac{d}{dx}[2x]$$   Product Rule

$$= (2x)(2x + 3) + (x^2 + 3x)(2)$$

$$= 4x^2 + 6x + 2x^2 + 6x$$

$$= 6x^2 + 12x.$$

**b.** By the Constant Multiple Rule,

$$\frac{dy}{dx} = 2\frac{d}{dx}[x^2 + 3x]$$   Constant Multiple Rule

$$= 2(2x + 3)$$

$$= 4x + 6.$$

✓ **CHECKPOINT 3**

Find the derivative of each function.

**a.**  $y = 3x(2x^2 + 5x)$

**b.**  $y = 3(2x^2 + 5x)$ ■

The Product Rule can be extended to products that have more than two factors. For example, if $f$, $g$, and $h$ are differentiable functions of $x$, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

## The Quotient Rule

In Section 2.2, you saw that by using the Constant Rule, the Power Rule, the Constant Multiple Rule, and the Sum and Difference Rules, you were able to differentiate any polynomial function. By combining these rules with the Quotient Rule, you can now differentiate any *rational* function.

### The Quotient Rule

The derivative of the quotient of two differentiable functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

**STUDY TIP**

From this differentiation rule, you can see that the derivative of a quotient is not, in general, the quotient of the derivatives. That is,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}.$$

**PROOF**   Let $F(x) = f(x)/g(x)$. As in the proof of the Product Rule, a key step in this proof is adding and subtracting the same quantity.

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{f(x + \Delta x)}{g(x + \Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x + \Delta x) - f(x)g(x + \Delta x)}{\Delta x g(x)g(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x + \Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x + \Delta x)}{\Delta x g(x)g(x + \Delta x)}$$

$$= \frac{\displaystyle\lim_{\Delta x \to 0}\frac{g(x)[f(x + \Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \to 0}\frac{f(x)[g(x + \Delta x) - g(x)]}{\Delta x}}{\displaystyle\lim_{\Delta x \to 0}[g(x)g(x + \Delta x)]}$$

$$= \frac{g(x)\left[\displaystyle\lim_{\Delta x \to 0}\frac{f(x + \Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\displaystyle\lim_{\Delta x \to 0}\frac{g(x + \Delta x) - g(x)}{\Delta x}\right]}{\displaystyle\lim_{\Delta x \to 0}[g(x)g(x + \Delta x)]}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

**STUDY TIP**

As suggested for the Product Rule, it can be more helpful to remember the verbal statement of the Quotient Rule rather than trying to remember the formula for the rule.

## Algebra Review

When applying the Quotient Rule, it is suggested that you enclose all factors and derivatives in symbols of grouping, such as parentheses. Also, pay special attention to the subtraction required in the numerator. For help in evaluating expressions like the one in Example 4, see the *Chapter 2 Algebra Review* on page 197, Example 2(d).

*Example 4* **Finding the Derivative of a Quotient**

Find the derivative of $y = \dfrac{x - 1}{2x + 3}$.

**SOLUTION** Apply the Quotient Rule, as shown.

$$\frac{dy}{dx} = \frac{(2x + 3)\dfrac{d}{dx}[x - 1] - (x - 1)\dfrac{d}{dx}[2x + 3]}{(2x + 3)^2}$$

$$= \frac{(2x + 3)(1) - (x - 1)(2)}{(2x + 3)^2}$$

$$= \frac{2x + 3 - 2x + 2}{(2x + 3)^2}$$

$$= \frac{5}{(2x + 3)^2}$$

✓**CHECKPOINT 4**

Find the derivative of $y = \dfrac{x + 4}{5x - 2}$. ■

*Example 5* **Finding an Equation of a Tangent Line**

Find an equation of the tangent line to the graph of

$$y = \frac{2x^2 - 4x + 3}{2 - 3x}$$

when $x = 1$.

**SOLUTION** Apply the Quotient Rule, as shown.

$$\frac{dy}{dx} = \frac{(2 - 3x)\dfrac{d}{dx}[2x^2 - 4x + 3] - (2x^2 - 4x + 3)\dfrac{d}{dx}[2 - 3x]}{(2 - 3x)^2}$$

$$= \frac{(2 - 3x)(4x - 4) - (2x^2 - 4x + 3)(-3)}{(2 - 3x)^2}$$

$$= \frac{-12x^2 + 20x - 8 - (-6x^2 + 12x - 9)}{(2 - 3x)^2}$$

$$= \frac{-12x^2 + 20x - 8 + 6x^2 - 12x + 9}{(2 - 3x)^2}$$

$$= \frac{-6x^2 + 8x + 1}{(2 - 3x)^2}$$



$y = \dfrac{2x^2 - 4x + 3}{2 - 3x}$

**FIGURE 2.27**

When $x = 1$, the value of the function is $y = -1$ and the slope is $m = 3$. Using the point-slope form of a line, you can find the equation of the tangent line to be $y = 3x - 4$. The graph of the function and the tangent line is shown in Figure 2.27.

✓**CHECKPOINT 5**

Find an equation of the tangent line to the graph of

$$y = \frac{x^2 - 4}{2x + 5} \quad \text{when} \quad x = 0.$$

Sketch the line tangent to the graph at $x = 0$. ■

**Example 6**    **Finding the Derivative of a Quotient**

Find the derivative of

$$y = \frac{3 - (1/x)}{x + 5}.$$

**SOLUTION**    Begin by rewriting the original function. Then apply the Quotient Rule and simplify the result.

$$y = \frac{3 - (1/x)}{x + 5}$$    Write original function.

$$= \frac{3x - 1}{x(x + 5)}$$    Multiply numerator and denominator by $x$.

$$= \frac{3x - 1}{x^2 + 5x}$$    Rewrite.

$$\frac{dy}{dx} = \frac{(x^2 + 5x)(3) - (3x - 1)(2x + 5)}{(x^2 + 5x)^2}$$    Apply Quotient Rule.

$$= \frac{(3x^2 + 15x) - (6x^2 + 13x - 5)}{(x^2 + 5x)^2}$$

$$= \frac{-3x^2 + 2x + 5}{(x^2 + 5x)^2}$$    Simplify.

**✓CHECKPOINT 6**

Find the derivative of $y = \dfrac{3 - (2/x)}{x + 4}$. ■

Not every quotient needs to be differentiated by the Quotient Rule. For instance, each of the quotients in the next example can be considered as the product of a constant and a function of $x$. In such cases, the Constant Multiple Rule is more efficient than the Quotient Rule.

**STUDY TIP**

To see the efficiency of using the Constant Multiple Rule in Example 7, try using the Quotient Rule to find the derivatives of the four functions.

**✓CHECKPOINT 7**

Find the derivative of each function.

**a.** $y = \dfrac{x^2 + 4x}{5}$    **b.** $y = \dfrac{3x^4}{4}$ ■

**Example 7**    **Rewriting Before Differentiating**

Find the derivative of each function.

| Original Function | Rewrite | Differentiate | Simplify |
|---|---|---|---|
| **a.** $y = \dfrac{x^2 + 3x}{6}$ | $y = \dfrac{1}{6}(x^2 + 3x)$ | $y' = \dfrac{1}{6}(2x + 3)$ | $y' = \dfrac{1}{3}x + \dfrac{1}{2}$ |
| **b.** $y = \dfrac{5x^4}{8}$ | $y = \dfrac{5}{8}x^4$ | $y' = \dfrac{5}{8}(4x^3)$ | $y' = \dfrac{5}{2}x^3$ |
| **c.** $y = \dfrac{-3(3x - 2x^2)}{7x}$ | $y = -\dfrac{3}{7}(3 - 2x)$ | $y' = -\dfrac{3}{7}(-2)$ | $y' = \dfrac{6}{7}$ |
| **d.** $y = \dfrac{9}{5x^2}$ | $y = \dfrac{9}{5}(x^{-2})$ | $y' = \dfrac{9}{5}(-2x^{-3})$ | $y' = -\dfrac{18}{5x^3}$ |

## Simplifying Derivatives

**Example 8**   **Combining the Product and Quotient Rules**

Find the derivative of

$$y = \frac{(1 - 2x)(3x + 2)}{5x - 4}.$$

**SOLUTION**   This function contains a product within a quotient. You could first multiply the factors in the numerator and then apply the Quotient Rule. However, to gain practice in using the Product Rule within the Quotient Rule, try differentiating as shown.

$$y' = \frac{(5x - 4) \frac{d}{dx}[(1 - 2x)(3x + 2)] - (1 - 2x)(3x + 2) \frac{d}{dx}[5x - 4]}{(5x - 4)^2}$$

$$= \frac{(5x - 4)[(1 - 2x)(3) + (3x + 2)(-2)] - (1 - 2x)(3x + 2)(5)}{(5x - 4)^2}$$

$$= \frac{(5x - 4)(-12x - 1) - (1 - 2x)(15x + 10)}{(5x - 4)^2}$$

$$= \frac{(-60x^2 + 43x + 4) - (-30x^2 - 5x + 10)}{(5x - 4)^2}$$

$$= \frac{-30x^2 + 48x - 6}{(5x - 4)^2}$$

✓ **CHECKPOINT 8**

Find the derivative of $y = \frac{(1 + x)(2x - 1)}{x - 1}.$ ■

In the examples in this section, much of the work in obtaining the final form of the derivative occurs *after* the differentiation. As summarized in the list below, direct application of differentiation rules often yields results that are not in simplified form. Note that two characteristics of simplified form are the absence of negative exponents and the combining of like terms.

| | *f'(x) After Differentiating* | *f'(x) After Simplifying* |
|---|---|---|
| *Example 1* | $(3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$ | $15 + 4x - 24x^2$ |
| *Example 2* | $(x^{-1} + 1)(1) + (x - 1)(-x^{-2})$ | $\dfrac{x^2 + 1}{x^2}$ |
| *Example 5* | $\dfrac{(2 - 3x)(4x - 4) - (2x^2 - 4x + 3)(-3)}{(2 - 3x)^2}$ | $\dfrac{-6x^2 + 8x + 1}{(2 - 3x)^2}$ |
| *Example 8* | $\dfrac{(5x - 4)[(1 - 2x)(3) + (3x + 2)(-2)] - (1 - 2x)(3x + 2)(5)}{(5x - 4)^2}$ | $\dfrac{-30x^2 + 48x - 6}{(5x - 4)^2}$ |
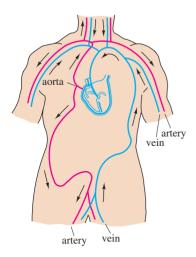
## Application



aorta

artery
vein

artery    vein

**Example 9**  **Rate of Change of Systolic Blood Pressure**  ®

As blood moves from the heart through the major arteries out to the capillaries and back through the veins, the systolic blood pressure continuously drops. Consider a person whose systolic blood pressure $P$ (in millimeters of mercury) is given by

$$P = \frac{25t^2 + 125}{t^2 + 1}, \quad 0 \le t \le 10$$

where $t$ is measured in seconds. At what rate is the blood pressure changing 5 seconds after blood leaves the heart?

**SOLUTION**  Begin by applying the Quotient Rule.

$$\frac{dP}{dt} = \frac{(t^2 + 1)(50t) - (25t^2 + 125)(2t)}{(t^2 + 1)^2} \qquad \text{Quotient Rule}$$

$$= \frac{50t^3 + 50t - 50t^3 - 250t}{(t^2 + 1)^2}$$

$$= -\frac{200t}{(t^2 + 1)^2} \qquad \text{Simplify.}$$

When $t = 5$, the rate of change is

$$-\frac{200(5)}{26^2} \approx -1.48 \text{ millimeters per second.}$$

So, the pressure is *dropping* at a rate of 1.48 millimeters per second when $t = 5$ seconds.

✓**CHECKPOINT 9**

In Example 9, find the rate at which systolic blood pressure is changing at each time shown in the table below. Describe the changes in blood pressure as the blood moves away from the heart.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\dfrac{dP}{dt}$ | | | | | | | | |

■

┌─ **CONCEPT CHECK** ─────────────────────────────┐

1. Write a verbal statement that represents the Product Rule.

2. Write a verbal statement that represents the Quotient Rule.

3. Is it possible to find the derivative of $f(x) = \dfrac{x^3 + 5x}{2}$ without using the Quotient Rule? If so, what differentiation rule can you use to find $f'$? (You do not need to find the derivative.)

4. Complete the following: In general, you can use the Product Rule to differentiate the _____ of two variable quantities and the Quotient Rule to differentiate any _____ function.

└──────────────────────────────────────────────────┘

**Skills Review 2.4**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4, 0.5, and 2.2.

In Exercises 1–10, simplify the expression.

**1.** $(x^2 + 1)(2) + (2x + 7)(2x)$

**2.** $(2x - x^3)(8x) + (4x^2)(2 - 3x^2)$

**3.** $x(4)(x^2 + 2)^3(2x) + (x^2 + 4)(1)$

**4.** $x^2(2)(2x + 1)(2) + (2x + 1)^4(2x)$

**5.** $\dfrac{(2x + 7)(5) - (5x + 6)(2)}{(2x + 7)^2}$

**6.** $\dfrac{(x^2 - 4)(2x + 1) - (x^2 + x)(2x)}{(x^2 - 4)^2}$

**7.** $\dfrac{(x^2 + 1)(2) - (2x + 1)(2x)}{(x^2 + 1)^2}$

**8.** $\dfrac{(1 - x^4)(4) - (4x - 1)(-4x^3)}{(1 - x^4)^2}$

**9.** $(x^{-1} + x)(2) + (2x - 3)(-x^{-2} + 1)$

**10.** $\dfrac{(1 - x^{-1})(1) - (x - 4)(x^{-2})}{(1 - x^{-1})^2}$

In Exercises 11–14, find $f'(2)$.

**11.** $f(x) = 3x^2 - x + 4$

**12.** $f(x) = -x^3 + x^2 + 8x$

**13.** $f(x) = \dfrac{1}{x}$

**14.** $f(x) = x^2 - \dfrac{1}{x^2}$

## Exercises 2.4

In Exercises 1–16, find the value of the derivative of the function at the given point. State which differentiation rule you used to find the derivative.

| *Function* | *Point* |
|---|---|
| **1.** $f(x) = x(x^2 + 3)$ | $(2, 14)$ |
| **2.** $g(x) = (x - 4)(x + 2)$ | $(4, 0)$ |
| **3.** $f(x) = x^2(3x^3 - 1)$ | $(1, 2)$ |
| **4.** $f(x) = (x^2 + 1)(2x + 5)$ | $(-1, 6)$ |
| **5.** $f(x) = \frac{1}{3}(2x^3 - 4)$ | $\left(0, -\frac{4}{3}\right)$ |
| **6.** $f(x) = \frac{1}{7}(5 - 6x^2)$ | $\left(1, -\frac{1}{7}\right)$ |
| **7.** $g(x) = (x^2 - 4x + 3)(x - 2)$ | $(4, 6)$ |
| **8.** $g(x) = (x^2 - 2x + 1)(x^3 - 1)$ | $(1, 0)$ |
| **9.** $h(x) = \dfrac{x}{x - 5}$ | $(6, 6)$ |
| **10.** $h(x) = \dfrac{x^2}{x + 3}$ | $\left(-1, \frac{1}{2}\right)$ |
| **11.** $f(t) = \dfrac{2t^2 - 3}{3t + 1}$ | $\left(3, \frac{3}{2}\right)$ |
| **12.** $f(x) = \dfrac{3x}{x^2 + 4}$ | $\left(-1, -\frac{3}{5}\right)$ |
| **13.** $g(x) = \dfrac{2x + 1}{x - 5}$ | $(6, 13)$ |
| **14.** $f(x) = \dfrac{x + 1}{x - 1}$ | $(2, 3)$ |

| *Function* | *Point* |
|---|---|
| **15.** $f(t) = \dfrac{t^2 - 1}{t + 4}$ | $(1, 0)$ |
| **16.** $g(x) = \dfrac{4x - 5}{x^2 - 1}$ | $(0, 5)$ |

In Exercises 17–24, find the derivative of the function. Use Example 7 as a model.

| *Function* | *Rewrite* | *Differentiate* | *Simplify* |
|---|---|---|---|
| **17.** $y = \dfrac{x^2 + 2x}{x}$ | | | |
| **18.** $y = \dfrac{4x^{3/2}}{x}$ | | | |
| **19.** $y = \dfrac{7}{3x^3}$ | | | |
| **20.** $y = \dfrac{4}{5x^2}$ | | | |
| **21.** $y = \dfrac{4x^2 - 3x}{8\sqrt{x}}$ | | | |
| **22.** $y = \dfrac{3x^2 - 4x}{6x}$ | | | |
| **23.** $y = \dfrac{x^2 - 4x + 3}{x - 1}$ | | | |
| **24.** $y = \dfrac{x^2 - 4}{x + 2}$ | | | |
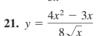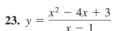
In Exercises 25–40, find the derivative of the function. State which differentiation rule(s) you used to find the derivative,

**25.** $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$

**26.** $h(t) = (t^5 - 1)(4t^2 - 7t - 3)$

**27.** $g(t) = (2t^3 - 1)^2$

**28.** $h(p) = (p^3 - 2)^2$

**29.** $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$

**30.** $f(x) = \sqrt[3]{x}(x + 1)$

**31.** $f(x) = \dfrac{3x - 2}{2x - 3}$

**32.** $f(x) = \dfrac{x^3 + 3x + 2}{x^2 - 1}$

**33.** $f(x) = \dfrac{3 - 2x - x^2}{x^2 - 1}$

**34.** $f(x) = (x^5 - 3x)\left(\dfrac{1}{x^2}\right)$

**35.** $f(x) = x\left(1 - \dfrac{2}{x + 1}\right)$

**36.** $h(t) = \dfrac{t + 2}{t^2 + 5t + 6}$

**37.** $g(s) = \dfrac{s^2 - 2s + 5}{\sqrt{s}}$

**38.** $f(x) = \dfrac{x + 1}{\sqrt{x}}$

**39.** $g(x) = \left(\dfrac{x - 3}{x + 4}\right)(x^2 + 2x + 1)$

**40.** $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$

ⓣ In Exercises 41–46, find an equation of the tangent line to the graph of the function at the given point. Then use a graphing utility to graph the function and the tangent line in the same viewing window.

|      *Function* | *Point* |
|-----------------|---------|
| **41.** $f(x) = (x - 1)^2(x - 2)$ | $(0, -2)$ |
| **42.** $h(x) = (x^2 - 1)^2$ | $(-2, 9)$ |
| **43.** $f(x) = \dfrac{x - 2}{x + 1}$ | $\left(1, -\frac{1}{2}\right)$ |
| **44.** $f(x) = \dfrac{2x + 1}{x - 1}$ | $(2, 5)$ |
| **45.** $f(x) = \left(\dfrac{x + 5}{x - 1}\right)(2x + 1)$ | $(0, -5)$ |
| **46.** $g(x) = (x + 2)\left(\dfrac{x - 5}{x + 1}\right)$ | $(0, -10)$ |

In Exercises 47–50, find the point(s), if any, at which the graph of $f$ has a horizontal tangent.

**47.** $f(x) = \dfrac{x^2}{x - 1}$

**48.** $f(x) = \dfrac{x^2}{x^2 + 1}$

**49.** $f(x) = \dfrac{x^4}{x^3 + 1}$

**50.** $f(x) = \dfrac{x^4 + 3}{x^2 + 1}$

ⓣ In Exercises 51–54, use a graphing utility to graph $f$ and $f'$ on the interval $[-2, 2]$.

**51.** $f(x) = x(x + 1)$

**52.** $f(x) = x^2(x + 1)$

**53.** $f(x) = x(x + 1)(x - 1)$

**54.** $f(x) = x^2(x + 1)(x - 1)$

**Demand**  In Exercises 55 and 56, use the demand function to find the rate of change in the demand $x$ for the given price $p$.

**55.** $x = 275\left(1 - \dfrac{3p}{5p + 1}\right)$, $p = \$4$

**56.** $x = 300 - p - \dfrac{2p}{p + 1}$, $p = \$3$

**57. Environment**  The model

$$f(t) = \dfrac{t^2 - t + 1}{t^2 + 1}$$

measures the level of oxygen in a pond, where $t$ is the time (in weeks) after organic waste is dumped into the pond. Find the rates of change of $f$ with respect to $t$ when (a) $t = 0.5$, (b) $t = 2$, and (c) $t = 8$.

**58. Physical Science**  The temperature $T$ (in degrees Fahrenheit) of food placed in a refrigerator is modeled by

$$T = 10\left(\dfrac{4t^2 + 16t + 75}{t^2 + 4t + 10}\right)$$

where $t$ is the time (in hours). What is the initial temperature of the food? Find the rates of change of $T$ with respect to $t$ when (a) $t = 1$, (b) $t = 3$, (c) $t = 5$, and (d) $t = 10$.

**59. Population Growth**  A population of bacteria is introduced into a culture. The number of bacteria $P$ can be modeled by

$$P = 500\left(1 + \dfrac{4t}{50 + t^2}\right)$$

where $t$ is the time (in hours). Find the rate of change of the population when $t = 2$.

**60. Quality Control**  The percent $P$ of defective parts produced by a new employee $t$ days after the employee starts work can be modeled by

$$P = \dfrac{t + 1750}{50(t + 2)}.$$

Find the rates of change of $P$ when (a) $t = 1$ and (b) $t = 10$.

**61. *MAKE A DECISION: NEGOTIATING A PRICE***  You decide to form a partnership with another business. Your business determines that the demand $x$ for your product is inversely proportional to the square of the price for $x \geq 5$.

(a) The price is $1000 and the demand is 16 units. Find the demand function.

(b) Your partner determines that the product costs $250 per unit and the fixed cost is $10,000. Find the cost function.

(c) Find the profit function and use a graphing utility to graph it. From the graph, what price would you negotiate with your partner for this product? Explain your reasoning.

(T) **62. Managing a Store** You are managing a store and have been adjusting the price of an item. You have found that you make a profit of $50 when 10 units are sold, $60 when 12 units are sold, and $65 when 14 units are sold.

(a) Fit these data to the model $P = ax^2 + bx + c$.

(b) Use a graphing utility to graph $P$.

(c) Find the point on the graph at which the marginal profit is zero. Interpret this point in the context of the problem.

(T) **63. Demand Function** Given $f(x) = x + 1$, which function would most likely represent a demand function? Explain your reasoning. Use a graphing utility to graph each function, and use each graph as part of your explanation.

(a) $p = f(x)$ (b) $p = xf(x)$ (c) $p = -f(x) + 5$

(T) **64. Cost** The cost of producing $x$ units of a product is given by

$C = x^3 - 15x^2 + 87x - 73, \quad 4 \le x \le 9$.

(a) Use a graphing utility to graph the marginal cost function and the average cost function, $C/x$, in the same viewing window.

(b) Find the point of intersection of the graphs of $dC/dx$ and $C/x$. Does this point have any significance?

**65. *MAKE A DECISION: INVENTORY REPLENISHMENT*** The ordering and transportation cost $C$ per unit (in thousands of dollars) of the components used in manufacturing a product is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad 1 \le x$$

where $x$ is the order size (in hundreds). Find the rate of change of $C$ with respect to $x$ for each order size. What do these rates of change imply about increasing the size of an order? Of the given order sizes, which would you choose? Explain.

(a) $x = 10$ (b) $x = 15$ (c) $x = 20$

**66. Inventory Replenishment** The ordering and transportation cost $C$ per unit for the components used in manufacturing a product is

$C = (375,000 + 6x^2)/x, \quad x \ge 1$

where $C$ is measured in dollars and $x$ is the order size. Find the rate of change of $C$ with respect to $x$ when (a) $x = 200$, (b) $x = 250$, and (c) $x = 300$. Interpret the meaning of these values.

**67. Consumer Awareness** The prices per pound of lean and extra lean ground beef in the United States from 1998 to 2005 can be modeled by

$$P = \frac{1.755 - 0.2079t + 0.00673t^2}{1 - 0.1282t + 0.00434t^2}, \quad 8 \le t \le 15$$

where $t$ is the year, with $t = 8$ corresponding to 1998. Find $dP/dt$ and evaluate it for $t = 8, 10, 12,$ and 14. Interpret the meaning of these values. *(Source: U.S. Bureau of Labor Statistics)*

**68. Sales Analysis** The monthly sales of memberships $M$ at a newly built fitness center are modeled by

$$M(t) = \frac{300t}{t^2 + 1} + 8$$

where $t$ is the number of months since the center opened.

(a) Find $M'(t)$.

(b) Find $M(3)$ and $M'(3)$ and interpret the results.

(c) Find $M(24)$ and $M'(24)$ and interpret the results.

In Exercises 69–72, use the given information to find $f'(2)$.

$g(2) = 3$ and $g'(2) = -2$

$h(2) = -1$ and $h'(2) = 4$

**69.** $f(x) = 2g(x) + h(x)$ **70.** $f(x) = 3 - g(x)$

**71.** $f(x) = g(x) + h(x)$ **72.** $f(x) = \dfrac{g(x)}{h(x)}$

**73. Research Project** Use your school's library, the Internet, or some other reference source to find information on a company that is noted for its philanthropy and community commitment. (One such business is described above.) Write a short paper about the company.

## Mid-Chapter Quiz

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, use the limit definition to find the derivative of the function. Then find the slope of the tangent line to the graph of $f$ at the given point.

**1.** $f(x) = -x + 2; (2, 0)$     **2.** $f(x) = \sqrt{x + 3}; (1, 2)$     **3.** $f(x) = \dfrac{4}{x}; (1, 4)$

In Exercises 4–12, find the derivative of the function.

**4.** $f(x) = 12$

**5.** $f(x) = 19x + 9$

**6.** $f(x) = 5 - 3x^2$

**7.** $f(x) = 12x^{1/4}$

**8.** $f(x) = 4x^{-2}$

**9.** $f(x) = 2\sqrt{x}$

**10.** $f(x) = \dfrac{2x + 3}{3x + 2}$

**11.** $f(x) = (x^2 + 1)(-2x + 4)$

**12.** $f(x) = \dfrac{4 - x}{x + 5}$

(T) In Exercises 13–16, use a graphing utility to graph the function and find its average rate of change on the interval. Compare this rate with the instantaneous rates of change at the endpoints of the interval.

**13.** $f(x) = x^2 - 3x + 1; [0, 3]$

**14.** $f(x) = 2x^3 + x^2 - x + 4; [-1, 1]$

**15.** $f(x) = \dfrac{1}{2x}; [2, 5]$

**16.** $f(x) = \sqrt[3]{x}; [8, 27]$

**17.** The profit (in dollars) from selling $x$ units of a product is given by

$$P = -0.0125x^2 + 16x - 600$$

  (a) Find the additional profit when the sales increase from 175 to 176 units.

  (b) Find the marginal profit when $x = 175$.

  (c) Compare the results of parts (a) and (b).

(T) In Exercises 18 and 19, find an equation of the tangent line to the graph of $f$ at the given point. Then use a graphing utility to graph the function and the equation of the tangent line in the same viewing window.

**18.** $f(x) = 5x^2 + 6x - 1; (-1, -2)$

**19.** $f(x) = (x - 1)(x + 1); (0, -1)$

**20.** From 2000 through 2005, the sales per share $S$ (in dollars) for CVS Corporation can be modeled by

$$S = 0.18390t^3 - 0.8242t^2 + 3.492t + 25.60, 0 \le t \le 5$$

  where $t$ represents the year, with $t = 0$ corresponding to 2000. *(Source: CVS Corporation)*

  (a) Find the rate of change of the sales per share with respect to the year.

  (b) At what rate were the sales per share changing in 2001? in 2004? in 2005?

## Section 2.5

# The Chain Rule

- Find derivatives using the Chain Rule.
- Find derivatives using the General Power Rule.
- Write derivatives in simplified form.
- Use derivatives to answer questions about real-life situations.
- Use the differentiation rules to differentiate algebraic functions.

## The Chain Rule

In this section, you will study one of the most powerful rules of differential calculus—the **Chain Rule.** This differentiation rule deals with composite functions and adds versatility to the rules presented in Sections 2.2 and 2.4. For example, compare the functions below. Those on the left can be differentiated without the Chain Rule, whereas those on the right are best done with the Chain Rule.

| *Without the Chain Rule* | *With the Chain Rule* |
|---|---|
| $y = x^2 + 1$ | $y = \sqrt{x^2 + 1}$ |
| $y = x + 1$ | $y = (x + 1)^{-1/2}$ |
| $y = 3x + 2$ | $y = (3x + 2)^5$ |
| $y = \dfrac{x + 5}{x^2 + 2}$ | $y = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$ |

### The Chain Rule

If $y = f(u)$ is a differentiable function of $u$, and $u = g(x)$ is a differentiable function of $x$, then $y = f(g(x))$ is a differentiable function of $x$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

Basically, the Chain Rule states that if $y$ changes $dy/du$ times as fast as $u$, and $u$ changes $du/dx$ times as fast as $x$, then $y$ changes

$$\frac{dy}{du} \cdot \frac{du}{dx}$$

times as fast as $x$, as illustrated in Figure 2.28. One advantage of the $dy/dx$ notation for derivatives is that it helps you remember differentiation rules, such as the Chain Rule. For instance, in the formula

$$dy/dx = (dy/du)(du/dx)$$

you can imagine that the $du$'s divide out.



$x$

Rate of change of $u$ with respect to $x$ is $\dfrac{du}{dx}$.

Input

Function $g$

Output

$u = g(x)$

$u$

Rate of change of $y$ with respect to $u$ is $\dfrac{dy}{du}$.

Input

Function $f$

Output

$y = f(u) = f(g(x))$

Rate of change of $y$ with respect to $x$ is $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.
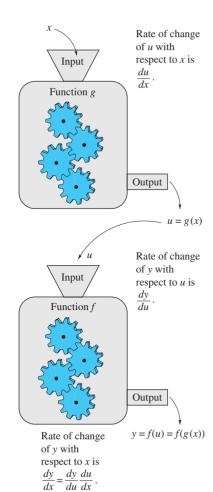
**FIGURE 2.28**

When applying the Chain Rule, it helps to think of the composite function $y = f(g(x))$ or $y = f(u)$ as having two parts—an *inside* and an *outside*—as illustrated below.

Inside

$$y = f(g(x)) = f(u)$$

Outside

The Chain Rule tells you that the derivative of $y = f(u)$ is the derivative of the outer function (at the inner function $u$) *times* the derivative of the inner function. That is,

$$y' = f'(u) \cdot u'.$$

✓**CHECKPOINT 1**

Write each function as the composition of two functions, where $y = f(g(x))$.

**a.** $y = \dfrac{1}{\sqrt{x+1}}$

**b.** $y = (x^2 + 2x + 5)^3$ ■

*Example 1*  **Decomposing Composite Functions**

Write each function as the composition of two functions.

**a.** $y = \dfrac{1}{x+1}$    **b.** $y = \sqrt{3x^2 - x + 1}$

**SOLUTION**    There is more than one correct way to decompose each function. One way for each is shown below.

| $y = f(g(x))$ | $u = g(x)$ *(inside)* | $y = f(u)$ *(outside)* |
|---|---|---|
| **a.** $y = \dfrac{1}{x+1}$ | $u = x + 1$ | $y = \dfrac{1}{u}$ |
| **b.** $y = \sqrt{3x^2 - x + 1}$ | $u = 3x^2 - x + 1$ | $y = \sqrt{u}$ |

*Example 2*  **Using the Chain Rule**

Find the derivative of $y = (x^2 + 1)^3$.

**SOLUTION**    To apply the Chain Rule, you need to identify the inside function $u$.

$$y = \overbrace{(x^2 + 1)}^{u}{}^3 = u^3$$

By the Chain Rule, you can write the derivative as shown.

$$\frac{dy}{dx} = \overbrace{3(x^2 + 1)^2}^{\frac{dy}{du}}\overbrace{(2x)}^{\frac{du}{dx}} = 6x(x^2 + 1)^2$$

**STUDY TIP**

Try checking the result of Example 2 by expanding the function to obtain

$$y = x^6 + 3x^4 + 3x^2 + 1$$

and finding the derivative. Do you obtain the same answer?

✓**CHECKPOINT 2**

Find the derivative of $y = (x^3 + 1)^2$. ■

## The General Power Rule

The function in Example 2 illustrates one of the most common types of composite functions—a power function of the form

$$y = [u(x)]^n.$$

The rule for differentiating such functions is called the **General Power Rule,** and it is a special case of the Chain Rule.

---

**The General Power Rule**

If $y = [u(x)]^n$, where $u$ is a differentiable function of $x$ and $n$ is a real number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}u'.$$

---

**PROOF**  Apply the Chain Rule and the Simple Power Rule as shown.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}[u^n]\frac{du}{dx}$$

$$= nu^{n-1}\frac{du}{dx}$$

**TECHNOLOGY**

(T) If you have access to a symbolic differentiation utility, try using it to confirm the result of Example 3.

*Example 3*  **Using the General Power Rule**

Find the derivative of

$$f(x) = (3x - 2x^2)^3.$$

**SOLUTION**  The inside function is $u = 3x - 2x^2$. So, by the General Power Rule,

$$f'(x) = \overset{n}{3}\overbrace{(3x - 2x^2)^2}^{u^{n-1}}\overbrace{\frac{d}{dx}[3x - 2x^2]}^{u'}$$

$$= 3(3x - 2x^2)^2(3 - 4x)$$

$$= (9 - 12x)(3x - 2x^2)^2.$$

✓**CHECKPOINT 3**

Find the derivative of $y = (x^2 + 3x)^4$. ■

**Example 4** **Rewriting Before Differentiating**

Find the tangent line to the graph of

$$y = \sqrt[3]{(x^2 + 4)^2}$$    Original function
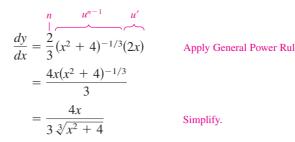
when $x = 2$.

**SOLUTION**    Begin by rewriting the function in rational exponent form.

$$y = (x^2 + 4)^{2/3}$$    Rewrite original function.

Then, using the inside function, $u = x^2 + 4$, apply the General Power Rule.

$$\frac{dy}{dx} = \overset{n}{\frac{2}{3}}(x^2 + 4)^{-1/3}\overbrace{(2x)}^{u'}$$    Apply General Power Rule.

$$= \frac{4x(x^2 + 4)^{-1/3}}{3}$$

$$= \frac{4x}{3\sqrt[3]{x^2 + 4}}$$    Simplify.

When $x = 2$, $y = 4$ and the slope of the line tangent to the graph at $(2, 4)$ is $\frac{4}{3}$. Using the point-slope form, you can find the equation of the tangent line to be $y = \frac{4}{3}x + \frac{4}{3}$. The graph of the function and the tangent line is shown in Figure 2.29.



$y = \sqrt[3]{(x^2 + 4)^2}$

**FIGURE 2.29**

✓ **CHECKPOINT 4**

Find the tangent line to the graph of $y = \sqrt[3]{(x + 4)^2}$ when $x = 4$. Sketch the line tangent to the graph at $x = 4$. ■

**STUDY TIP**
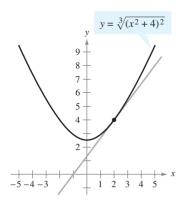
The derivative of a quotient can sometimes be found more easily with the General Power Rule than with the Quotient Rule. This is especially true when the numerator is a constant, as shown in Example 5.

**Example 5** **Finding the Derivative of a Quotient**

Find the derivative of each function.

**a.** $y = \dfrac{3}{x^2 + 1}$    **b.** $y = \dfrac{3}{(x + 1)^2}$

**SOLUTION**

**a.** Begin by rewriting the function as

$$y = 3(x^2 + 1)^{-1}.$$    Rewrite original function.

Then apply the General Power Rule to obtain

$$\frac{dy}{dx} = -3(x^2 + 1)^{-2}(2x) = -\frac{6x}{(x^2 + 1)^2}.$$    Apply General Power Rule.

**b.** Begin by rewriting the function as

$$y = 3(x + 1)^{-2}.$$    Rewrite original function.

Then apply the General Power Rule to obtain

$$\frac{dy}{dx} = -6(x + 1)^{-3}(1) = -\frac{6}{(x + 1)^3}.$$    Apply General Power Rule.

✓ **CHECKPOINT 5**

Find the derivative of each function.

**a.** $y = \dfrac{4}{2x + 1}$

**b.** $y = \dfrac{2}{(x - 1)^3}$ ■

## Simplification Techniques

Throughout this chapter, writing derivatives in simplified form has been emphasized. The reason for this is that most applications of derivatives require a simplified form. The next two examples illustrate some useful simplification techniques.

**Example 6**  **Simplifying by Factoring Out Least Powers**

Find the derivative of $y = x^2\sqrt{1 - x^2}$.

$$y = x^2\sqrt{1 - x^2}$$  Write original function.

$$= x^2(1 - x^2)^{1/2}$$  Rewrite function.

$$y' = x^2\frac{d}{dx}\left[(1 - x^2)^{1/2}\right] + (1 - x^2)^{1/2}\frac{d}{dx}\left[x^2\right]$$  Product Rule

$$= x^2\left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] + (1 - x^2)^{1/2}(2x)$$  Power Rule

$$= -x^3(1 - x^2)^{-1/2} + 2x(1 - x^2)^{1/2}$$

$$= x(1 - x^2)^{-1/2}\left[-x^2(1) + 2(1 - x^2)\right]$$  Factor.

$$= x(1 - x^2)^{-1/2}(2 - 3x^2)$$

$$= \frac{x(2 - 3x^2)}{\sqrt{1 - x^2}}$$  Simplify.

✓**CHECKPOINT 6**

Find and simplify the derivative of $y = x^2\sqrt{x^2 + 1}$. ■

**Example 7**  **Differentiating a Quotient Raised to a Power**

Find the derivative of

$$f(x) = \left(\frac{3x - 1}{x^2 + 3}\right)^2.$$

**SOLUTION**

$$f'(x) = \overset{n}{2}\left(\frac{3x - 1}{x^2 + 3}\right)^{\overset{u^{n-1}}{}}\frac{d}{dx}\overset{u'}{\left[\frac{3x - 1}{x^2 + 3}\right]}$$

$$= \left[\frac{2(3x - 1)}{x^2 + 3}\right]\left[\frac{(x^2 + 3)(3) - (3x - 1)(2x)}{(x^2 + 3)^2}\right]$$

$$= \frac{2(3x - 1)(3x^2 + 9 - 6x^2 + 2x)}{(x^2 + 3)^3}$$

$$= \frac{2(3x - 1)(-3x^2 + 2x + 9)}{(x^2 + 3)^3}$$

✓**CHECKPOINT 7**

Find the derivative of

$$f(x) = \left(\frac{x + 1}{x - 5}\right)^2. ■$$

**Example 8** **Finding Rates of Change** Ⓡ

From 1996 through 2005, the revenue per share $R$ (in dollars) for U.S. Cellular can be modeled by $R = (-0.009t^2 + 0.54t - 0.1)^2$ for $6 \le t \le 15$, where $t$ is the year, with $t = 6$ corresponding to 1996. Use the model to approximate the rates of change in the revenue per share in 1997, 1999, and 2003. If you had been a U.S. Cellular stockholder from 1996 through 2005, would you have been satisfied with the performance of this stock? *(Source: U.S. Cellular)*

**SOLUTION**    The rate of change in $R$ is given by the derivative $dR/dt$. You can use the General Power Rule to find the derivative.

$$\frac{dR}{dt} = 2(-0.009t^2 + 0.54t - 0.1)^1(-0.018t + 0.54)$$

$$= (-0.036t + 1.08)(-0.009t^2 + 0.54t - 0.1)$$

In 1997, the revenue per share was changing at a rate of

$$[-0.036(7) + 1.08][-0.009(7)^2 + 0.54(7) - 0.1] \approx \$2.68 \text{ per year.}$$

In 1999, the revenue per share was changing at a rate of

$$[-0.036(9) + 1.08][-0.009(9)^2 + 0.54(9) - 0.1] \approx \$3.05 \text{ per year.}$$

In 2003, the revenue per share was changing at a rate of

$$[-0.036(13) + 1.08][-0.009(13)^2 + 0.54(13) - 0.1] \approx \$3.30 \text{ per year.}$$

The graph of the revenue per share function is shown in Figure 2.30. For most investors, the performance of U.S. Cellular stock would be considered to be good.
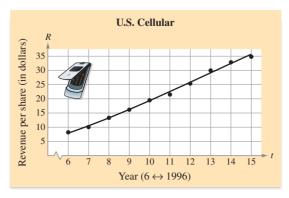


**FIGURE 2.30**

**✓ CHECKPOINT 8**

From 1996 through 2005, the sales per share (in dollars) for Dollar Tree can be modeled by $S = (-0.002t^2 + 0.39t + 0.1)^2$ for $6 \le t \le 15$, where $t$ is the year, with $t = 6$ corresponding to 1996. Use the model to approximate the rate of change in sales per share in 2003. *(Source: Dollar Tree Stores, Inc.)* ■

## Summary of Differentiation Rules

You now have all the rules you need to differentiate *any* algebraic function. For your convenience, they are summarized below.

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### Summary of Differentiation Rules

Let $u$ and $v$ be differentiable functions of $x$.

**1.** Constant Rule

$$\frac{d}{dx}[c] = 0, \quad c \text{ is a constant.}$$

**2.** Constant Multiple Rule

$$\frac{d}{dx}[cu] = c\frac{du}{dx}, \quad c \text{ is a constant.}$$

**3.** Sum and Difference Rules

$$\frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx}$$

**4.** Product Rule

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

**5.** Quotient Rule

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

**6.** Power Rules

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$$

**7.** Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

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### CONCEPT CHECK

1. Write a verbal statement that represents the Chain Rule.

2. Write a verbal statement that represents the General Power Rule.

3. Complete the following: When the numerator of a quotient is a constant, you may be able to find the derivative of the quotient more easily with the _____ _____ Rule than with the Quotient Rule.

4. In the expression $f(g(x))$, $f$ is the outer function and $g$ is the inner function. Write a verbal statement of the Chain Rule using the words "inner" and "outer."

**Skills Review 2.5**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 0.4.

In Exercises 1–6, rewrite the expression with rational exponents.

**1.** $\sqrt[5]{(1 - 5x)^2}$

**2.** $\sqrt[4]{(2x - 1)^3}$

**3.** $\dfrac{1}{\sqrt{4x^2 + 1}}$

**4.** $\dfrac{1}{\sqrt[3]{x - 6}}$

**5.** $\dfrac{\sqrt{x}}{\sqrt[3]{1 - 2x}}$

**6.** $\dfrac{\sqrt{(3 - 7x)^3}}{2x}$

In Exercises 7–10, factor the expression.

**7.** $3x^3 - 6x^2 + 5x - 10$

**8.** $5x\sqrt{x} - x - 5\sqrt{x} + 1$

**9.** $4(x^2 + 1)^2 - x(x^2 + 1)^3$

**10.** $-x^5 + 3x^3 + x^2 - 3$

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## Exercises 2.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, identify the inside function, $u = g(x)$, and the outside function, $y = f(u)$.

| $y = f(g(x))$ | $u = g(x)$ | $y = f(u)$ |
|---|---|---|
| **1.** $y = (6x - 5)^4$ | | |
| **2.** $y = (x^2 - 2x + 3)^3$ | | |
| **3.** $y = (4 - x^2)^{-1}$ | | |
| **4.** $y = (x^2 + 1)^{4/3}$ | | |
| **5.** $y = \sqrt{5x - 2}$ | | |
| **6.** $y = \sqrt{1 - x^2}$ | | |
| **7.** $y = (3x + 1)^{-1}$ | | |
| **8.** $y = (x + 2)^{-1/2}$ | | |

In Exercises 9–14, find $dy/du$, $du/dx$, and $dy/dx$.

**9.** $y = u^2, u = 4x + 7$

**10.** $y = u^3, u = 3x^2 - 2$

**11.** $y = \sqrt{u}, u = 3 - x^2$

**12.** $y = 2\sqrt{u}, u = 5x + 9$

**13.** $y = u^{2/3}, u = 5x^4 - 2x$

**14.** $y = u^{-1}, u = x^3 + 2x^2$

In Exercises 15–22, match the function with the rule that you would use to find the derivative *most efficiently*.

(a) Simple Power Rule     (b) Constant Rule

(c) General Power Rule    (d) Quotient Rule

**15.** $f(x) = \dfrac{2}{1 - x^3}$

**16.** $f(x) = \dfrac{2x}{1 - x^3}$

**17.** $f(x) = \sqrt[3]{8^2}$

**18.** $f(x) = \sqrt[3]{x^2}$

**19.** $f(x) = \dfrac{x^2 + 2}{x}$

**20.** $f(x) = \dfrac{x^4 - 2x + 1}{\sqrt{x}}$

**21.** $f(x) = \dfrac{2}{x - 2}$

**22.** $f(x) = \dfrac{5}{x^2 + 1}$

In Exercises 23–40, use the General Power Rule to find the derivative of the function.

**23.** $y = (2x - 7)^3$

**24.** $y = (2x^3 + 1)^2$

**25.** $g(x) = (4 - 2x)^3$

**26.** $h(t) = (1 - t^2)^4$

**27.** $h(x) = (6x - x^3)^2$

**28.** $f(x) = (4x - x^2)^3$

**29.** $f(x) = (x^2 - 9)^{2/3}$

**30.** $f(t) = (9t + 2)^{2/3}$

**31.** $f(t) = \sqrt{t + 1}$

**32.** $g(x) = \sqrt{5 - 3x}$

**33.** $s(t) = \sqrt{2t^2 + 5t + 2}$

**34.** $y = \sqrt[3]{3x^3 + 4x}$

**35.** $y = \sqrt[3]{9x^2 + 4}$

**36.** $y = 2\sqrt{4 - x^2}$

**37.** $f(x) = -3\sqrt[4]{2 - 9x}$

**38.** $f(x) = (25 + x^2)^{-1/2}$

**39.** $h(x) = (4 - x^3)^{-4/3}$

**40.** $f(x) = (4 - 3x)^{-5/2}$

In Exercises 41–46, find an equation of the tangent line to the graph of $f$ at the point $(2, f(2))$. Use a graphing utility to check your result by graphing the original function and the tangent line in the same viewing window.

**41.** $f(x) = 2(x^2 - 1)^3$

**42.** $f(x) = 3(9x - 4)^4$

**43.** $f(x) = \sqrt{4x^2 - 7}$

**44.** $f(x) = x\sqrt{x^2 + 5}$

**45.** $f(x) = \sqrt{x^2 - 2x + 1}$

**46.** $f(x) = (4 - 3x^2)^{-2/3}$

Ⓣ In Exercises 47–50, use a symbolic differentiation utility to find the derivative of the function. Graph the function and its derivative in the same viewing window. Describe the behavior of the function when the derivative is zero.

**47.** $f(x) = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

**48.** $f(x) = \sqrt{\dfrac{2x}{x + 1}}$

**49.** $f(x) = \sqrt{\dfrac{x + 1}{x}}$

**50.** $f(x) = \sqrt{x}(2 - x^2)$

In Exercises 51–66, find the derivative of the function. State which differentiation rule(s) you used to find the derivative.

**51.** $y = \dfrac{1}{x - 2}$

**52.** $s(t) = \dfrac{1}{t^2 + 3t - 1}$

**53.** $y = -\dfrac{4}{(t + 2)^2}$

**54.** $f(x) = \dfrac{3}{(x^3 - 4)^2}$

**55.** $f(x) = \dfrac{1}{(x^2 - 3x)^2}$

**56.** $y = \dfrac{1}{\sqrt{x + 2}}$

**57.** $g(t) = \dfrac{1}{t^2 - 2}$

**58.** $g(x) = \dfrac{3}{\sqrt[3]{x^3 - 1}}$

**59.** $f(x) = x(3x - 9)^3$

**60.** $f(x) = x^3(x - 4)^2$

**61.** $y = x\sqrt{2x + 3}$

**62.** $y = t\sqrt{t + 1}$

**63.** $y = t^2\sqrt{t - 2}$

**64.** $y = \sqrt{x}(x - 2)^2$

**65.** $y = \left(\dfrac{6 - 5x}{x^2 - 1}\right)^2$

**66.** $y = \left(\dfrac{4x^2}{3 - x}\right)^3$

Ⓣ In Exercises 67–72, find an equation of the tangent line to the graph of the function at the given point. Then use a graphing utility to graph the function and the tangent line in the same viewing window.

| Function | Point |
|---|---|

**67.** $f(t) = \dfrac{36}{(3 - t)^2}$  $(0, 4)$

**68.** $s(x) = \dfrac{1}{\sqrt{x^2 - 3x + 4}}$  $\left(3, \frac{1}{2}\right)$

**69.** $f(t) = (t^2 - 9)\sqrt{t + 2}$  $(-1, -8)$

**70.** $y = \dfrac{2x}{\sqrt{x + 1}}$  $(3, 3)$

**71.** $f(x) = \dfrac{x + 1}{\sqrt{2x - 3}}$  $(2, 3)$

**72.** $y = \dfrac{x}{\sqrt{25 + x^2}}$  $(0, 0)$

**73. Compound Interest**   You deposit $1000 in an account with an annual interest rate of $r$ (in decimal form) compounded monthly. At the end of 5 years, the balance is

$$A = 1000\left(1 + \dfrac{r}{12}\right)^{60}.$$

Find the rates of change of $A$ with respect to $r$ when (a) $r = 0.08$, (b) $r = 0.10$, and (c) $r = 0.12$.

**74. Environment**   An environmental study indicates that the average daily level $P$ of a certain pollutant in the air, in parts per million, can be modeled by the equation

$$P = 0.25\sqrt{0.5n^2 + 5n + 25}$$

where $n$ is the number of residents of the community, in thousands. Find the rate at which the level of pollutant is increasing when the population of the community is 12,000.

**75. Biology**   The number $N$ of bacteria in a culture after $t$ days is modeled by

$$N = 400\left[1 - \dfrac{3}{(t^2 + 2)^2}\right].$$

Complete the table. What can you conclude?

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $dN/dt$ | | | | | |

**76. Depreciation**   The value $V$ of a machine $t$ years after it is purchased is inversely proportional to the square root of $t + 1$. The initial value of the machine is $10,000.

(a) Write $V$ as a function of $t$.

(b) Find the rate of depreciation when $t = 1$.

(c) Find the rate of depreciation when $t = 3$.

**77. Depreciation**   Repeat Exercise 76 given that the value of the machine $t$ years after it is purchased is inversely proportional to the cube root of $t + 1$.

Ⓣ **78. Credit Card Rate**   The average annual rate $r$ (in percent form) for commercial bank credit cards from 2000 through 2005 can be modeled by

$$r = \sqrt{-1.7409t^4 + 18.070t^3 - 52.68t^2 + 10.9t + 249}$$

where $t$ represents the year, with $t = 0$ corresponding to 2000.   *(Source: Federal Reserve Bulletin)*

(a) Find the derivative of this model. Which differentiation rule(s) did you use?

(b) Use a graphing utility to graph the derivative on the interval $0 \le t \le 5$.

(c) Use the *trace* feature to find the years during which the finance rate was changing the most.

(d) Use the *trace* feature to find the years during which the finance rate was changing the least.

**True or False?**   In Exercises 79 and 80, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**79.** If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

**80.** If $y$ is a differentiable function of $u$, $u$ is a differentiable function of $v$, and $v$ is a differentiable function of $x$, then

$$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}.$$

**81.** Given that $f(x) = h(g(x))$, find $f'(2)$ for each of the following.

(a) $g(2) = -6$ and $g'(2) = 5$, $h(5) = 4$ and $h'(-6) = 3$

(b) $g(2) = -1$ and $g'(2) = -2$, $h(2) = 4$ and $h'(-1) = 5$

# Higher-Order Derivatives

■ Find higher-order derivatives.
■ Find and use the position functions to determine the velocity and acceleration of moving objects.

## Second, Third, and Higher-Order Derivatives

The derivative of $f'$ is the **second derivative** of $f$ and is denoted by $f''$.

$$\frac{d}{dx}[f'(x)] = f''(x) \qquad \text{Second derivative}$$

The derivative of $f''$ is the **third derivative** of $f$ and is denoted by $f'''$.

$$\frac{d}{dx}[f''(x)] = f'''(x) \qquad \text{Third derivative}$$

By continuing this process, you obtain **higher-order derivatives** of $f$. Higher-order derivatives are denoted as follows.

**STUDY TIP**

In the context of higher-order derivatives, the "standard" derivative $f'$ is often called the **first derivative** of $f$.

**DISCOVERY**

For each function, find the indicated higher-order derivative.

**a.** $y = x^2$  **b.** $y = x^3$

$y''$    $y'''$

**c.** $y = x^4$  **d.** $y = x^n$

$y^{(4)}$    $y^{(n)}$

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**Notation for Higher-Order Derivatives**

**1.** 1st derivative:  $y'$,  $f'(x)$,  $\dfrac{dy}{dx}$,  $\dfrac{d}{dx}[f(x)]$,  $D_x[y]$

**2.** 2nd derivative:  $y''$,  $f''(x)$,  $\dfrac{d^2y}{dx^2}$,  $\dfrac{d^2}{dx^2}[f(x)]$,  $D_x^2[y]$

**3.** 3rd derivative:  $y'''$,  $f'''(x)$,  $\dfrac{d^3y}{dx^3}$,  $\dfrac{d^3}{dx^3}[f(x)]$,  $D_x^3[y]$

**4.** 4th derivative:  $y^{(4)}$,  $f^{(4)}(x)$,  $\dfrac{d^4y}{dx^4}$,  $\dfrac{d^4}{dx^4}[f(x)]$,  $D_x^4[y]$

**5.** $n$th derivative:  $y^{(n)}$,  $f^{(n)}(x)$,  $\dfrac{d^ny}{dx^n}$,  $\dfrac{d^n}{dx^n}[f(x)]$,  $D_x^n[y]$

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*Example 1*  **Finding Higher-Order Derivatives**

Find the first five derivatives of $f(x) = 2x^4 - 3x^2$.

$$f(x) = 2x^4 - 3x^2 \qquad \text{Write original function.}$$
$$f'(x) = 8x^3 - 6x \qquad \text{First derivative}$$
$$f''(x) = 24x^2 - 6 \qquad \text{Second derivative}$$
$$f'''(x) = 48x \qquad \text{Third derivative}$$
$$f^{(4)}(x) = 48 \qquad \text{Fourth derivative}$$
$$f^{(5)}(x) = 0 \qquad \text{Fifth derivative}$$

✓ **CHECKPOINT 1**

Find the first four derivatives of $f(x) = 6x^3 - 2x^2 + 1$. ■

### Example 2    Finding Higher-Order Derivatives

Find the value of $g'''(2)$ for the function

$$g(t) = -t^4 + 2t^3 + t + 4.$$    Original function

**SOLUTION**    Begin by differentiating three times.

$$g'(t) = -4t^3 + 6t^2 + 1$$    First derivative
$$g''(t) = -12t^2 + 12t$$    Second derivative
$$g'''(t) = -24t + 12$$    Third derivative

Then, evaluate the third derivative of $g$ at $t = 2$.

$$g'''(2) = -24(2) + 12$$
$$= -36$$    Value of third derivative

✓**CHECKPOINT 2**

Find the value of $g'''(1)$ for $g(x) = x^4 - x^3 + 2x$. ■

Examples 1 and 2 show how to find higher-order derivatives of *polynomial* functions. Note that with each successive differentiation, the degree of the polynomial drops by one. Eventually, higher-order derivatives of polynomial functions degenerate to a constant function. Specifically, the $n$th-order derivative of an $n$th-degree polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is the constant function

$$f^{(n)}(x) = n! a_n$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$. Each derivative of order higher than $n$ is the zero function. Polynomial functions are the *only* functions with this characteristic. For other functions, successive differentiation never produces a constant function.

### Example 3    Finding Higher-Order Derivatives

Find the first four derivatives of $y = x^{-1}$.

$$y = x^{-1} = \frac{1}{x}$$    Write original function.

$$y' = (-1)x^{-2} = -\frac{1}{x^2}$$    First derivative

$$y'' = (-1)(-2)x^{-3} = \frac{2}{x^3}$$    Second derivative

$$y''' = (-1)(-2)(-3)x^{-4} = -\frac{6}{x^4}$$    Third derivative

$$y^{(4)} = (-1)(-2)(-3)(-4)x^{-5} = \frac{24}{x^5}$$    Fourth derivative

✓**CHECKPOINT 3**

Find the fourth derivative of

$$y = \frac{1}{x^2}.$$ ■
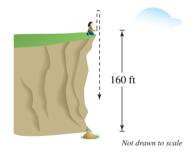
## Acceleration

In Section 2.3, you saw that the velocity of an object moving in a straight path (neglecting air resistance) is given by the derivative of its position function. In other words, the rate of change of the position with respect to time is defined to be the velocity. In a similar way, the rate of change of the velocity with respect to time is defined to be the **acceleration** of the object.

$$s = f(t) \qquad \text{Position function}$$

$$\frac{ds}{dt} = f'(t) \qquad \text{Velocity function}$$

$$\frac{d^2s}{dt^2} = f''(t) \qquad \text{Acceleration function}$$

To find the position, velocity, or acceleration at a particular time $t$, substitute the given value of $t$ into the appropriate function, as illustrated in Example 4.

**Example 4** **Finding Acceleration** ®

A ball is thrown upward from the top of a 160-foot cliff, as shown in Figure 2.31. The initial velocity of the ball is 48 feet per second, which implies that the position function is

$$s = -16t^2 + 48t + 160$$

where the time $t$ is measured in seconds. Find the height, the velocity, and the acceleration of the ball when $t = 3$.

**SOLUTION** Begin by differentiating to find the velocity and acceleration functions.

$$s = -16t^2 + 48t + 160 \qquad \text{Position function}$$

$$\frac{ds}{dt} = -32t + 48 \qquad \text{Velocity function}$$

$$\frac{d^2s}{dt^2} = -32 \qquad \text{Acceleration function}$$

To find the height, velocity, and acceleration when $t = 3$, substitute $t = 3$ into each of the functions above.
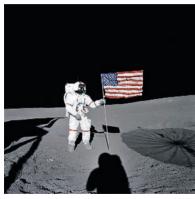
Height $= -16(3)^2 + 48(3) + 160 = 160$ feet

Velocity $= -32(3) + 48 = -48$ feet per second

Acceleration $= -32$ feet per second squared



**FIGURE 2.31**

160 ft

*Not drawn to scale*

✓**CHECKPOINT 4**

A ball is thrown upward from the top of an 80-foot cliff with an initial velocity of 64 feet per second. Give the position function. Then find the velocity and acceleration functions. ∎

The acceleration due to gravity on the surface of the moon is only about one-sixth that exerted by Earth. So, if you were on the moon and threw an object into the air, it would rise to a greater height than it would on Earth's surface.

In Example 4, notice that the acceleration of the ball is $-32$ feet per second squared at any time $t$. This constant acceleration is due to the gravitational force of Earth and is called the **acceleration due to gravity.** Note that the negative value indicates that the ball is being pulled *down*—toward Earth.

Although the acceleration exerted on a falling object is relatively constant near Earth's surface, it varies greatly throughout our solar system. Large planets exert a much greater gravitational pull than do small planets or moons. The next example describes the motion of a free-falling object on the moon.

**Example 5** **Finding Acceleration on the Moon** Ⓡ

An astronaut standing on the surface of the moon throws a rock into the air. The height $s$ (in feet) of the rock is given by

$$s = -\frac{27}{10}t^2 + 27t + 6$$

where $t$ is measured in seconds. How does the acceleration due to gravity on the moon compare with that on Earth?

**SOLUTION**

$$s = -\frac{27}{10}t^2 + 27t + 6 \qquad \text{Position function}$$

$$\frac{ds}{dt} = -\frac{27}{5}t + 27 \qquad \text{Velocity function}$$

$$\frac{d^2s}{dt^2} = -\frac{27}{5} \qquad \text{Acceleration function}$$

So, the acceleration at any time is

$$-\frac{27}{5} = -5.4 \text{ feet per second squared}$$

—about one-sixth of the acceleration due to gravity on Earth.

✓ **CHECKPOINT 5**

The position function on Earth, where $s$ is measured in meters, $t$ is measured in seconds, $v_0$ is the initial velocity in meters per second, and $h_0$ is the initial height in meters, is

$$s = -4.9t^2 + v_0 t + h_0.$$

If the initial velocity is 2.2 and the initial height is 3.6, what is the acceleration due to gravity on Earth in meters per second per second? ∎

The position function described in Example 5 neglects air resistance, which is appropriate because the moon has no atmosphere—and *no air resistance*. This means that the position function for any free-falling object on the moon is given by

$$s = -\frac{27}{10}t^2 + v_0 t + h_0$$

where $s$ is the height (in feet), $t$ is the time (in seconds), $v_0$ is the initial velocity, and $h_0$ is the initial height. For instance, the rock in Example 5 was thrown upward with an initial velocity of 27 feet per second and had an initial height of 6 feet. This position function is valid for all objects, whether heavy ones such as hammers or light ones such as feathers.

In 1971, astronaut David R. Scott demonstrated the lack of atmosphere on the moon by dropping a hammer and a feather from the same height. Both took exactly the same time to fall to the ground. If they were dropped from a height of 6 feet, how long did each take to hit the ground?

**Example 6**   **Finding Velocity and Acceleration**   Ⓡ
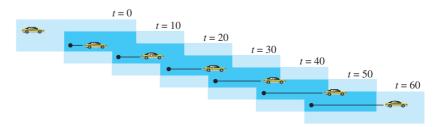
The velocity $v$ (in feet per second) of a certain automobile starting from rest is

$$v = \frac{80t}{t + 5}$$   Velocity function

where $t$ is the time (in seconds). The positions of the automobile at 10-second intervals are shown in Figure 2.32. Find the velocity and acceleration of the automobile at 10-second intervals from $t = 0$ to $t = 60$.



**FIGURE 2.32**

**SOLUTION**   To find the acceleration function, differentiate the velocity function.

$$\frac{dv}{dt} = \frac{(t + 5)(80) - (80t)(1)}{(t + 5)^2}$$

$$= \frac{400}{(t + 5)^2}$$   Acceleration function

| $t$ (seconds) | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| $v$ (ft/sec) | 0 | 53.5 | 64.0 | 68.6 | 71.1 | 72.7 | 73.8 |
| $\dfrac{dv}{dt}$ (ft/sec²) | 16 | 1.78 | 0.64 | 0.33 | 0.20 | 0.13 | 0.09 |

✓ **CHECKPOINT 6**

Use a graphing utility to graph the velocity function and acceleration function in Example 6 in the same viewing window. Compare the graphs with the table at the right. As the velocity levels off, what does the acceleration approach? ■

In the table, note that the acceleration approaches zero as the velocity levels off. This observation should agree with your experience—when riding in an accelerating automobile, you do not feel the velocity, but you do feel the acceleration. In other words, you feel changes in velocity.

**CONCEPT CHECK**

1. Use mathematical notation to write the third derivative of $f(x)$.

2. Give a verbal description of what is meant by $\dfrac{d^2y}{dx^2}$.

3. Complete the following:  If $f(x)$ is an $n$th-degree polynomial, then $f^{(n+1)}(x)$ is equal to _____.

4. If the velocity of an object is constant, what is its acceleration?

**Skills Review 2.6**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.4 and 2.5.

In Exercises 1–4, solve the equation.

**1.** $-16t^2 + 24t = 0$

**2.** $-16t^2 + 80t + 224 = 0$

**3.** $-16t^2 + 128t + 320 = 0$

**4.** $-16t^2 + 9t + 1440 = 0$

In Exercises 5–8, find $dy/dx$.

**5.** $y = x^2(2x + 7)$

**6.** $y = (x^2 + 3x)(2x^2 - 5)$

**7.** $y = \dfrac{x^2}{2x + 7}$

**8.** $y = \dfrac{x^2 + 3x}{2x^2 - 5}$

In Exercises 9 and 10, find the domain and range of $f$.

**9.** $f(x) = x^2 - 4$

**10.** $f(x) = \sqrt{x - 7}$

## Exercises 2.6

In Exercises 1–16, find the second derivative of the function.

**1.** $f(x) = 9 - 2x$

**2.** $f(x) = 4x + 15$

**3.** $f(x) = x^2 + 7x - 4$

**4.** $f(x) = 3x^2 + 4x$

**5.** $g(t) = \frac{1}{3}t^3 - 4t^2 + 2t$

**6.** $f(x) = 4(x^2 - 1)^2$

**7.** $f(t) = \dfrac{3}{4t^2}$

**8.** $g(t) = 32t^{-2}$

**9.** $f(x) = 3(2 - x^2)^3$

**10.** $f(x) = x\sqrt[3]{x}$

**11.** $y = (x^3 - 2x)^4$

**12.** $y = 4(x^2 + 5x)^3$

**13** $f(x) = \dfrac{x + 1}{x - 1}$

**14.** $g(t) = -\dfrac{4}{(t + 2)^2}$

**15.** $y = x^2(x^2 + 4x + 8)$

**16.** $h(s) = s^3(s^2 - 2s + 1)$

In Exercises 17–22, find the third derivative of the function.

**17.** $f(x) = x^5 - 3x^4$

**18.** $f(x) = x^4 - 2x^3$

**19.** $f(x) = 5x(x + 4)^3$

**20.** $f(x) = (x^3 - 6)^4$

**21.** $f(x) = \dfrac{3}{16x^2}$

**22.** $f(x) = \dfrac{1}{x}$

In Exercises 23–28, find the given value.

| Function | Value |
|---|---|
| **23.** $g(t) = 5t^4 + 10t^2 + 3$ | $g''(2)$ |
| **24.** $f(x) = 9 - x^2$ | $f''\left(-\sqrt{5}\right)$ |
| **25.** $f(x) = \sqrt{4 - x}$ | $f'''(-5)$ |
| **26.** $f(t) = \sqrt{2t + 3}$ | $f'''\left(\frac{1}{2}\right)$ |
| **27.** $f(x) = x^2(3x^2 + 3x - 4)$ | $f'''(-2)$ |
| **28.** $g(x) = 2x^3(x^2 - 5x + 4)$ | $g'''(0)$ |

In Exercises 29–34, find the higher-order derivative.

| Given | Derivative |
|---|---|
| **29.** $f'(x) = 2x^2$ | $f''(x)$ |
| **30.** $f''(x) = 20x^3 - 36x^2$ | $f'''(x)$ |
| **31.** $f'''(x) = (3x - 1)/x$ | $f^{(4)}(x)$ |
| **32.** $f'''(x) = 2\sqrt{x - 1}$ | $f^{(4)}(x)$ |
| **33.** $f^{(4)}(x) = (x^2 + 1)^2$ | $f^{(6)}(x)$ |
| **34.** $f''(x) = 2x^2 + 7x - 12$ | $f^{(5)}(x)$ |

In Exercises 35–42, find the second derivative and solve the equation $f''(x) = 0$.

**35.** $f(x) = x^3 - 9x^2 + 27x - 27$

**36.** $f(x) = 3x^3 - 9x + 1$

**37.** $f(x) = (x + 3)(x - 4)(x + 5)$

**38.** $f(x) = (x + 2)(x - 2)(x + 3)(x - 3)$

**39.** $f(x) = x\sqrt{x^2 - 1}$

**40.** $f(x) = x\sqrt{4 - x^2}$

**41.** $f(x) = \dfrac{x}{x^2 + 3}$

**42.** $f(x) = \dfrac{x}{x - 1}$

**43. Velocity and Acceleration**   A ball is propelled straight upward from ground level with an initial velocity of 144 feet per second.

(a) Write the position, velocity, and acceleration functions of the ball.

(b) When is the ball at its highest point? How high is this point?

(c) How fast is the ball traveling when it hits the ground? How is this speed related to the initial velocity?

**44. Velocity and Acceleration**   A brick becomes dislodged from the top of the Empire State Building (at a height of 1250 feet) and falls to the sidewalk below.

(a) Write the position, velocity, and acceleration functions of the brick.

(b) How long does it take the brick to hit the sidewalk?

(c) How fast is the brick traveling when it hits the sidewalk?

**45. Velocity and Acceleration**   The velocity (in feet per second) of an automobile starting from rest is modeled by

$$\frac{ds}{dt} = \frac{90t}{t + 10}.$$

Create a table showing the velocity and acceleration at 10-second intervals during the first minute of travel. What can you conclude?
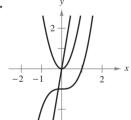
**46. Stopping Distance**   A car is traveling at a rate of 66 feet per second (45 miles per hour) when the brakes are applied. The position function for the car is given by $s = -8.25t^2 + 66t$, where $s$ is measured in feet and $t$ is measured in seconds. Create a table showing the position, velocity, and acceleration for each given value of $t$. What can you conclude?

(T) In Exercises 47 and 48, use a graphing utility to graph $f$, $f'$, and $f''$ in the same viewing window. What is the relationship among the degree of $f$ and the degrees of its successive derivatives? In general, what is the relationship among the degree of a polynomial function and the degrees of its successive derivatives?
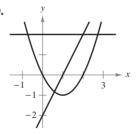
**47.** $f(x) = x^2 - 6x + 6$      **48.** $f(x) = 3x^3 - 9x$

In Exercises 49 and 50, the graphs of $f$, $f'$, and $f''$ are shown on the same set of coordinate axes. Which is which? Explain your reasoning.

**49.**



**50.**



(T) **51. Modeling Data**   The table shows the retail values $y$ (in billions of dollars) of motor homes sold in the United States for 2000 to 2005, where $t$ is the year, with $t = 0$ corresponding to 2000. *(Source: Recreation Vehicle Industry Association)*

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | 9.5 | 8.6 | 11.0 | 12.1 | 14.7 | 14.4 |

(a) Use a graphing utility to find a cubic model for the total retail value $y(t)$ of the motor homes.

(b) Use a graphing utility to graph the model and plot the data in the same viewing window. How well does the model fit the data?

(c) Find the first and second derivatives of the function.

(d) Show that the retail value of motor homes was increasing from 2001 to 2004.

(e) Find the year when the retail value was increasing at the greatest rate by solving $y''(t) = 0$.

(f) Explain the relationship among your answers for parts (c), (d), and (e).

**52. Projectile Motion**   An object is thrown upward from the top of a 64-foot building with an initial velocity of 48 feet per second.

(a) Write the position, velocity, and acceleration functions of the object.

(b) When will the object hit the ground?

(c) When is the velocity of the object zero?

(d) How high does the object go?

(T) (e) Use a graphing utility to graph the position, velocity, and acceleration functions in the same viewing window. Write a short paragraph that describes the relationship among these functions.

**True or False?**   In Exercises 53–56, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**53.** If $y = f(x)g(x)$, then $y' = f'(x)g'(x)$.

**54.** If $y = (x + 1)(x + 2)(x + 3)(x + 4)$, then $\dfrac{d^5y}{dx^5} = 0$.

**55.** If $f'(c)$ and $g'(c)$ are zero and $h(x) = f(x)g(x)$, then $h'(c) = 0$.

**56.** The second derivative represents the rate of change of the first derivative.

**57. Finding a Pattern**   Develop a general rule for $[x f(x)]^{(n)}$ where $f$ is a differentiable function of $x$.

**58. Extended Application**   To work an extended application analyzing the median prices of new privately owned U.S. homes in the South for 1980 through 2005, visit this text's website at *college.hmco.com*. *(Data Source: U.S. Census Bureau)*

# Implicit Differentiation

■ Find derivatives explicitly.
■ Find derivatives implicitly.
■ Use derivatives to answer questions about real-life situations.

## Explicit and Implicit Functions

So far in this text, most functions involving two variables have been expressed in the **explicit form** $y = f(x)$. That is, one of the two variables has been explicitly given in terms of the other. For example, in the equation

$$y = 3x - 5 \qquad \text{Explicit form}$$

the variable $y$ is explicitly written as a function of $x$. Some functions, however, are not given explicitly and are only implied by a given equation, as shown in Example 1.

### Example 1 Finding a Derivative Explicitly

Find $dy/dx$ for the equation

$$xy = 1.$$

**SOLUTION** In this equation, $y$ is **implicitly** defined as a function of $x$. One way to find $dy/dx$ is first to solve the equation for $y$, then differentiate as usual.

$$xy = 1 \qquad \text{Write original equation.}$$

$$y = \frac{1}{x} \qquad \text{Solve for } y.$$

$$= x^{-1} \qquad \text{Rewrite.}$$

$$\frac{dy}{dx} = -x^{-2} \qquad \text{Differentiate with respect to } x.$$

$$= -\frac{1}{x^2} \qquad \text{Simplify.}$$

### ✓ CHECKPOINT 1

Find $dy/dx$ for the equation $x^2y = 1$. ■

The procedure shown in Example 1 works well whenever you can easily write the given function explicitly. You cannot, however, use this procedure when you are unable to solve for $y$ as a function of $x$. For instance, how would you find $dy/dx$ in the equation

$$x^2 - 2y^3 + 4y = 2$$

where it is very difficult to express $y$ as a function of $x$ explicitly? To do this, you can use a procedure called **implicit differentiation.**

## Implicit Differentiation

To understand how to find $dy/dx$ implicitly, you must realize that the differentiation is taking place *with respect to x*. This means that when you differentiate terms involving $x$ alone, you can differentiate as usual. *But* when you differentiate terms involving $y$, you must apply the Chain Rule because you are assuming that $y$ is defined implicitly as a differentiable function of $x$. Study the next example carefully. Note in particular how the Chain Rule is used to introduce the $dy/dx$ factors in Examples 2(b) and 2(d).

**Example 2    Applying the Chain Rule**

Differentiate each expression with respect to $x$.

**a.** $3x^2$     **b.** $2y^3$     **c.** $x + 3y$     **d.** $xy^2$

**SOLUTION**

**a.** The only variable in this expression is $x$. So, to differentiate with respect to $x$, you can use the Simple Power Rule and the Constant Multiple Rule to obtain

$$\frac{d}{dx}[3x^2] = 6x.$$

**b.** This case is different. The variable in the expression is $y$, and yet you are asked to differentiate with respect to $x$. To do this, assume that $y$ is a differentiable function of $x$ and use the Chain Rule.

$$\frac{d}{dx}[2y^3] = \overset{c}{2}\ \overset{n}{(3)}\ \overset{u^{n-1}}{y^2}\ \overset{u'}{\frac{dy}{dx}} \qquad \text{Chain Rule}$$

$$= 6y^2\frac{dy}{dx}$$

**c.** This expression involves both $x$ and $y$. By the Sum Rule and the Constant Multiple Rule, you can write

$$\frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}.$$

**d.** By the Product Rule and the Chain Rule, you can write

$$\frac{d}{dx}[xy^2] = x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x] \qquad \text{Product Rule}$$

$$= x\left(2y\frac{dy}{dx}\right) + y^2(1) \qquad \text{Chain Rule}$$

$$= 2xy\frac{dy}{dx} + y^2.$$

✓**CHECKPOINT 2**

Differentiate each expression with respect to $x$.

**a.** $4x^3$     **b.** $3y^2$     **c.** $x + 5y$     **d.** $xy^3$ ◼

## Implicit Differentiation

Consider an equation involving $x$ and $y$ in which $y$ is a differentiable function of $x$. You can use the steps below to find $dy/dx$.

1. Differentiate both sides of the equation *with respect to x*.

2. Write the result so that all terms involving $dy/dx$ are on the left side of the equation and all other terms are on the right side of the equation.

3. Factor $dy/dx$ out of the terms on the left side of the equation.

4. Solve for $dy/dx$ by dividing both sides of the equation by the left-hand factor that does not contain $dy/dx$.

In Example 3, note that implicit differentiation can produce an expression for $dy/dx$ that contains both $x$ and $y$.

### Example 3  Finding the Slope of a Graph Implicitly

Find the slope of the tangent line to the ellipse given by $x^2 + 4y^2 = 4$ at the point $\left(\sqrt{2}, -1/\sqrt{2}\right)$, as shown in Figure 2.33.

**SOLUTION**

$$x^2 + 4y^2 = 4 \qquad \text{Write original equation.}$$

$$\frac{d}{dx}[x^2 + 4y^2] = \frac{d}{dx}[4] \qquad \text{Differentiate with respect to } x.$$

$$2x + 8y\left(\frac{dy}{dx}\right) = 0 \qquad \text{Implicit differentiation}$$

$$8y\left(\frac{dy}{dx}\right) = -2x \qquad \text{Subtract } 2x \text{ from each side.}$$

$$\frac{dy}{dx} = \frac{-2x}{8y} \qquad \text{Divide each side by } 8y.$$

$$\frac{dy}{dx} = -\frac{x}{4y} \qquad \text{Simplify.}$$

To find the slope at the given point, substitute $x = \sqrt{2}$ and $y = -1/\sqrt{2}$ into the derivative, as shown below.

$$-\frac{\sqrt{2}}{4\left(-1/\sqrt{2}\right)} = \frac{1}{2}$$



**FIGURE 2.33**  Slope of tangent line is $\frac{1}{2}$.

### ✓ CHECKPOINT 3
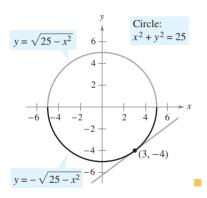
Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.



### STUDY TIP

To see the benefit of implicit differentiation, try reworking Example 3 using the explicit function

$$y = -\frac{1}{2}\sqrt{4 - x^2}.$$

The graph of this function is the lower half of the ellipse.

$y^3 + y^2 - 5y - x^2 = -4$

**FIGURE 2.34**



$2x^2 - y^2 = 1$

(1, 1)

**FIGURE 2.35**    Hyperbola

**Example 4**    Using Implicit Differentiation

Find $dy/dx$ for the equation $y^3 + y^2 - 5y - x^2 = -4$.

**SOLUTION**

$$y^3 + y^2 - 5y - x^2 = -4 \qquad \text{Write original equation.}$$

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4] \qquad \text{Differentiate with respect to } x.$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0 \qquad \text{Implicit differentiation}$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x \qquad \text{Collect } dy/dx \text{ terms.}$$

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x \qquad \text{Factor.}$$

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

The graph of the original equation is shown in Figure 2.34. What are the slopes of the graph at the points $(1, -3)$, $(2, 0)$, and $(1, 1)$?
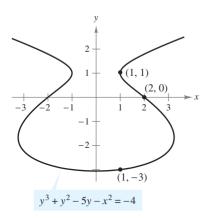
✓ **CHECKPOINT 4**

Find $dy/dx$ for the equation $y^2 + x^2 - 2y - 4x = 4$. ■

**Example 5**    Finding the Slope of a Graph Implicitly

Find the slope of the graph of $2x^2 - y^2 = 1$ at the point $(1, 1)$.

**SOLUTION**    Begin by finding $dy/dx$ implicitly.

$$2x^2 - y^2 = 1 \qquad \text{Write original equation.}$$

$$4x - 2y\left(\frac{dy}{dx}\right) = 0 \qquad \text{Differentiate with respect to } x.$$

$$-2y\left(\frac{dy}{dx}\right) = -4x \qquad \text{Subtract } 4x \text{ from each side.}$$

$$\frac{dy}{dx} = \frac{2x}{y} \qquad \text{Divide each side by } -2y.$$

At the point $(1, 1)$, the slope of the graph is

$$\frac{2(1)}{1} = 2$$

as shown in Figure 2.35. The graph is called a **hyperbola.**
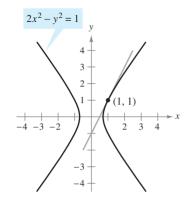
✓ **CHECKPOINT 5**

Find the slope of the graph of $x^2 - 9y^2 = 16$ at the point $(5, 1)$. ■

## Application



**Demand Function**

Price (in dollars)

Demand (in thousands of units)

**FIGURE 2.36**

**Example 6**  **Using a Demand Function** ®

The demand function for a product is modeled by

$$p = \frac{3}{0.000001x^3 + 0.01x + 1}$$

where $p$ is measured in dollars and $x$ is measured in thousands of units, as shown in Figure 2.36. Find the rate of change of the demand $x$ with respect to the price $p$ when $x = 100$.

**SOLUTION**   To simplify the differentiation, begin by rewriting the function. Then, differentiate *with respect to p*.

$$p = \frac{3}{0.000001x^3 + 0.01x + 1}$$

$$0.000001x^3 + 0.01x + 1 = \frac{3}{p}$$

$$0.000003x^2 \frac{dx}{dp} + 0.01\frac{dx}{dp} = -\frac{3}{p^2}$$

$$(0.000003x^2 + 0.01)\frac{dx}{dp} = -\frac{3}{p^2}$$

$$\frac{dx}{dp} = -\frac{3}{p^2(0.000003x^2 + 0.01)}$$

When $x = 100$, the price is

$$p = \frac{3}{0.000001(100)^3 + 0.01(100) + 1} = \$1.$$

So, when $x = 100$ and $p = 1$, the rate of change of the demand with respect to the price is

$$-\frac{3}{(1)^2[0.000003(100)^2 + 0.01]} = -75.$$

This means that when $x = 100$, the demand is dropping at the rate of 75 thousand units for each dollar increase in price.

**✓ CHECKPOINT 6**

The demand function for a product is given by

$$p = \frac{2}{0.001x^2 + x + 1}.$$

Find $dx/dp$ implicitly. ■

**CONCEPT CHECK**

1. Complete the following: The equation $x + y = 1$ is written in _____ form and the equation $y = 1 - x$ is written in _____ form.

2. Complete the following: When you are asked to find $dy/dt$, you are being asked to find the derivative of _____ with respect to _____.

3. Describe the difference between the explicit form of a function and an implicit equation. Give an example of each.

4. In your own words, state the guidelines for implicit differentiation.

In Exercises 1–6, solve the equation for $y$.

**1.** $x - \dfrac{y}{x} = 2$

**2.** $\dfrac{4}{x - 3} = \dfrac{1}{y}$

**3.** $xy - x + 6y = 6$

**4.** $12 + 3y = 4x^2 + x^2 y$

**5.** $x^2 + y^2 = 5$

**6.** $x = \pm\sqrt{6 - y^2}$

In Exercises 7–10, evaluate the expression at the given point.

**7.** $\dfrac{3x^2 - 4}{3y^2}$,　$(2, 1)$

**8.** $\dfrac{x^2 - 2}{1 - y}$,　$(0, -3)$

**9.** $\dfrac{5x}{3y^2 - 12y + 5}$,　$(-1, 2)$

**10.** $\dfrac{1}{y^2 - 2xy + x^2}$,　$(4, 3)$

## Exercises 2.7

In Exercises 1–12, find $dy/dx$.

**1.** $xy = 4$

**2.** $3x^2 - y = 8x$

**3.** $y^2 = 1 - x^2,\ 0 \le x \le 1$

**4.** $4x^2 y - \dfrac{3}{y} = 0$

**5.** $x^2 y^2 - 2x = 3$

**6.** $xy^2 + 4xy = 10$

**7.** $4y^2 - xy = 2$

**8.** $2xy^3 - x^2 y = 2$

**9.** $\dfrac{2y - x}{y^2 - 3} = 5$

**10.** $\dfrac{xy - y^2}{y - x} = 1$

**11.** $\dfrac{x + y}{2x - y} = 1$

**12.** $\dfrac{2x + y}{x - 5y} = 1$

In Exercises 13–24, find $dy/dx$ by implicit differentiation and evaluate the derivative at the given point.

|  | *Equation* | *Point* |
|---|---|---|
| **13.** | $x^2 + y^2 = 16$ | $(0, 4)$ |
| **14.** | $x^2 - y^2 = 25$ | $(5, 0)$ |
| **15.** | $y + xy = 4$ | $(-5, -1)$ |
| **16.** | $x^3 - y^2 = 0$ | $(1, 1)$ |
| **17.** | $x^3 - xy + y^2 = 4$ | $(0, -2)$ |
| **18.** | $x^2 y + y^2 x = -2$ | $(2, -1)$ |
| **19.** | $x^3 y^3 - y = x$ | $(0, 0)$ |
| **20.** | $x^3 + y^3 = 2xy$ | $(1, 1)$ |
| **21.** | $x^{1/2} + y^{1/2} = 9$ | $(16, 25)$ |
| **22.** | $\sqrt{xy} = x - 2y$ | $(4, 1)$ |
| **23.** | $x^{2/3} + y^{2/3} = 5$ | $(8, 1)$ |
| **24.** | $(x + y)^3 = x^3 + y^3$ | $(-1, 1)$ |
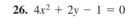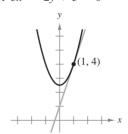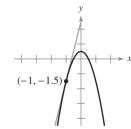
In Exercises 25–30, find the slope of the graph at the given point.

**25.** $3x^2 - 2y + 5 = 0$



**26.** $4x^2 + 2y - 1 = 0$



**27.** $x^2 + y^2 = 4$



**28.** $4x^2 + y^2 = 4$



**29.** $4x^2 + 9y^2 = 36$



**30.** $x^2 - y^3 = 0$
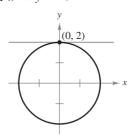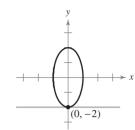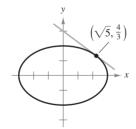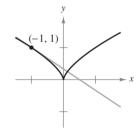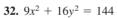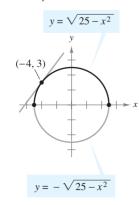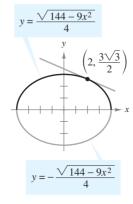
In Exercises 31–34, find $dy/dx$ implicitly and explicitly (the explicit functions are shown on the graph) and show that the results are equivalent. Use the graph to estimate the slope of the tangent line at the labeled point. Then verify your result analytically by evaluating $dy/dx$ at the point.

**31.** $x^2 + y^2 = 25$

**32.** $9x^2 + 16y^2 = 144$



$y = \sqrt{25 - x^2}$

$(-4, 3)$

$y = -\sqrt{25 - x^2}$

$y = \dfrac{\sqrt{144 - 9x^2}}{4}$

$\left(2, \dfrac{3\sqrt{3}}{2}\right)$

$y = -\dfrac{\sqrt{144 - 9x^2}}{4}$

**33.** $x - y^2 - 1 = 0$

**34.** $4y^2 - x^2 = 7$



$y = \sqrt{x - 1}$

$(2, -1)$

$y = -\sqrt{x - 1}$

$y = \dfrac{\sqrt{x^2 + 7}}{2}$

$(3, 2)$

$y = -\dfrac{\sqrt{x^2 + 7}}{2}$
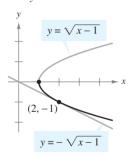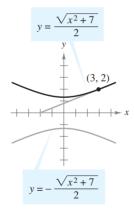
**T** In Exercises 35–40, find equations of the tangent lines to the graph at the given points. Use a graphing utility to graph the equation and the tangent lines in the same viewing window.

| *Equation* | *Points* |
|---|---|
| **35.** $x^2 + y^2 = 100$ | $(8, 6)$ and $(-6, 8)$ |
| **36.** $x^2 + y^2 = 9$ | $(0, 3)$ and $\left(2, \sqrt{5}\right)$ |
| **37.** $y^2 = 5x^3$ | $\left(1, \sqrt{5}\right)$ and $\left(1, -\sqrt{5}\right)$ |
| **38.** $4xy + x^2 = 5$ | $(1, 1)$ and $(5, -1)$ |
| **39.** $x^3 + y^3 = 8$ | $(0, 2)$ and $(2, 0)$ |
| **40.** $y^2 = \dfrac{x^3}{4 - x}$ | $(2, 2)$ and $(2, -2)$ |

**Demand** In Exercises 41–44, find the rate of change of $x$ with respect to $p$.

**41.** $p = \dfrac{2}{0.00001x^3 + 0.1x}$   $x \geq 0$
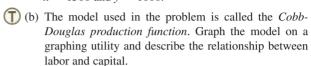
**42.** $p = \dfrac{4}{0.000001x^2 + 0.05x + 1}$   $x \geq 0$

**43.** $p = \sqrt{\dfrac{200 - x}{2x}}$,   $0 < x \leq 200$

**44.** $p = \sqrt{\dfrac{500 - x}{2x}}$,   $0 < x \leq 500$

**45. Production** Let $x$ represent the units of labor and $y$ the capital invested in a manufacturing process. When 135,540 units are produced, the relationship between labor and capital can be modeled by $100x^{0.75}y^{0.25} = 135,540$.

   (a) Find the rate of change of $y$ with respect to $x$ when $x = 1500$ and $y = 1000$.

   **T** (b) The model used in the problem is called the *Cobb-Douglas production function*. Graph the model on a graphing utility and describe the relationship between labor and capital.

**46. Production** Repeat Exercise 45(a) by finding the rate of change of $y$ with respect to $x$ when $x = 3000$ and $y = 125$.

**47. Health: U.S. HIV/AIDS Epidemic** The numbers (in thousands) of cases $y$ of HIV/AIDS reported in the years 2001 through 2005 can be modeled by

$$y^2 - 1141.6 = 24.9099t^3 - 183.045t^2 + 452.79t$$

where $t$ represents the year, with $t = 1$ corresponding to 2001. *(Source: U.S. Centers for Disease Control and Prevention)*

   **T** (a) Use a graphing utility to graph the model and describe the results.

   (b) Use the graph to estimate the year during which the number of reported cases was increasing at the greatest rate.

   (c) Complete the table to estimate the year during which the number of reported cases was increasing at the greatest rate. Compare this estimate with your answer in part (b).

| $t$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | | | | | |
| $y'$ | | | | | |

**Section 2.8**
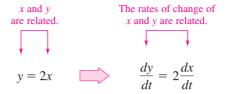
**Related Rates**

■ Examine related variables.
■ Solve related-rate problems.

## Related Variables

In this section, you will study problems involving variables that are changing with respect to time. If two or more such variables are related to each other, then their rates of change with respect to time are also related.

For instance, suppose that $x$ and $y$ are related by the equation $y = 2x$. If both variables are changing with respect to time, then their rates of change will also be related.

$x$ and $y$ are related.

The rates of change of $x$ and $y$ are related.

$$y = 2x \qquad \Longrightarrow \qquad \frac{dy}{dt} = 2\frac{dx}{dt}$$

In this simple example, you can see that because $y$ always has twice the value of $x$, it follows that the rate of change of $y$ with respect to time is always twice the rate of change of $x$ with respect to time.

**Example 1**    **Examining Two Rates That Are Related**

The variables $x$ and $y$ are differentiable functions of $t$ and are related by the equation

$$y = x^2 + 3.$$

When $x = 1$, $dx/dt = 2$. Find $dy/dt$ when $x = 1$.

**SOLUTION**    Use the Chain Rule to differentiate both sides of the equation with respect to $t$.

$$y = x^2 + 3 \qquad \text{Write original equation.}$$

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2 + 3] \qquad \text{Differentiate with respect to } t.$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt} \qquad \text{Apply Chain Rule.}$$

When $x = 1$ and $dx/dt = 2$, you have

$$\frac{dy}{dt} = 2(1)(2)$$

$$= 4.$$

**✓CHECKPOINT 1**

When $x = 1$, $dx/dt = 3$. Find $dy/dt$ when $x = 1$ if $y = x^3 + 2$. ■

## Solving Related-Rate Problems

In Example 1, you were *given* the mathematical model.

*Given equation:* $y = x^2 + 3$

*Given rate:* $\dfrac{dx}{dt} = 2$ when $x = 1$

*Find:* $\dfrac{dy}{dt}$ when $x = 1$

In the next example, you are asked to *create* a similar mathematical model.

**Example 2**  **Changing Area**  Ⓡ

A pebble is dropped into a calm pool of water, causing ripples in the form of concentric circles, as shown in the photo. The radius $r$ of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area $A$ of the disturbed water changing?

**SOLUTION**  The variables $r$ and $A$ are related by the equation for the area of a circle, $A = \pi r^2$. To solve this problem, use the fact that the rate of change of the radius is given by $dr/dt$.

*Equation:* $A = \pi r^2$

*Given rate:* $\dfrac{dr}{dt} = 1$ when $r = 4$

*Find:* $\dfrac{dA}{dt}$ when $r = 4$

Using this model, you can proceed as in Example 1.

$A = \pi r^2$      Write original equation.

$\dfrac{d}{dt}[A] = \dfrac{d}{dt}[\pi r^2]$    Differentiate with respect to *t*.

$\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$     Apply Chain Rule.

When $r = 4$ and $dr/dt = 1$, you have

$\dfrac{dA}{dt} = 2\pi(4)(1) = 8\pi$    Substitute 4 for *r* and 1 for *dr/dt*.

When the radius is 4 feet, the area is changing at a rate of $8\pi$ square feet per second.



© Randy Faris/Corbis

Total area increases as the outer radius increases.

✓**CHECKPOINT 2**

If the radius $r$ of the outer ripple in Example 2 is increasing at a rate of 2 feet per second, at what rate is the total area changing when the radius is 3 feet? ■

---

**STUDY TIP**

In Example 2, note that the radius changes at a *constant* rate ($dr/dt = 1$ for all *t*), but the area changes at a *nonconstant* rate.

| When $r = 1$ ft | When $r = 2$ ft | When $r = 3$ ft | When $r = 4$ ft |
|---|---|---|---|
| $\dfrac{dA}{dt} = 2\pi$ ft²/sec | $\dfrac{dA}{dt} = 4\pi$ ft²/sec | $\dfrac{dA}{dt} = 6\pi$ ft²/sec | $\dfrac{dA}{dt} = 8\pi$ ft²/sec |

The solution shown in Example 2 illustrates the steps for solving a related-rate problem.

**Guidelines for Solving a Related-Rate Problem**

1. Identify all *given* quantities and all quantities *to be determined*. If possible, make a sketch and label the quantities.

2. Write an equation that relates all variables whose rates of change are either given or to be determined.

3. Use the Chain Rule to differentiate both sides of the equation *with respect to time*.

4. Substitute into the resulting equation all known values of the variables and their rates of change. Then solve for the required rate of change.

**STUDY TIP**

Be sure you notice the order of Steps 3 and 4 in the guidelines. Do not substitute the known values for the variables until after you have differentiated.

In Step 2 of the guidelines, note that you must write an equation that relates the given variables. To help you with this step, reference tables that summarize many common formulas are included in the appendices. For instance, the volume of a sphere of radius $r$ is given by the formula

$$V = \frac{4}{3}\pi r^3$$
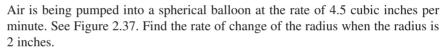
as listed in Appendix D.

The table below shows the mathematical models for some common rates of change that can be used in the first step of the solution of a related-rate problem.

| Verbal statement | Mathematical model |
|---|---|
| The velocity of a car after traveling for 1 hour is 50 miles per hour. | $x$ = distance traveled <br> $\dfrac{dx}{dt} = 50$ when $t = 1$ |
| Water is being pumped into a swimming pool at the rate of 10 cubic feet per minute. | $V$ = volume of water in pool <br> $\dfrac{dV}{dt} = 10$ ft$^3$/min |
| A population of bacteria is increasing at the rate of 2000 per hour. | $x$ = number in population <br> $\dfrac{dx}{dt} = 2000$ bacteria per hour |
| Revenue is increasing at the rate of $4000 per month. | $R$ = revenue <br> $\dfrac{dR}{dt} = 4000$ dollars per month |

*Example 3*  **Changing Volume**  ®

Air is being pumped into a spherical balloon at the rate of 4.5 cubic inches per minute. See Figure 2.37. Find the rate of change of the radius when the radius is 2 inches.

**SOLUTION**   Let $V$ represent the volume of the balloon and let $r$ represent the radius. Because the volume is increasing at the rate of 4.5 cubic inches per minute, you know that $dV/dt = 4.5$. An equation that relates $V$ and $r$ is $V = \frac{4}{3}\pi r^3$. So, the problem can be represented by the model shown below.

$$\text{Equation: } V = \frac{4}{3}\pi r^3$$

$$\text{Given rate: } \frac{dV}{dt} = 4.5$$

$$\text{Find: } \frac{dr}{dt} \text{ when } r = 2$$

By differentiating the equation, you obtain

$$V = \frac{4}{3}\pi r^3 \qquad \text{Write original equation.}$$

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] \qquad \text{Differentiate with respect to } t.$$

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} \qquad \text{Apply Chain Rule.}$$

$$\frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{dr}{dt}. \qquad \text{Solve for } dr/dt.$$

When $r = 2$ and $dV/dt = 4.5$, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{4\pi(2^2)}(4.5)$$

$$\approx 0.09 \text{ inch per minute.}$$

In Example 3, note that the volume is increasing at a *constant rate* but the radius is increasing at a *variable* rate. In this particular example, the radius is increasing more and more slowly as $t$ increases. This is illustrated in the table below.

| $t$ | 1 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|
| $V = 4.5t$ | 4.5 | 13.5 | 22.5 | 31.5 | 40.5 | 49.5 |
| $t = \sqrt[3]{\dfrac{3V}{4\pi}}$ | 1.02 | 1.48 | 1.75 | 1.96 | 2.13 | 2.28 |
| $\dfrac{dr}{dt}$ | 0.34 | 0.16 | 0.12 | 0.09 | 0.08 | 0.07 |

**FIGURE 2.37**   Expanding Balloon

✓ **CHECKPOINT 3**

If the radius of a spherical balloon increases at a rate of 1.5 inches per minute, find the rate at which the surface area changes when the radius is 6 inches. (Formula for surface area of a sphere: $S = 4\pi r^2$) ■

*Example 4*  **Analyzing a Profit Function**   Ⓡ

A company's profit $P$ (in dollars) from selling $x$ units of a product can be modeled by

$$P = 500x - \left(\frac{1}{4}\right)x^2.$$     <span style="color:magenta">Model for profit</span>

The sales are increasing at a rate of 10 units per day. Find the rate of change in the profit (in dollars per day) when 500 units have been sold.

**SOLUTION**   Because you are asked to find the rate of change in dollars per day, you should differentiate the given equation with respect to the time $t$.

$$P = 500x - \left(\frac{1}{4}\right)x^2$$     <span style="color:magenta">Write model for profit.</span>

$$\frac{dP}{dt} = 500\left(\frac{dx}{dt}\right) - 2\left(\frac{1}{4}\right)(x)\left(\frac{dx}{dt}\right)$$     <span style="color:magenta">Differentiate with respect to $t$.</span>

The sales are increasing at a constant rate of 10 units per day, so

$$\frac{dx}{dt} = 10.$$

When $x = 500$ units and $dx/dt = 10$, the rate of change in the profit is

$$\frac{dP}{dt} = 500(10) - 2\left(\frac{1}{4}\right)(500)(10)$$

$$= 5000 - 2500$$

$$= \$2500 \text{ per day.}$$     <span style="color:magenta">Simplify.</span>

The graph of the profit function (in terms of $x$) is shown in Figure 2.38.

**STUDY TIP**

In Example 4, note that one of the keys to successful use of calculus in applied problems is the interpretation of a rate of change as a derivative.
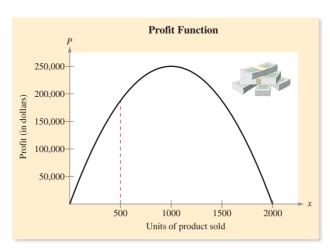


**FIGURE 2.38**

✓**CHECKPOINT 4**

Find the rate of change in profit (in dollars per day) when 50 units have been sold, sales have increased at a rate of 10 units per day, and $P = 200x - \frac{1}{2}x^2$.  ∎

*Example 5*
MAKE A DECISION  **Increasing Production**  (R)

A company is increasing the production of a product at the rate of 200 units per week. The weekly demand function is modeled by

$$p = 100 - 0.001x$$

where $p$ is the price per unit and $x$ is the number of units produced in a week. Find the rate of change of the revenue with respect to time when the weekly production is 2000 units. Will the rate of change of the revenue be greater than $20,000 per week?

**SOLUTION**

*Equation:* $R = xp = x(100 - 0.001x) = 100x - 0.001x^2$

*Given rate:* $\dfrac{dx}{dt} = 200$

*Find:* $\dfrac{dR}{dt}$ when $x = 2000$

By differentiating the equation, you obtain

$R = 100x - 0.001x^2$   Write original equation.

$\dfrac{d}{dt}[R] = \dfrac{d}{dt}[100x - 0.001x^2]$   Differentiate with respect to $t$.

$\dfrac{dR}{dt} = (100 - 0.002x)\dfrac{dx}{dt}.$   Apply Chain Rule.

Using $x = 2000$ and $dx/dt = 200$, you have

$\dfrac{dR}{dt} = [100 - 0.002(2000)](200)$

$= \$19,200$ per week.

No, the rate of change of the revenue will not be greater than $20,000 per week.

✓**CHECKPOINT 5**

Find the rate of change of revenue with respect to time for the company in Example 5 if the weekly demand function is

$$p = 150 - 0.002x. \ \blacksquare$$

**CONCEPT CHECK**

1. Complete the following. Two variables $x$ and $y$ are changing with respect to _____. If $x$ and $y$ are related to each other, then their rates of change with respect to time are also _____.

2. The volume $V$ of an object is a differentiable function of time $t$. Describe what $dV/dt$ represents.

3. The area $A$ of an object is a differentiable function of time $t$. Describe what $dA/dt$ represents.

4. In your own words, state the guidelines for solving related-rate problems.

---

**Skills Review 2.8**    The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 2.7.

In Exercises 1–6, write a formula for the given quantity.

**1.** Area of a circle

**2.** Volume of a sphere

**3.** Surface area of a cube

**4.** Volume of a cube

**5.** Volume of a cone

**6.** Area of a triangle

In Exercises 7–10, find $dy/dx$ by implicit differentiation.

**7.** $x^2 + y^2 = 9$

**8.** $3xy - x^2 = 6$

**9.** $x^2 + 2y + xy = 12$

**10.** $x + xy^2 - y^2 = xy$

---

## Exercises 2.8

In Exercises 1–4, use the given values to find $dy/dt$ and $dx/dt$.

| Equation | Find | Given |
|---|---|---|
| **1.** $y = \sqrt{x}$ | (a) $\dfrac{dy}{dt}$ | $x = 4, \dfrac{dx}{dt} = 3$ |
| | (b) $\dfrac{dx}{dt}$ | $x = 25, \dfrac{dy}{dt} = 2$ |
| **2.** $y = 2(x^2 - 3x)$ | (a) $\dfrac{dy}{dt}$ | $x = 3, \dfrac{dx}{dt} = 2$ |
| | (b) $\dfrac{dx}{dt}$ | $x = 1, \dfrac{dy}{dt} = 5$ |
| **3.** $xy = 4$ | (a) $\dfrac{dy}{dt}$ | $x = 8, \dfrac{dx}{dt} = 10$ |
| | (b) $\dfrac{dx}{dt}$ | $x = 1, \dfrac{dy}{dt} = -6$ |
| **4.** $x^2 + y^2 = 25$ | (a) $\dfrac{dy}{dt}$ | $x = 3, y = 4, \dfrac{dx}{dt} = 8$ |
| | (b) $\dfrac{dx}{dt}$ | $x = 4, y = 3, \dfrac{dy}{dt} = -2$ |

**5. Area**   The radius $r$ of a circle is increasing at a rate of 3 inches per minute. Find the rates of change of the area when (a) $r = 6$ inches and (b) $r = 24$ inches.

**6. Volume**   The radius $r$ of a sphere is increasing at a rate of 3 inches per minute. Find the rates of change of the volume when (a) $r = 6$ inches and (b) $r = 24$ inches.

**7. Area**   Let $A$ be the area of a circle of radius $r$ that is changing with respect to time. If $dr/dt$ is constant, is $dA/dt$ constant? Explain your reasoning.

**8. Volume**   Let $V$ be the volume of a sphere of radius $r$ that is changing with respect to time. If $dr/dt$ is constant, is $dV/dt$ constant? Explain your reasoning.

**9. Volume**   A spherical balloon is inflated with gas at a rate of 10 cubic feet per minute. How fast is the radius of the balloon changing at the instant the radius is (a) 1 foot and (b) 2 feet?

**10. Volume**   The radius $r$ of a right circular cone is increasing at a rate of 2 inches per minute. The height $h$ of the cone is related to the radius by $h = 3r$. Find the rates of change of the volume when (a) $r = 6$ inches and (b) $r = 24$ inches.

**11. Cost, Revenue, and Profit**   A company that manufactures sport supplements calculates that its costs and revenue can be modeled by the equations

$$C = 125{,}000 + 0.75x \quad \text{and} \quad R = 250x - \frac{1}{10}x^2$$

where $x$ is the number of units of sport supplements produced in 1 week. If production in one particular week is 1000 units and is increasing at a rate of 150 units per week, find:

(a) the rate at which the cost is changing.

(b) the rate at which the revenue is changing.

(c) the rate at which the profit is changing.

**12. Cost, Revenue, and Profit**   A company that manufactures pet toys calculates that its costs and revenue can be modeled by the equations

$$C = 75{,}000 + 1.05x \quad \text{and} \quad R = 500x - \frac{x^2}{25}$$

where $x$ is the number of toys produced in 1 week. If production in one particular week is 5000 toys and is increasing at a rate of 250 toys per week, find:

(a) the rate at which the cost is changing.

(b) the rate at which the revenue is changing.

(c) the rate at which the profit is changing.

**13. Volume**   All edges of a cube are expanding at a rate of 3 centimeters per second. How fast is the volume changing when each edge is (a) 1 centimeter and (b) 10 centimeters?

**14. Surface Area**   All edges of a cube are expanding at a rate of 3 centimeters per second. How fast is the surface area changing when each edge is (a) 1 centimeter and (b) 10 centimeters?

**15. Moving Point**   A point is moving along the graph of $y = x^2$ such that $dx/dt$ is 2 centimeters per minute. Find $dy/dt$ for each value of $x$.

(a) $x = -3$   (b) $x = 0$   (c) $x = 1$   (d) $x = 3$

**16. Moving Point**   A point is moving along the graph of $y = 1/(1 + x^2)$ such that $dx/dt$ is 2 centimeters per minute. Find $dy/dt$ for each value of $x$.

(a) $x = -2$   (b) $x = 2$   (c) $x = 0$   (d) $x = 10$

**17. Moving Ladder**   A 25-foot ladder is leaning against a house (see figure). The base of the ladder is pulled away from the house at a rate of 2 feet per second. How fast is the top of the ladder moving down the wall when the base is (a) 7 feet, (b) 15 feet, and (c) 24 feet from the house?



Figure for 17           Figure for 18

**18. Boating**   A boat is pulled by a winch on a dock, and the winch is 12 feet above the deck of the boat (see figure). The winch pulls the rope at a rate of 4 feet per second. Find the speed of the boat when 13 feet of rope is out. What happens to the speed of the boat as it gets closer and closer to the dock?

**19. Air Traffic Control**   An air traffic controller spots two airplanes at the same altitude converging to a point as they fly at right angles to each other. One airplane is 150 miles from the point and has a speed of 450 miles per hour. The other is 200 miles from the point and has a speed of 600 miles per hour.

(a) At what rate is the distance between the planes changing?

(b) How much time does the controller have to get one of the airplanes on a different flight path?

**20. Air Traffic Control**   An airplane flying at an altitude of 6 miles passes directly over a radar antenna (see figure). When the airplane is 10 miles away ($s = 10$), the radar detects that the distance $s$ is changing at a rate of 240 miles per hour. What is the speed of the airplane?
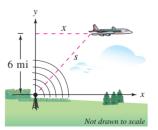


Figure for 20           Figure for 21

**21. Baseball**   A (square) baseball diamond has sides that are 90 feet long (see figure). A player 26 feet from third base is running at a speed of 30 feet per second. At what rate is the player's distance from home plate changing?

**22. Advertising Costs**   A retail sporting goods store estimates that weekly sales $S$ and weekly advertising costs $x$ are related by the equation $S = 2250 + 50x + 0.35x^2$. The current weekly advertising costs are \$1500, and these costs are increasing at a rate of \$125 per week. Find the current rate of change of weekly sales.

**23. Environment**   An accident at an oil drilling platform is causing a circular oil slick. The slick is 0.08 foot thick, and when the radius of the slick is 150 feet, the radius is increasing at the rate of 0.5 foot per minute. At what rate (in cubic feet per minute) is oil flowing from the site of the accident?

**Ⓣ 24. Profit**   A company is increasing the production of a product at the rate of 25 units per week. The demand and cost functions for the product are given by $p = 50 - 0.01x$ and $C = 4000 + 40x - 0.02x^2$. Find the rate of change of the profit with respect to time when the weekly sales are $x = 800$ units. Use a graphing utility to graph the profit function, and use the *zoom* and *trace* features of the graphing utility to verify your result.

**25. Sales**   The profit for a product is increasing at a rate of \$5600 per week. The demand and cost functions for the product are given by $p = 6000 - 25x$ and $C = 2400x + 5200$. Find the rate of change of sales with respect to time when the weekly sales are $x = 44$ units.

**Ⓣ 26. Cost**   The annual cost (in millions of dollars) for a government agency to seize $p\%$ of an illegal drug is given by

$$C = \frac{528p}{100 - p}, \quad 0 \le p < 100.$$

The agency's goal is to increase $p$ by 5% per year. Find the rates of change of the cost when (a) $p = 30\%$ and (b) $p = 60\%$. Use a graphing utility to graph $C$. What happens to the graph of $C$ as $p$ approaches 100?

# Algebra Review

## Simplifying Algebraic Expressions

To be successful in using derivatives, you must be good at simplifying algebraic expressions. Here are some helpful simplification techniques.

1. Combine *like terms*. This may involve expanding an expression by multiplying factors.

2. Divide out *like factors* in the numerator and denominator of an expression.

3. Factor an expression.

4. Rationalize a denominator.

5. Add, subtract, multiply, or divide fractions.

### *Example 1*   Simplifying a Fractional Expression

**a.** $\dfrac{(x + \Delta x)^2 - x^2}{\Delta x} = \dfrac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x}$    Expand expression.

$= \dfrac{2x(\Delta x) + (\Delta x)^2}{\Delta x}$    Combine like terms.

$= \dfrac{\Delta x(2x + \Delta x)}{\Delta x}$    Factor.

$= 2x + \Delta x, \qquad \Delta x \neq 0$    Divide out like factors.

**b.** $\dfrac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2)}{(x^2 - 1)^2}$

$= \dfrac{(-2x^2 - 2x^3 + 2 + 2x) - (6 - 4x - 2x^2)}{(x^2 - 1)^2}$    Expand expression.

$= \dfrac{-2x^2 - 2x^3 + 2 + 2x - 6 + 4x + 2x^2}{(x^2 - 1)^2}$    Remove parentheses.

$= \dfrac{-2x^3 + 6x - 4}{(x^2 - 1)^2}$    Combine like terms.

**c.** $2\left(\dfrac{2x + 1}{3x}\right)\left[\dfrac{3x(2) - (2x + 1)(3)}{(3x)^2}\right]$

$= 2\left(\dfrac{2x + 1}{3x}\right)\left[\dfrac{6x - (6x + 3)}{(3x)^2}\right]$    Multiply factors.

$= \dfrac{2(2x + 1)(6x - 6x - 3)}{(3x)^3}$    Multiply fractions and remove parentheses.

$= \dfrac{2(2x + 1)(-3)}{3(9)x^3}$    Combine like terms and factor.

$= \dfrac{-2(2x + 1)}{9x^3}$    Divide out like factors.

### Example 2   Simplifying an Expression with Powers or Radicals

**a.** $(2x + 1)^2(6x + 1) + (3x^2 + x)(2)(2x + 1)(2)$

$= (2x + 1)[(2x + 1)(6x + 1) + (3x^2 + x)(2)(2)]$     Factor.

$= (2x + 1)[12x^2 + 8x + 1 + (12x^2 + 4x)]$     Multiply factors.

$= (2x + 1)(12x^2 + 8x + 1 + 12x^2 + 4x)$     Remove parentheses.

$= (2x + 1)(24x^2 + 12x + 1)$     Combine like terms.

**b.** $(-1)(6x^2 - 4x)^{-2}(12x - 4)$

$= \dfrac{(-1)(12x - 4)}{(6x^2 - 4x)^2}$     Rewrite as a fraction.

$= \dfrac{(-1)(4)(3x - 1)}{(6x^2 - 4x)^2}$     Factor.

$= \dfrac{-4(3x - 1)}{(6x^2 - 4x)^2}$     Multiply factors.

**c.** $(x)\left(\dfrac{1}{2}\right)(2x + 3)^{-1/2} + (2x + 3)^{1/2}(1)$

$= (2x + 3)^{-1/2}\left(\dfrac{1}{2}\right)[x + (2x + 3)(2)]$     Factor.

$= \dfrac{x + 4x + 6}{(2x + 3)^{1/2}(2)}$     Rewrite as a fraction.

$= \dfrac{5x + 6}{2(2x + 3)^{1/2}}$     Combine like terms.

**d.** $\dfrac{x^2\left(\frac{1}{2}\right)(2x)(x^2 + 1)^{-1/2} - (x^2 + 1)^{1/2}(2x)}{x^4}$

$= \dfrac{(x^3)(x^2 + 1)^{-1/2} - (x^2 + 1)^{1/2}(2x)}{x^4}$     Multiply factors.

$= \dfrac{(x^2 + 1)^{-1/2}(x)[x^2 - (x^2 + 1)(2)]}{x^4}$     Factor.

$= \dfrac{x[x^2 - (2x^2 + 2)]}{(x^2 + 1)^{1/2}x^4}$     Write with positive exponents.

$= \dfrac{x^2 - 2x^2 - 2}{(x^2 + 1)^{1/2}x^3}$     Divide out like factors and remove parentheses.

$= \dfrac{-x^2 - 2}{(x^2 + 1)^{1/2}x^3}$     Combine like terms.

All but one of the expressions in this Algebra Review are derivatives. Can you see what the original function is? Explain your reasoning.

# Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 200.
Answers to odd-numbered Review Exercises are given in the back of the text.*

| Section 2.1 | Review Exercises |
|---|---|
| ■ Approximate the slope of the tangent line to a graph at a point. | *1–4* |
| ■ Interpret the slope of a graph in a real-life setting. | *5–8* |
| ■ Use the limit definition to find the derivative of a function and the slope of a graph at a point. | *9–16* |

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

| | |
|---|---|
| ■ Use the derivative to find the slope of a graph at a point. | *17–24* |
| ■ Use the graph of a function to recognize points at which the function is not differentiable. | *25–28* |

| Section 2.2 | |
|---|---|
| ■ Use the Constant Multiple Rule for differentiation. | *29, 30* |

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

| | |
|---|---|
| ■ Use the Sum and Difference Rules for differentiation. | *31–38* |

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

| Section 2.3 | |
|---|---|
| ■ Find the average rate of change of a function over an interval and the instantaneous rate of change at a point. | *39, 40* |

$$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$

$$\text{Instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

| | |
|---|---|
| ■ Find the average and instantaneous rates of change of a quantity in a real-life problem. | *41–44* |
| ■ Find the velocity of an object that is moving in a straight line. | *45, 46* |
| ■ Create mathematical models for the revenue, cost, and profit for a product. | *47, 48* |

$$P = R - C, \quad R = xp$$

| | |
|---|---|
| ■ Find the marginal revenue, marginal cost, and marginal profit for a product. | *49–58* |

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* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

## Section 2.4                                                                    Review Exercises

- Use the Product Rule for differentiation.                                         *59–62*

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

- Use the Quotient Rule for differentiation.                                        *63, 64*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

## Section 2.5

- Use the General Power Rule for differentiation.                                   *65–68*

$$\frac{d}{dx}[u^n] = nu^{n-1}u'$$

- Use differentiation rules efficiently to find the derivative of any algebraic function,   *69–78*
  then simplify the result.
- Use derivatives to answer questions about real-life situations.   (Sections 2.1–2.5)   *79, 80*

## Section 2.6

- Find higher-order derivatives.                                                    *81–88*
- Find and use the position function to determine the velocity and acceleration of a   *89, 90*
  moving object.

## Section 2.7

- Find derivatives implicitly.                                                      *91–98*

## Section 2.8

- Solve related-rate problems.                                                      *99, 100*

## Study Strategies

- **Simplify Your Derivatives**   Often our students ask if they have to simplify their derivatives. Our answer is "Yes, if you expect to use them." In the next chapter, you will see that almost all applications of derivatives require that the derivatives be written in simplified form. It is not difficult to see the advantage of a derivative in simplified form. Consider, for instance, the derivative of

$$f(x) = x/\sqrt{x^2 + 1}.$$

  The "raw form" produced by the Quotient and Chain Rules
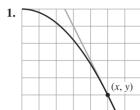
$$f'(x) = \frac{(x^2 + 1)^{1/2}(1) - (x)\left(\frac{1}{2}\right)(x^2 + 1)^{-1/2}(2x)}{\left(\sqrt{x^2 + 1}\right)^2}$$
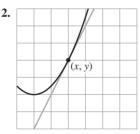
  is obviously much more difficult to use than the simplified form
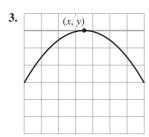
$$f'(x) = \frac{1}{(x^2 + 1)^{3/2}}.$$
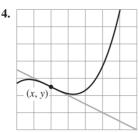
- **List Units of Measure in Applied Problems**   When using derivatives in real-life applications, be sure to list the units of measure for each variable. For instance, if $R$ is measured in dollars and $t$ is measured in years, then the derivative $dR/dt$ is measured in dollars per year.

## Review Exercises

In Exercises 1–4, approximate the slope of the tangent line to the graph at $(x, y)$.

**1.**

**2.**

**3.**

**4.**

**5. Sales**  The graph approximates the annual sales $S$ (in millions of dollars per year) of Home Depot for the years 1999 through 2005, where $t$ is the year, with $t = 9$ corresponding to 1999. Estimate the slopes of the graph when $t = 10$, $t = 13$, and $t = 15$. Interpret each slope in the context of the problem.  *(Source: The Home Depot, Inc.)*

**Home Depot Sales**

**6. Consumer Trends**  The graph approximates the number of subscribers $S$ (in millions per year) of cellular telephones for the years 1996 through 2005, where $t$ is the year, with $t = 6$ corresponding to 1996. Estimate the slopes of the graph when $t = 7$, $t = 11$, and $t = 15$. Interpret each slope in the context of the problem.  *(Source: Cellular Telecommunications & Internet Association)*

**Cellular Phone Subscribers**
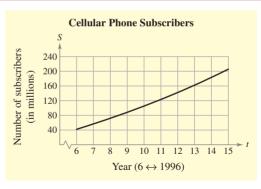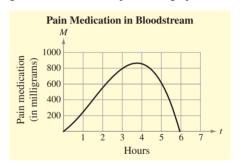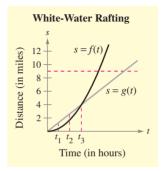
Figure for 6

**7. Medicine**  The graph shows the estimated number of milligrams of a pain medication $M$ in the bloodstream $t$ hours after a 1000-milligram dose of the drug has been given. Estimate the slopes of the graph at $t = 0, 4$, and 6.

**Pain Medication in Bloodstream**

**8. White-Water Rafting**  Two white-water rafters leave a campsite simultaneously and start downstream on a 9-mile trip. Their distances from the campsite are given by $s = f(t)$ and $s = g(t)$, where $s$ is measured in miles and $t$ is measured in hours.

**White-Water Rafting**

(a) Which rafter is traveling at a greater rate at $t_1$?

(b) What can you conclude about their rates at $t_2$?

(c) What can you conclude about their rates at $t_3$?

(d) Which rafter finishes the trip first? Explain your reasoning.

In Exercises 9–16, use the limit definition to find the derivative of the function. Then use the limit definition to find the slope of the tangent line to the graph of $f$ at the given point.

9. $f(x) = -3x - 5$; $(-2, 1)$    10. $f(x) = 7x + 3$; $(-1, 4)$

11. $f(x) = x^2 - 4x$; $(1, -3)$    12. $f(x) = x^2 + 10$; $(2, 14)$

13. $f(x) = \sqrt{x + 9}$; $(-5, 2)$    14. $f(x) = \sqrt{x - 1}$; $(10, 3)$

15. $f(x) = \dfrac{1}{x - 5}$; $(6, 1)$    16. $f(x) = \dfrac{1}{x + 4}$; $(-3, 1)$

In Exercises 17–24, find the slope of the graph of $f$ at the given point.

17. $f(x) = 5 - 3x$; $(1, -2)$    18. $f(x) = 1 - 4x$; $(2, -7)$

19. $f(x) = -\frac{1}{2}x^2 + 2x$; $(2, 2)$    20. $f(x) = 4 - x^2$; $(-1, 3)$

21. $f(x) = \sqrt{x} + 2$; $(9, 5)$    22. $f(x) = 2\sqrt{x} + 1$; $(4, 5)$
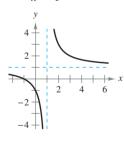
23. $f(x) = \dfrac{5}{x}$; $(1, 5)$    24. $f(x) = \dfrac{2}{x} - 1$; $\left(\dfrac{1}{2}, 3\right)$
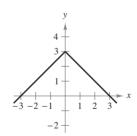
In Exercises 25–28, determine the $x$-value at which the function is not differentiable.

25. $y = \dfrac{x + 1}{x - 1}$    26. $y = -|x| + 3$
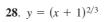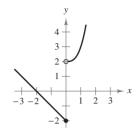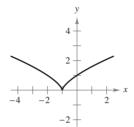


27. $y = \begin{cases} -x - 2, & x \le 0 \\ x^3 + 2, & x > 0 \end{cases}$    28. $y = (x + 1)^{2/3}$



(T) In Exercises 29–38, find the equation of the tangent line at the given point. Then use a graphing utility to graph the function and the equation of the tangent line in the same viewing window.

29. $g(t) = \dfrac{2}{3t^2}$, $\left(1, \dfrac{2}{3}\right)$

30. $h(x) = \dfrac{2}{(3x)^2}$, $\left(2, \dfrac{1}{18}\right)$

31. $f(x) = x^2 + 3$, $(1, 4)$

32. $f(x) = 2x^2 - 3x + 1$, $(2, 3)$

33. $y = 11x^4 - 5x^2 + 1$, $(-1, 7)$

34. $y = x^3 - 5 + \dfrac{3}{x^3}$, $(-1, -9)$

35. $f(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}}$, $(1, 0)$

36. $f(x) = 2x^{-3} + 4 - \sqrt{x}$, $(1, 5)$

37. $f(x) = \dfrac{x^2 + 3}{x}$, $(1, 4)$

38. $f(x) = -x^2 - 4x - 4$, $(-4, -4)$

In Exercises 39 and 40, find the average rate of change of the function over the indicated interval. Then compare the average rate of change with the instantaneous rates of change at the endpoints of the interval.

39. $f(x) = x^2 + 3x - 4$; $[0, 1]$

40. $f(x) = x^3 + x$; $[-2, 2]$

41. **Sales**  The annual sales $S$ (in millions of dollars per year) of Home Depot for the years 1999 through 2005 can be modeled by

$$S = 123.833t^3 - 4319.55t^2 + 56{,}278.0t - 208{,}517$$

where $t$ is the time in years, with $t = 9$ corresponding to 1999. A graph of this model appears in Exercise 5. *(Source: The Home Depot, Inc.)*

(a) Find the average rate of change for the interval from 1999 through 2005.

(b) Find the instantaneous rates of change of the model for 1999 and 2005.

(c) Interpret the results of parts (a) and (b) in the context of the problem.

42. **Consumer Trends**  The numbers of subscribers $S$ (in millions per year) of cellular telephones for the years 1996 through 2005 can be modeled by

$$S = \dfrac{-33.2166 + 11.6732t}{1 - 0.0207t}$$

where $t$ is the time in years, with $t = 6$ corresponding to 1996. A graph of this model appears in Exercise 6. *(Source: Cellular Telecommunications & Internet Association)*

(a) Find the average rate of change for the interval from 2000 through 2005.

(b) Find the instantaneous rates of change of the model for 2000 and 2005.

(c) Interpret the results of parts (a) and (b) in the context of the problem.

**43. Retail Price** The average retail price $P$ (in dollars) of a half-gallon of prepackaged ice cream from 1992 through 2006 can be modeled by the equation

$$P = -0.00149t^3 + 0.0340t^2 - 0.086t + 2.53$$

where $t$ is the year, with $t = 2$ corresponding to 1992. *(Source: U.S. Bureau of Labor Statistics)*

(a) Find the rate of change of the price with respect to the year.

(b) At what rate was the price of a half gallon of prepackaged ice cream changing in 1997? in 2003? in 2005?

(T) (c) Use a graphing utility to graph the function for $2 \le t \le 16$. During which years was the price increasing? decreasing?

(d) For what years do the slopes of the tangent lines appear to be positive? negative?

(e) Compare your answers for parts (c) and (d).

**44. Recycling** The amount $T$ of recycled paper products in millions of tons from 1997 through 2005 can be modeled by the equation

$$T = \sqrt{1.3150t^3 - 42.747t^2 + 522.28t - 885.2}$$

where $t$ is the year, with $t = 7$ corresponding to 1997. *(Source: Franklin Associates, Ltd.)*

(T) (a) Use a graphing utility to graph the equation. Be sure to choose an appropriate window.

(b) Determine $dT/dt$. Evaluate $dT/dt$ for 1997, 2002, and 2005.

(c) Is $dT/dt$ positive for $t \ge 7$? Does this agree with the graph of the function? What does this tell you about this situation? Explain your reasoning.

**45. Velocity** A rock is dropped from a tower on the Brooklyn Bridge, 276 feet above the East River. Let $t$ represent the time in seconds.

(a) Write a model for the position function (assume that air resistance is negligible).

(b) Find the average velocity during the first 2 seconds.

(c) Find the instantaneous velocities when $t = 2$ and $t = 3$.

(d) How long will it take for the rock to hit the water?

(e) When it hits the water, what is the rock's speed?

**46. Velocity** The straight-line distance $s$ (in feet) traveled by an accelerating bicyclist can be modeled by

$$s = 2t^{3/2}, \quad 0 \le t \le 8$$

where $t$ is the time (in seconds). Complete the table, showing the velocity of the bicyclist at two-second intervals.

| Time, $t$ | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| Velocity | | | | | |

**47. Cost, Revenue, and Profit** The fixed cost of operating a small flower shop is $2500 per month. The average cost of a floral arrangement is $15 and the average price is $27.50. Write the monthly revenue, cost, and profit functions for the floral shop in terms of $x$, the number of arrangements sold.

**48. Profit** The weekly demand and cost functions for a product are given by

$$p = 1.89 - 0.0083x \quad \text{and} \quad C = 21 + 0.65x.$$

Write the profit function for this product.

**Marginal Cost** In Exercises 49–52, find the marginal cost function.

**49.** $C = 2500 + 320x$      **50.** $C = 225x + 4500$

**51.** $C = 370 + 2.55\sqrt{x}$      **52.** $C = 475 + 5.25x^{2/3}$

**Marginal Revenue** In Exercises 53–56, find the marginal revenue function.

**53.** $R = 200x - \frac{1}{5}x^2$      **54.** $R = 150x - \frac{3}{4}x^2$

**55.** $R = \frac{35x}{\sqrt{x-2}}, \quad x \ge 6$      **56.** $R = x\left(5 + \frac{10}{\sqrt{x}}\right)$

**Marginal Profit** In Exercises 57 and 58, find the marginal profit function.

**57.** $P = -0.0002x^3 + 6x^2 - x - 2000$

**58.** $P = -\frac{1}{15}x^3 + 4000x^2 - 120x - 144,000$

In Exercises 59–78, find the derivative of the function. Simplify your result. State which differentiation rule(s) you used to find the derivative.

**59.** $f(x) = x^3(5 - 3x^2)$      **60.** $y = (3x^2 + 7)(x^2 - 2x)$

**61.** $y = (4x - 3)(x^3 - 2x^2)$      **62.** $s = \left(4 - \frac{1}{t^2}\right)(t^2 - 3t)$

**63.** $f(x) = \frac{6x - 5}{x^2 + 1}$      **64.** $f(x) = \frac{x^2 + x - 1}{x^2 - 1}$

**65.** $f(x) = (5x^2 + 2)^3$      **66.** $f(x) = \sqrt[3]{x^2 - 1}$

**67.** $h(x) = \frac{2}{\sqrt{x+1}}$

**68.** $g(x) = \sqrt{x^6 - 12x^3 + 9}$

**69.** $g(x) = x\sqrt{x^2 + 1}$

**70.** $g(t) = \frac{t}{(1 - t)^3}$

**71.** $f(x) = x(1 - 4x^2)^2$

**72.** $f(x) = \left(x^2 + \frac{1}{x}\right)^5$

**73.** $h(x) = [x^2(2x + 3)]^3$

**74.** $f(x) = [(x - 2)(x + 4)]^2$

**75.** $f(x) = x^2(x - 1)^5$

**76.** $f(s) = s^3(s^2 - 1)^{5/2}$

**77.** $h(t) = \dfrac{\sqrt{3t + 1}}{(1 - 3t)^2}$

**78.** $g(x) = \dfrac{(3x + 1)^2}{(x^2 + 1)^2}$

**79. Physical Science**  The temperature $T$ (in degrees Fahrenheit) of food placed in a freezer can be modeled by

$$T = \dfrac{1300}{t^2 + 2t + 25}$$

where $t$ is the time (in hours).

(a) Find the rates of change of $T$ when $t = 1$, $t = 3$, $t = 5$, and $t = 10$.

(T) (b) Graph the model on a graphing utility and describe the rate at which the temperature is changing.

**80. Forestry**  According to the *Doyle Log Rule*, the volume $V$ (in board-feet) of a log of length $L$ (feet) and diameter $D$ (inches) at the small end is

$$V = \left(\dfrac{D - 4}{4}\right)^2 L.$$

Find the rates at which the volume is changing with respect to $D$ for a 12-foot-long log whose smallest diameter is (a) 8 inches, (b) 16 inches, (c) 24 inches, and (d) 36 inches.

In Exercises 81–88, find the higher-order derivative.

**81.** Given $f(x) = 3x^2 + 7x + 1$, find $f''(x)$.

**82.** Given $f'(x) = 5x^4 - 6x^2 + 2x$, find $f'''(x)$.

**83.** Given $f'''(x) = -\dfrac{6}{x^4}$, find $f^{(5)}(x)$.

**84.** Given $f(x) = \sqrt{x}$, find $f^{(4)}(x)$.

**85.** Given $f'(x) = 7x^{5/2}$, find $f''(x)$.

**86.** Given $f(x) = x^2 + \dfrac{3}{x}$, find $f''(x)$.

**87.** Given $f''(x) = 6\sqrt[3]{x}$, find $f'''(x)$.

**88.** Given $f'''(x) = 20x^4 - \dfrac{2}{x^3}$, find $f^{(5)}(x)$.

**89. Athletics**  A person dives from a 30-foot platform with an initial velocity of 5 feet per second (upward).

(a) Find the position function of the diver.

(b) How long will it take for the diver to hit the water?

(c) What is the diver's velocity at impact?

(d) What is the diver's acceleration at impact?

**90. Velocity and Acceleration**  The position function of a particle is given by

$$s = \dfrac{1}{t^2 + 2t + 1}$$
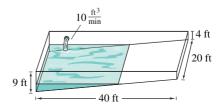
where $s$ is the height (in feet) and $t$ is the time (in seconds). Find the velocity and acceleration functions.

In Exercises 91–94, use implicit differentiation to find $dy/dx$.

**91.** $x^2 + 3xy + y^3 = 10$

**92.** $x^2 + 9xy + y^2 = 0$

**93.** $y^2 - x^2 + 8x - 9y - 1 = 0$

**94.** $y^2 + x^2 - 6y - 2x - 5 = 0$

In Exercises 95–98, use implicit differentiation to find an equation of the tangent line at the given point.

|  | *Equation* | *Point* |
|---|---|---|
| **95.** | $y^2 = x - y$ | $(2, 1)$ |
| **96.** | $2\sqrt[3]{x} + 3\sqrt{y} = 10$ | $(8, 4)$ |
| **97.** | $y^2 - 2x = xy$ | $(1, 2)$ |
| **98.** | $y^3 - 2x^2 y + 3xy^2 = -1$ | $(0, -1)$ |

**99. Water Level**  A swimming pool is 40 feet long, 20 feet wide, 4 feet deep at the shallow end, and 9 feet deep at the deep end (see figure). Water is being pumped into the pool at the rate of 10 cubic feet per minute. How fast is the water level rising when there is 4 feet of water in the deep end?



**100. Profit**  The demand and cost functions for a product can be modeled by

$$p = 211 - 0.002x$$

and

$$C = 30x + 1,500,000$$

where $x$ is the number of units produced.

(a) Write the profit function for this product.

(b) Find the marginal profit when 80,000 units are produced.

(T) (c) Graph the profit function on a graphing utility and use the graph to determine the price you would charge for the product. Explain your reasoning.

## Chapter Test

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1 and 2, use the limit definition to find the derivative of the function. Then find the slope of the tangent line to the graph of *f* at the given point.

**1.** $f(x) = x^2 + 1;\ (2, 5)$

**2.** $f(x) = \sqrt{x} - 2;\ (4, 0)$

In Exercises 3–11, find the derivative of the function. Simplify your result.

**3.** $f(t) = t^3 + 2t$

**4.** $f(x) = 4x^2 - 8x + 1$

**5.** $f(x) = x^{3/2}$

**6.** $f(x) = (x + 3)(x - 3)$

**7.** $f(x) = -3x^{-3}$

**8.** $f(x) = \sqrt{x}(5 + x)$

**9.** $f(x) = (3x^2 + 4)^2$

**10.** $f(x) = \sqrt{1 - 2x}$

**11.** $f(x) = \dfrac{(5x - 1)^3}{x}$

(T) **12.** Find an equation of the tangent line to the graph of $f(x) = x - \dfrac{1}{x}$ at the point $(1, 0)$. Then use a graphing utility to graph the function and the tangent line in the same viewing window.

**13.** The annual sales *S* (in millions of dollars per year) of Bausch & Lomb for the years 1999 through 2005 can be modeled by

$$S = -2.9667t^3 + 135.008t^2 - 1824.42t + 9426.3,\ 9 \le t \le 15$$

where *t* represents the year, with $t = 9$ corresponding to 1999.    *(Source: Bausch & Lomb, Inc.)*

(a) Find the average rate of change for the interval from 2001 through 2005.

(b) Find the instantaneous rates of change of the model for 2001 and 2005.

(c) Interpret the results of parts (a) and (b) in the context of the problem.

**14.** The monthly demand and cost functions for a product are given by

$$p = 1700 - 0.016x \qquad \text{and} \qquad C = 715{,}000 + 240x.$$

Write the profit function for this product.

In Exercises 15–17, find the third derivative of the function. Simplify your result.

**15.** $f(x) = 2x^2 + 3x + 1$

**16.** $f(x) = \sqrt{3 - x}$

**17.** $f(x) = \dfrac{2x + 1}{2x - 1}$

In Exercises 18–20, use implicit differentiation to find *dy/dx*.

**18.** $x + xy = 6$

**19.** $y^2 + 2x - 2y + 1 = 0$

**20.** $x^2 - 2y^2 = 4$

**21.** The radius *r* of a right circular cylinder is increasing at a rate of 0.25 centimeter per minute. The height *h* of the cylinder is related to the radius by $h = 20r$. Find the rate of change of the volume when (a) $r = 0.5$ centimeter and (b) $r = 1$ centimeter.

# Applications of the Derivative

<div style="text-align:right">

**3**

</div>



Still Images/Getty Images

Designers use the derivative to find the dimensions of a container that will minimize cost. (See Section 3.4, Exercise 28.)

## Applications

Derivatives have many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Profit Analysis, Exercise 43, page 214
- Phishing, Exercise 75, page 234
- Average Cost, Exercises 61 and 62, page 265
- Make a Decision: Social Security, Exercise 55, page 274
- Economics: Gross Domestic Product, Exercise 41, page 282

# Increasing and Decreasing Functions

■ Test for increasing and decreasing functions.
■ Find the critical numbers of functions and find the open intervals on which functions are increasing or decreasing.
■ Use increasing and decreasing functions to model and solve real-life problems.

## Increasing and Decreasing Functions

A function is **increasing** if its graph moves up as $x$ moves to the right and **decreasing** if its graph moves down as $x$ moves to the right. The following definition states this more formally.

---

### Definition of Increasing and Decreasing Functions

A function $f$ is **increasing** on an interval if for any two numbers $x_1$ and $x_2$ in the interval

$$x_2 > x_1 \quad \text{implies} \quad f(x_2) > f(x_1).$$

A function $f$ is **decreasing** on an interval if for any two numbers $x_1$ and $x_2$ in the interval

$$x_2 > x_1 \quad \text{implies} \quad f(x_2) < f(x_1).$$

---

The function in Figure 3.1 is decreasing on the interval $(-\infty, a)$, constant on the interval $(a, b)$, and increasing on the interval $(b, \infty)$. Actually, from the definition of increasing and decreasing functions, the function shown in Figure 3.1 is decreasing on the interval $(-\infty, a]$ and increasing on the interval $[b, \infty)$. This text restricts the discussion to finding *open* intervals on which a function is increasing or decreasing.

The derivative of a function can be used to determine whether the function is increasing or decreasing on an interval.
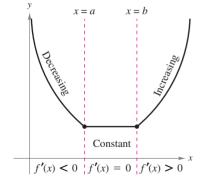
**FIGURE 3.1**

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### Test for Increasing and Decreasing Functions

Let $f$ be differentiable on the interval $(a, b)$.

**1.** If $f'(x) > 0$ for all $x$ in $(a, b)$, then $f$ is increasing on $(a, b)$.

**2.** If $f'(x) < 0$ for all $x$ in $(a, b)$, then $f$ is decreasing on $(a, b)$.

**3.** If $f'(x) = 0$ for all $x$ in $(a, b)$, then $f$ is constant on $(a, b)$.

---

### STUDY TIP

The conclusions in the first two cases of testing for increasing and decreasing functions are valid even if $f'(x) = 0$ at a finite number of $x$-values in $(a, b)$.

$f(x) = x^2$

$f'(x) < 0$     $f'(x) > 0$

$(-\infty, 0)$   $(0, \infty)$
Decreasing   Increasing
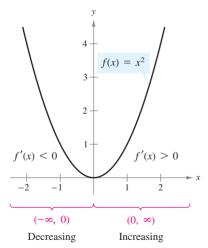
**FIGURE 3.2**

---

**DISCOVERY**

Use a graphing utility to graph $f(x) = 2 - x^2$ and $f'(x) = -2x$ in the same viewing window. On what interval is $f$ increasing? On what interval is $f'$ positive? Describe how the first derivative can be used to determine where a function is increasing and decreasing. Repeat this analysis for $g(x) = x^3 - x$ and $g'(x) = 3x^2 - 1$.

---

✓**CHECKPOINT 2**

From 1995 through 2004, the consumption $W$ of bottled water in the United States (in gallons per person per year) can be modeled by

$$W = 0.058t^2 + 0.19t + 9.2,$$

$$5 \le t \le 14$$

where $t = 5$ corresponds to 1995. Show that the consumption of bottled water was increasing from 1995 to 2004.   *(Source: U.S. Department of Agriculture)* ∎

---

*Example 1*   **Testing for Increasing and Decreasing Functions**

Show that the function

$$f(x) = x^2$$

is decreasing on the open interval $(-\infty, 0)$ and increasing on the open interval $(0, \infty)$.

**SOLUTION**   The derivative of $f$ is

$$f'(x) = 2x.$$

On the open interval $(-\infty, 0)$, the fact that $x$ is negative implies that $f'(x) = 2x$ is also negative. So, by the test for a decreasing function, you can conclude that $f$ is *decreasing* on this interval. Similarly, on the open interval $(0, \infty)$, the fact that $x$ is positive implies that $f'(x) = 2x$ is also positive. So, it follows that $f$ is *increasing* on this interval, as shown in Figure 3.2.

✓**CHECKPOINT 1**

Show that the function $f(x) = x^4$ is decreasing on the open interval $(-\infty, 0)$ and increasing on the open interval $(0, \infty)$. ∎

*Example 2*   **Modeling Consumption**   ®

From 1997 through 2004, the consumption $C$ of Italian cheeses in the United States (in pounds per person per year) can be modeled by

$$C = -0.0333t^2 + 0.996t + 5.40, \quad 7 \le t \le 14$$

where $t = 7$ corresponds to 1997 (see Figure 3.3). Show that the consumption of Italian cheeses was increasing from 1997 to 2004.   *(Source: U.S. Department of Agriculture)*

**SOLUTION**   The derivative of this model is $dC/dt = -0.0666t + 0.996$. For the open interval $(7, 14)$, the derivative is positive. So, the function is increasing, which implies that the consumption of Italian cheeses was increasing during the given time period.
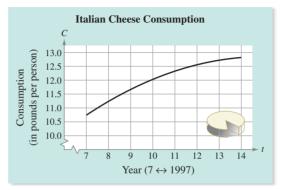


**FIGURE 3.3**

## Critical Numbers and Their Use

In Example 1, you were given two intervals: one on which the function was decreasing and one on which it was increasing. Suppose you had been asked to determine these intervals. To do this, you could have used the fact that for a continuous function, $f'(x)$ can change signs only at $x$-values where $f'(x) = 0$ or at $x$-values where $f'(x)$ is undefined, as shown in Figure 3.4. These two types of numbers are called the **critical numbers** of $f$.
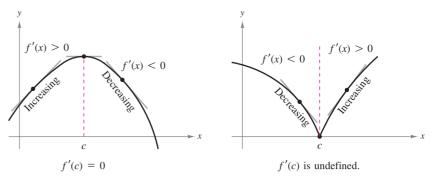


**FIGURE 3.4**

---

**Definition of Critical Number**

If $f$ is defined at $c$, then $c$ is a critical number of $f$ if $f'(c) = 0$ or if $f'(c)$ is undefined.

---

**STUDY TIP**

This definition requires that a critical number be in the domain of the function. For example, $x = 0$ is not a critical number of the function $f(x) = 1/x$.

---

To determine the intervals on which a continuous function is increasing or decreasing, you can use the guidelines below.

---

**Guidelines for Applying Increasing/Decreasing Test**

1. Find the derivative of $f$.

2. Locate the critical numbers of $f$ and use these numbers to determine test intervals. That is, find all $x$ for which $f'(x) = 0$ or $f'(x)$ is undefined.

3. Test the sign of $f'(x)$ at an arbitrary number in each of the test intervals.

4. Use the test for increasing and decreasing functions to decide whether $f$ is increasing or decreasing on each interval.

*Example 3*   **Finding Increasing and Decreasing Intervals**

Find the open intervals on which the function is increasing or decreasing.

$$f(x) = x^3 - \frac{3}{2}x^2$$

**SOLUTION**   Begin by finding the derivative of $f$. Then set the derivative equal to zero and solve for the critical numbers.

$$f'(x) = 3x^2 - 3x$$   Differentiate original function.

$$3x^2 - 3x = 0$$   Set derivative equal to 0.

$$3(x)(x - 1) = 0$$   Factor.

$$x = 0, x = 1$$   Critical numbers

Because there are no $x$-values for which $f'$ is undefined, it follows that $x = 0$ and $x = 1$ are the *only* critical numbers. So, the intervals that need to be tested are $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. The table summarizes the testing of these three intervals.

$f(x) = x^3 - \frac{3}{2}x^2$

**FIGURE 3.5**

| Interval | $-\infty < x < 0$ | $0 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Test value | $x = -1$ | $x = \frac{1}{2}$ | $x = 2$ |
| Sign of $f'(x)$ | $f'(-1) = 6 > 0$ | $f'\left(\frac{1}{2}\right) = -\frac{3}{4} < 0$ | $f'(2) = 6 > 0$ |
| Conclusion | Increasing | Decreasing | Increasing |

✓**CHECKPOINT 3**

Find the open intervals on which the function $f(x) = x^3 - 12x$ is increasing or decreasing. ■

The graph of $f$ is shown in Figure 3.5. Note that the test values in the intervals were chosen for convenience—other $x$-values could have been used.

**TECHNOLOGY**

Ⓣ You can use the *trace* feature of a graphing utility to confirm the result of Example 3. Begin by graphing the function, as shown at the right. Then activate the *trace* feature and move the cursor from left to right. In intervals on which the function is increasing, note that the $y$-values increase as the $x$-values increase, whereas in intervals on which the function is decreasing, the $y$-values decrease as the $x$-values increase.*

On this interval, the $y$-values increase as the $x$-values increase.

On this interval, the $y$-values decrease as the $x$-values increase.

On this interval, the $y$-values increase as the $x$-values increase.

*Specific calculator keystroke instructions for operations in this and other technology boxes can be found at *college.hmco.com/info/larsonapplied*.

Not only is the function in Example 3 continuous on the entire real line, it is also differentiable there. For such functions, the only critical numbers are those for which $f'(x) = 0$. The next example considers a continuous function that has *both* types of critical numbers—those for which $f'(x) = 0$ and those for which $f'(x)$ is undefined.

*Example 4*   **Finding Increasing and Decreasing Intervals**

Find the open intervals on which the function

$$f(x) = (x^2 - 4)^{2/3}$$

is increasing or decreasing.

**SOLUTION**   Begin by finding the derivative of the function.

$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3}(2x) \qquad \text{Differentiate.}$$

$$= \frac{4x}{3(x^2 - 4)^{1/3}} \qquad \text{Simplify.}$$

From this, you can see that the derivative is zero when $x = 0$ and the derivative is undefined when $x = \pm 2$. So, the critical numbers are

$$x = -2, \quad x = 0, \quad \text{and} \quad x = 2. \qquad \text{Critical numbers}$$

This implies that the test intervals are

$$(-\infty, -2), \quad (-2, 0), \quad (0, 2), \quad \text{and} \quad (2, \infty). \qquad \text{Test intervals}$$

The table summarizes the testing of these four intervals, and the graph of the function is shown in Figure 3.6.



FIGURE 3.6

| Interval | $-\infty < x < -2$ | $-2 < x < 0$ | $0 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|---|
| Test value | $x = -3$ | $x = -1$ | $x = 1$ | $x = 3$ |
| Sign of $f'(x)$ | $f'(-3) < 0$ | $f'(-1) > 0$ | $f'(1) < 0$ | $f'(3) > 0$ |
| Conclusion | Decreasing | Increasing | Decreasing | Increasing |
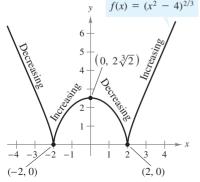
✓**CHECKPOINT 4**

Find the open intervals on which the function $f(x) = x^{2/3}$ is increasing or decreasing. ▪

**STUDY TIP**

To test the intervals in the table, it is not necessary to *evaluate* $f'(x)$ at each test value—you only need to determine its sign. For example, you can determine the sign of $f'(-3)$ as shown.

$$f'(-3) = \frac{4(-3)}{3(9 - 4)^{1/3}} = \frac{\text{negative}}{\text{positive}} = \text{negative}$$

The functions in Examples 1 through 4 are continuous on the entire real line. If there are isolated $x$-values at which a function is not continuous, then these $x$-values should be used along with the critical numbers to determine the test intervals. For example, the function

$$f(x) = \frac{x^4 + 1}{x^2}$$

is not continuous when $x = 0$. Because the derivative of $f$

$$f'(x) = \frac{2(x^4 - 1)}{x^3}$$

is zero when $x = \pm 1$, you should use the following numbers to determine the test intervals.

$x = -1, x = 1$     Critical numbers

$x = 0$     Discontinuity

After testing $f'(x)$, you can determine that the function is decreasing on the intervals $(-\infty, -1)$ and $(0, 1)$, and increasing on the intervals $(-1, 0)$ and $(1, \infty)$, as shown in Figure 3.7.

The converse of the test for increasing and decreasing functions is *not* true. For instance, it is possible for a function to be increasing on an interval even though its derivative is not positive at every point in the interval.



$f(x) = \dfrac{x^4 + 1}{x^2}$
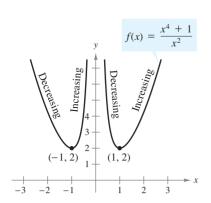
**FIGURE 3.7**

### Example 5  Testing an Increasing Function

Show that

$$f(x) = x^3 - 3x^2 + 3x$$

is increasing on the entire real line.

**SOLUTION**    From the derivative of $f$

$$f'(x) = 3x^2 - 6x + 3 = 3(x - 1)^2$$

you can see that the only critical number is $x = 1$. So, the test intervals are $(-\infty, 1)$ and $(1, \infty)$. The table summarizes the testing of these two intervals. From Figure 3.8, you can see that $f$ is increasing on the entire real line, even though $f'(1) = 0$. To convince yourself of this, look back at the definition of an increasing function.



$f(x) = x^3 - 3x^2 + 3x$

**FIGURE 3.8**

| Interval | $-\infty < x < 1$ | $1 < x < \infty$ |
|---|---|---|
| Test value | $x = 0$ | $x = 2$ |
| Sign of $f'(x)$ | $f'(0) = 3(-1)^2 > 0$ | $f'(2) = 3(1)^2 > 0$ |
| Conclusion | Increasing | Increasing |

### ✓ CHECKPOINT 5

Show that $f(x) = -x^3 + 2$ is decreasing on the entire real line.  ■

## Application

**Example 6**  **Profit Analysis**  ®

A national toy distributor determines the cost and revenue models for one of its games.

$$C = 2.4x - 0.0002x^2, \quad 0 \le x \le 6000$$
$$R = 7.2x - 0.001x^2, \quad 0 \le x \le 6000$$

Determine the interval on which the profit function is increasing.

**SOLUTION**  The profit for producing $x$ games is

$$P = R - C$$
$$= (7.2x - 0.001x^2) - (2.4x - 0.0002x^2)$$
$$= 4.8x - 0.0008x^2.$$

To find the interval on which the profit is increasing, set the marginal profit $P'$ equal to zero and solve for $x$.

| | |
|---|---|
| $P' = 4.8 - 0.0016x$ | Differentiate profit function. |
| $4.8 - 0.0016x = 0$ | Set $P'$ equal to 0. |
| $-0.0016x = -4.8$ | Subtract 4.8 from each side. |
| $x = \dfrac{-4.8}{-0.0016}$ | Divide each side by $-0.0016$. |
| $x = 3000$ games | Simplify. |

On the interval $(0, 3000)$, $P'$ is positive and the profit is *increasing*. On the interval $(3000, 6000)$, $P'$ is negative and the profit is *decreasing*. The graphs of the cost, revenue, and profit functions are shown in Figure 3.9.



**Profit Analysis**
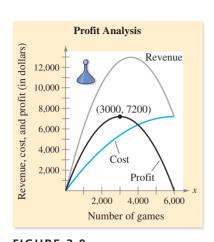
**FIGURE 3.9**

✓ **CHECKPOINT 6**

A national distributor of pet toys determines the cost and revenue functions for one of its toys.

$$C = 1.2x - 0.0001x^2, \quad 0 \le x \le 6000$$
$$R = 3.6x - 0.0005x^2, \quad 0 \le x \le 6000$$

Determine the interval on which the profit function is increasing.  ■

┌─ **CONCEPT CHECK** ─────────────────────────────

1. Write a verbal description of (a) the graph of an increasing function and (b) the graph of a decreasing function.

2. Complete the following: If $f'(x) > 0$ for all $x$ in $(a, b)$, then $f$ is _____ on $(a, b)$. [Assume $f$ is differentiable on $(a, b)$.]

3. If $f$ is defined at $c$, under what condition(s) is $c$ a critical number of $f$?

4. In your own words, state the guidelines for determining the intervals on which a continuous function is increasing or decreasing.

| Skills Review 3.1 | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 1.4. |

In Exercises 1–4, solve the equation.

**1.** $x^2 = 8x$

**2.** $15x = \dfrac{5}{8}x^2$

**3.** $\dfrac{x^2 - 25}{x^3} = 0$

**4.** $\dfrac{2x}{\sqrt{1 - x^2}} = 0$

In Exercises 5–8, find the domain of the expression.

**5.** $\dfrac{x + 3}{x - 3}$

**6.** $\dfrac{2}{\sqrt{1 - x}}$

**7.** $\dfrac{2x + 1}{x^2 - 3x - 10}$

**8.** $\dfrac{3x}{\sqrt{9 - 3x^2}}$

In Exercises 9–12, evaluate the expression when $x = -2$, 0, and 2.

**9.** $-2(x + 1)(x - 1)$

**10.** $4(2x + 1)(2x - 1)$

**11.** $\dfrac{2x + 1}{(x - 1)^2}$
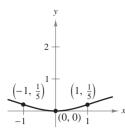
**12.** $\dfrac{-2(x + 1)}{(x - 4)^2}$

## Exercises 3.1
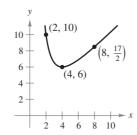
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, evaluate the derivative of the function at the indicated points on the graph.

**1.** $f(x) = \dfrac{x^2}{x^2 + 4}$
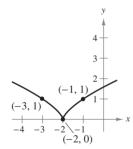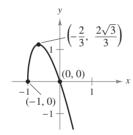


**2.** $f(x) = x + \dfrac{32}{x^2}$



**3.** $f(x) = (x + 2)^{2/3}$
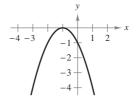


**4.** $f(x) = -3x\sqrt{x + 1}$
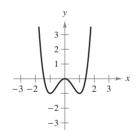


In Exercises 5–8, use the derivative to identify the open intervals on which the function is increasing or decreasing. Verify your result with the graph of the function.

**5.** $f(x) = -(x + 1)^2$



**6.** $f(x) = \dfrac{x^3}{4} - 3x$



**7.** $f(x) = x^4 - 2x^2$



**8.** $f(x) = \dfrac{x^2}{x + 1}$
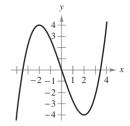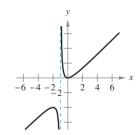
(T) In Exercises 9–32, find the critical numbers and the open intervals on which the function is increasing or decreasing. Then use a graphing utility to graph the function.

**9.** $f(x) = 2x - 3$

**10.** $f(x) = 5 - 3x$

**11.** $g(x) = -(x - 1)^2$

**12.** $g(x) = (x + 2)^2$

**13.** $y = x^2 - 6x$

**14.** $y = -x^2 + 2x$

**15.** $y = x^3 - 6x^2$

**16.** $y = (x - 2)^3$

**17.** $f(x) = \sqrt{x^2 - 1}$

**18.** $f(x) = \sqrt{9 - x^2}$

**19.** $y = x^{1/3} + 1$

**20.** $y = x^{2/3} - 4$

**21.** $g(x) = (x - 1)^{1/3}$

**22.** $g(x) = (x - 1)^{2/3}$

**23.** $f(x) = -2x^2 + 4x + 3$

**24.** $f(x) = x^2 + 8x + 10$

**25.** $y = 3x^3 + 12x^2 + 15x$

**26.** $y = x^3 - 3x + 2$

**27.** $f(x) = x\sqrt{x + 1}$

**28.** $h(x) = x\sqrt[3]{x - 1}$

**29.** $f(x) = x^4 - 2x^3$

**30.** $f(x) = \frac{1}{4}x^4 - 2x^2$

**31.** $f(x) = \dfrac{x}{x^2 + 4}$

**32.** $f(x) = \dfrac{x^2}{x^2 + 4}$

In Exercises 33–38, find the critical numbers and the open intervals on which the function is increasing or decreasing. (*Hint:* Check for discontinuities.) Sketch the graph of the function.

**33.** $f(x) = \dfrac{2x}{16 - x^2}$

**34.** $f(x) = \dfrac{x}{x + 1}$

**35.** $y = \begin{cases} 4 - x^2, & x \le 0 \\ -2x, & x > 0 \end{cases}$

**36.** $y = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$

**37.** $y = \begin{cases} 3x + 1, & x \le 1 \\ 5 - x^2, & x > 1 \end{cases}$

**38.** $y = \begin{cases} -x^3 + 1, & x \le 0 \\ -x^2 + 2x, & x > 0 \end{cases}$
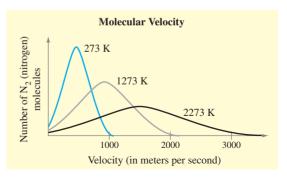
(T) **39. Cost**   The ordering and transportation cost $C$ (in hundreds of dollars) for an automobile dealership is modeled by

$$C = 10\left(\frac{1}{x} + \frac{x}{x + 3}\right), \quad x \ge 1$$

where $x$ is the number of automobiles ordered.

(a) Find the intervals on which $C$ is increasing or decreasing.

(b) Use a graphing utility to graph the cost function.

(c) Use the *trace* feature to determine the order sizes for which the cost is $900. Assuming that the revenue function is increasing for $x \ge 0$, which order size would you use? Explain your reasoning.

(B) **40. Chemistry: Molecular Velocity**   Plots of the relative numbers of $N_2$ (nitrogen) molecules that have a given velocity at each of three temperatures (in degrees Kelvin) are shown in the figure. Identify the differences in the

average velocities (indicated by the peaks of the curves) for the three temperatures, and describe the intervals on which the velocity is increasing and decreasing for each of the three temperatures. *(Source: Adapted from Zumdahl, Chemistry, Seventh Edition)*



Molecular Velocity

**41. Medical Degrees**   The number $y$ of medical degrees conferred in the United States from 1970 through 2004 can be modeled by

$$y = 0.813t^3 - 55.70t^2 + 1185.2t + 7752, \quad 0 \le t \le 34$$

where $t$ is the time in years, with $t = 0$ corresponding to 1970. *(Source: U.S. National Center for Education Statistics)*

(T) (a) Use a graphing utility to graph the model. Then graphically estimate the years during which the model is increasing and the years during which it is decreasing.

(b) Use the test for increasing and decreasing functions to verify the result of part (a).

**42. MAKE A DECISION:  PROFIT**   The profit $P$ made by a cinema from selling $x$ bags of popcorn can be modeled by

$$P = 2.36x - \frac{x^2}{25,000} - 3500, \quad 0 \le x \le 50,000.$$

(a) Find the intervals on which $P$ is increasing and decreasing.

(b) If you owned the cinema, what price would you charge to obtain a maximum profit for popcorn? Explain your reasoning.

**43. Profit Analysis**   A fast-food restaurant determines the cost and revenue models for its hamburgers.

$$C = 0.6x + 7500, \quad 0 \le x \le 50,000$$

$$R = \frac{1}{20,000}(65,000x - x^2), \quad 0 \le x \le 50,000$$

(a) Write the profit function for this situation.

(b) Determine the intervals on which the profit function is increasing and decreasing.

(c) Determine how many hamburgers the restaurant needs to sell to obtain a maximum profit. Explain your reasoning.

## Section 3.2

# Extrema and the First-Derivative Test

■ Recognize the occurrence of relative extrema of functions.
■ Use the First-Derivative Test to find the relative extrema of functions.
■ Find absolute extrema of continuous functions on a closed interval.
■ Find minimum and maximum values of real-life models and interpret the results in context.
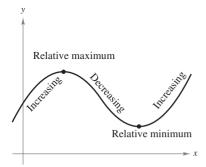
## Relative Extrema

You have used the derivative to determine the intervals on which a function is increasing or decreasing. In this section, you will examine the points at which a function changes from increasing to decreasing, or vice versa. At such a point, the function has a **relative extremum.** (The plural of extremum is *extrema.*) The **relative extrema** of a function include the **relative minima** and **relative maxima** of the function. For instance, the function shown in Figure 3.10 has a relative maximum at the left point and a relative minimum at the right point.

**FIGURE 3.10**

### Definition of Relative Extrema

Let $f$ be a function defined at $c$.

1. $f(c)$ is a **relative maximum** of $f$ if there exists an interval $(a, b)$ containing $c$ such that $f(x) \leq f(c)$ for all $x$ in $(a, b)$.
2. $f(c)$ is a **relative minimum** of $f$ if there exists an interval $(a, b)$ containing $c$ such that $f(x) \geq f(c)$ for all $x$ in $(a, b)$.

If $f(c)$ is a relative extremum of $f$, then the relative extremum is said to occur at $x = c$.

For a continuous function, the relative extrema must occur at critical numbers of the function, as shown in Figure 3.11.
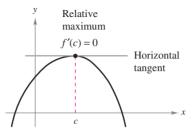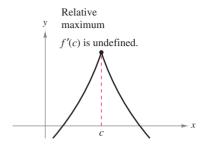
**FIGURE 3.11**

### Occurrences of Relative Extrema

If $f$ has a relative minimum or relative maximum when $x = c$, then $c$ is a critical number of $f$. That is, either $f'(c) = 0$ or $f'(c)$ is undefined.

## The First-Derivative Test
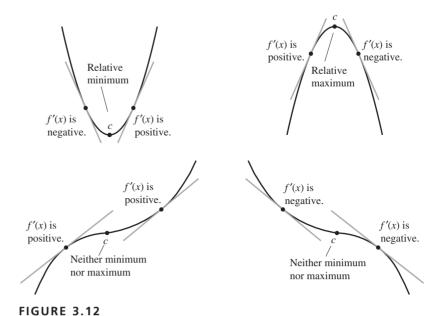
The discussion on the preceding page implies that in your search for relative extrema of a continuous function, you only need to test the critical numbers of the function. Once you have determined that $c$ is a critical number of a function $f$, the **First-Derivative Test** for relative extrema enables you to classify $f(c)$ as a relative minimum, a relative maximum, or neither.

---

**First-Derivative Test for Relative Extrema**

Let $f$ be continuous on the interval $(a, b)$ in which $c$ is the only critical number. If $f$ is differentiable on the interval (except possibly at $c$), then $f(c)$ can be classified as a relative minimum, a relative maximum, or neither, as shown.

**1.** On the interval $(a, b)$, if $f'(x)$ is negative to the left of $x = c$ and positive to the right of $x = c$, then $f(c)$ is a relative minimum.

**2.** On the interval $(a, b)$, if $f'(x)$ is positive to the left of $x = c$ and negative to the right of $x = c$, then $f(c)$ is a relative maximum.

**3.** On the interval $(a, b)$, if $f'(x)$ is positive on both sides of $x = c$ or negative on both sides of $x = c$, then $f(c)$ is not a relative extremum of $f$.

---

A graphical interpretation of the First-Derivative Test is shown in Figure 3.12.



**FIGURE 3.12**

*Example 1*  **Finding Relative Extrema**

Find all relative extrema of the function

$$f(x) = 2x^3 - 3x^2 - 36x + 14.$$

**SOLUTION**   Begin by finding the critical numbers of $f$.

$$f'(x) = 6x^2 - 6x - 36 \qquad \text{Find derivative of } f.$$
$$6x^2 - 6x - 36 = 0 \qquad \text{Set derivative equal to 0.}$$
$$6(x^2 - x - 6) = 0 \qquad \text{Factor out common factor.}$$
$$6(x - 3)(x + 2) = 0 \qquad \text{Factor.}$$
$$x = -2, x = 3 \qquad \text{Critical numbers}$$

Because $f'(x)$ is defined for all $x$, the only critical numbers of $f$ are $x = -2$ and $x = 3$. Using these numbers, you can form the three test intervals $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. The testing of the three intervals is shown in the table.

| Interval | $-\infty < x < -2$ | $-2 < x < 3$ | $3 < x < \infty$ |
|---|---|---|---|
| Test value | $x = -3$ | $x = 0$ | $x = 4$ |
| Sign of $f'(x)$ | $f'(-3) = 36 > 0$ | $f'(0) = -36 < 0$ | $f'(4) = 36 > 0$ |
| Conclusion | Increasing | Decreasing | Increasing |

Using the First-Derivative Test, you can conclude that the critical number $-2$ yields a relative maximum $[f'(x)$ changes sign from positive to negative$]$, and the critical number 3 yields a relative minimum $[f'(x)$ changes sign from negative to positive$]$.
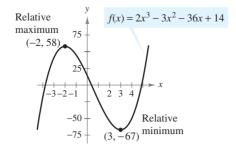
**FIGURE 3.13**

The graph of $f$ is shown in Figure 3.13. The relative maximum is $f(-2) = 58$ and the relative minimum is $f(3) = -67$.

✓ **CHECKPOINT 1**

Find all relative extrema of $f(x) = 2x^3 - 6x + 1$. ◼

In Example 1, both critical numbers yielded relative extrema. In the next example, only one of the two critical numbers yields a relative extremum.

### *Example 2*  Finding Relative Extrema
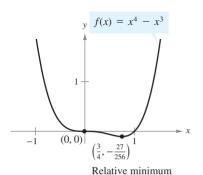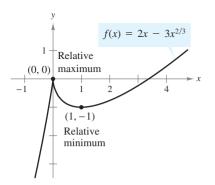
Find all relative extrema of the function $f(x) = x^4 - x^3$.

**SOLUTION**    From the derivative of the function

$$f'(x) = 4x^3 - 3x^2 = x^2(4x - 3)$$

you can see that the function has only two critical numbers: $x = 0$ and $x = \frac{3}{4}$. These numbers produce the test intervals $(-\infty, 0)$, $\left(0, \frac{3}{4}\right)$, and $\left(\frac{3}{4}, \infty\right)$, which are tested in the table.



FIGURE 3.14

| Interval | $-\infty < x < 0$ | $0 < x < \frac{3}{4}$ | $\frac{3}{4} < x < \infty$ |
|---|---|---|---|
| Test value | $x = -1$ | $x = \frac{1}{2}$ | $x = 1$ |
| Sign of $f'(x)$ | $f'(-1) = -7 < 0$ | $f'\left(\frac{1}{2}\right) = -\frac{1}{4} < 0$ | $f'(1) = 1 > 0$ |
| Conclusion | Decreasing | Decreasing | Increasing |

By the First-Derivative Test, it follows that $f$ has a relative minimum when $x = \frac{3}{4}$, as shown in Figure 3.14. The relative minimum is $f\left(\frac{3}{4}\right) = -\frac{27}{256}$. Note that the critical number $x = 0$ does not yield a relative extremum.

✓**CHECKPOINT 2**

Find all relative extrema of $f(x) = x^4 - 4x^3$. ■

### *Example 3*  Finding Relative Extrema

Find all relative extrema of the function

$$f(x) = 2x - 3x^{2/3}.$$

**SOLUTION**    From the derivative of the function

$$f'(x) = 2 - \frac{2}{x^{1/3}} = \frac{2(x^{1/3} - 1)}{x^{1/3}}$$

you can see that $f'(1) = 0$ and $f'$ is undefined at $x = 0$. So, the function has two critical numbers: $x = 1$ and $x = 0$. These numbers produce the test intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. By testing these intervals, you can conclude that $f$ has a relative maximum at $(0, 0)$ and a relative minimum at $(1, -1)$, as shown in Figure 3.15.



FIGURE 3.15

✓**CHECKPOINT 3**
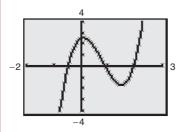
Find all relative extrema of $f(x) = 3x^{2/3} - 2x$. ■

## TECHNOLOGY
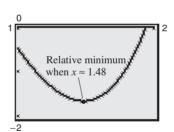
(T) There are several ways to use technology to find relative extrema of a function. One way is to use a graphing utility to graph the function, and then use the *zoom* and *trace* features to find the relative minimum and relative maximum points. For instance, consider the graph of
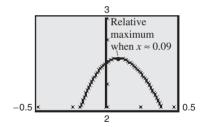
$$f(x) = 3.1x^3 - 7.3x^2 + 1.2x + 2.5$$

as shown below.



From the graph, you can see that the function has one relative maximum and one relative minimum. You can approximate these values by zooming in and using the *trace* feature, as shown below.



A second way to use technology to find relative extrema is to perform the First-Derivative Test with a symbolic differentiation utility. You can use the utility to differentiate the function, set the derivative equal to zero, and then solve the resulting equation. After obtaining the critical numbers, 1.48288 and 0.0870148, you can graph the function and observe that the first yields a relative minimum and the second yields a relative maximum. Compare the two ways shown above with doing the calculations by hand, as shown below.

| | |
|---|---|
| $f(x) = 3.1x^3 - 7.3x^2 + 1.2x + 2.5$ | Write original function. |
| $f'(x) = \dfrac{d}{dx}[3.1x^3 - 7.3x^2 + 1.2x + 2.5]$ | Differentiate with respect to $x$. |
| $f'(x) = 9.3x^2 - 14.6x + 1.2$ | First derivative |
| $9.3x^2 - 14.6x + 1.2 = 0$ | Set derivative equal to 0. |
| $x = \dfrac{73 \pm \sqrt{4213}}{93}$ | Solve for $x$. |
| $x \approx 1.48288, \ x \approx 0.0870148$ | Approximate. |

**STUDY TIP**

Some graphing calculators have a special feature that allows you to find the minimum or maximum of a function on an interval. Consult the user's manual for information on the *minimum value* and *maximum value* features of your graphing utility.

## Absolute Extrema

The terms *relative minimum* and *relative maximum* describe the *local* behavior of a function. To describe the *global* behavior of the function on an entire interval, you can use the terms **absolute maximum** and **absolute minimum.**

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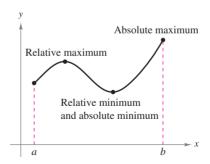
### Definition of Absolute Extrema

Let $f$ be defined on an interval $I$ containing $c$.

1. $f(c)$ is an **absolute minimum of $f$** on $I$ if $f(c) \leq f(x)$ for every $x$ in $I$.

2. $f(c)$ is an **absolute maximum of $f$** on $I$ if $f(c) \geq f(x)$ for every $x$ in $I$.

The absolute minimum and absolute maximum values of a function on an interval are sometimes simply called the **minimum** and **maximum** of $f$ on $I$.

---

Be sure that you understand the distinction between relative extrema and absolute extrema. For instance, in Figure 3.16, the function has a relative minimum that also happens to be an absolute minimum on the interval $[a, b]$. The relative maximum of $f$, however, is not the absolute maximum on the interval $[a, b]$. The next theorem points out that if a continuous function has a closed interval as its domain, then it *must* have both an absolute minimum and an absolute maximum on the interval. From Figure 3.16, note that these extrema can occur at endpoints of the interval.



**FIGURE 3.16**



**FIGURE 3.17**

---

### Extreme Value Theorem

If $f$ is continuous on $[a, b]$, then $f$ has both a minimum value and a maximum value on $[a, b]$.

---

Although a continuous function has just one minimum and one maximum value on a closed interval, either of these values can occur for more than one $x$-value. For instance, on the interval $[-3, 3]$, the function $f(x) = 9 - x^2$ has a minimum value of zero when $x = -3$ *and* when $x = 3$, as shown in Figure 3.17.
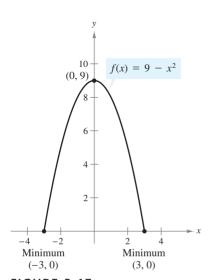
When looking for extrema of a function on a *closed* interval, remember that you must consider the values of the function at the endpoints as well as at the critical numbers of the function. You can use the guidelines below to find extrema on a closed interval.

**Guidelines for Finding Extrema on a Closed Interval**

To find the extrema of a continuous function $f$ on a closed interval $[a, b]$, use the steps below.

1. Evaluate $f$ at each of its critical numbers in $(a, b)$.

2. Evaluate $f$ at each endpoint, $a$ and $b$.

3. The least of these values is the minimum, and the greatest is the maximum.

*Example 4*  **Finding Extrema on a Closed Interval**

Find the minimum and maximum values of

$$f(x) = x^2 - 6x + 2$$

on the interval $[0, 5]$.
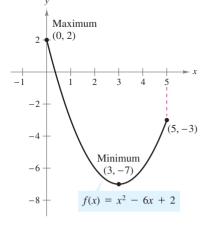
**SOLUTION**   Begin by finding the critical numbers of the function.

$$f'(x) = 2x - 6 \qquad \text{Find derivative of } f.$$
$$2x - 6 = 0 \qquad \text{Set derivative equal to 0.}$$
$$2x = 6 \qquad \text{Add 6 to each side.}$$
$$x = 3 \qquad \text{Solve for } x.$$

From this, you can see that the only critical number of $f$ is $x = 3$. Because this number lies in the interval under question, you should test the values of $f(x)$ at this number *and* at the endpoints of the interval, as shown in the table.

| $x$-value | Endpoint: $x = 0$ | Critical number: $x = 3$ | Endpoint: $x = 5$ |
|---|---|---|---|
| $f(x)$ | $f(0) = 2$ | $f(3) = -7$ | $f(5) = -3$ |
| Conclusion | Maximum is 2 | Minimum is $-7$ | Neither maximum nor minimum |
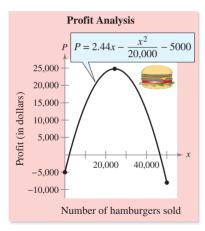
From the table, you can see that the minimum of $f$ on the interval $[0, 5]$ is $f(3) = -7$. Moreover, the maximum of $f$ on the interval $[0, 5]$ is $f(0) = 2$. This is confirmed by the graph of $f$, as shown in Figure 3.18.



**FIGURE 3.18**

✓**CHECKPOINT 4**

Find the minimum and maximum values of $f(x) = x^2 - 8x + 10$ on the interval $[0, 7]$. Sketch the graph of $f(x)$ and label the minimum and maximum values. ■

**Profit Analysis**



$P = 2.44x - \dfrac{x^2}{20,000} - 5000$

**FIGURE 3.19**

✓**CHECKPOINT 5**

Verify the results of Example 5 by completing the table.

| x (units) | 24,000 | 24,200 |
|-----------|--------|--------|
| P (profit) |        |        |

| x (units) | 24,300 | 24,400 |
|-----------|--------|--------|
| P (profit) |        |        |

| x (units) | 24,500 | 24,600 |
|-----------|--------|--------|
| P (profit) |        |        |

| x (units) | 24,800 | 25,000 |
|-----------|--------|--------|
| P (profit) |        |        |

# Applications of Extrema

Finding the minimum and maximum values of a function is one of the most common applications of calculus.

*Example 5*    **Finding the Maximum Profit**    ®

Recall the fast-food restaurant in Examples 7 and 8 in Section 2.3. The restaurant's profit function for hamburgers is given by

$$P = 2.44x - \frac{x^2}{20,000} - 5000, \quad 0 \le x \le 50,000.$$

Find the sales level that yields a maximum profit.

**SOLUTION**   To begin, find an equation for marginal profit. Then set the marginal profit equal to zero and solve for $x$.

$$P' = 2.44 - \frac{x}{10,000} \qquad \text{Find marginal profit.}$$

$$2.44 - \frac{x}{10,000} = 0 \qquad \text{Set marginal profit equal to 0.}$$

$$-\frac{x}{10,000} = -2.44 \qquad \text{Subtract 2.44 from each side.}$$

$$x = 24,400 \text{ hamburgers} \qquad \text{Critical number}$$

From Figure 3.19, you can see that the critical number $x = 24,400$ corresponds to the sales level that yields a maximum profit. To find the maximum profit, substitute $x = 24,400$ into the profit function.

$$P = 2.44x - \frac{x^2}{20,000} - 5000$$

$$= 2.44(24,400) - \frac{(24,400)^2}{20,000} - 5000$$

$$= \$24,768$$

**CONCEPT CHECK**

1. Complete the following: The relative extrema of a function include the relative _____ and the relative_____.

2. Let $f$ be continuous on the open interval $(a, b)$ in which $c$ is the only critical number and assume $f$ is differentiable on the interval (except possibly at $c$). According to the First-Derivative Test, what are the three possible classifications for $f(c)$?

3. Let $f$ be defined on an interval $I$ containing $c$. The value $f(c)$ is an absolute minimum of $f$ on $I$ if what is true?

4. In your own words, state the guidelines for finding the extrema of a continuous function $f$ on a closed interval $[a, b]$.

**Skills Review 3.2**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.2, 2.4, and 3.1.

In Exercises 1–6, solve the equation $f'(x) = 0$.

**1.** $f(x) = 4x^4 - 2x^2 + 1$

**2.** $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 10x$

**3.** $f(x) = 5x^{4/5} - 4x$

**4.** $f(x) = \frac{1}{2}x^2 - 3x^{5/3}$

**5.** $f(x) = \dfrac{x + 4}{x^2 + 1}$

**6.** $f(x) = \dfrac{x - 1}{x^2 + 4}$

In Exercises 7–10, use $g(x) = -x^5 - 2x^4 + 4x^3 + 2x - 1$ to determine the sign of the derivative.

**7.** $g'(-4)$

**8.** $g'(0)$

**9.** $g'(1)$

**10.** $g'(3)$

In Exercises 11 and 12, decide whether the function is increasing or decreasing on the given interval.

**11.** $f(x) = 2x^2 - 11x - 6,$ $(3, 6)$

**12.** $f(x) = x^3 + 2x^2 - 4x - 8,$ $(-2, 0)$

## Exercises 3.2

In Exercises 1–4, use a table similar to that in Example 1 to find all relative extrema of the function.

**1.** $f(x) = -2x^2 + 4x + 3$

**2.** $f(x) = x^2 + 8x + 10$

**3.** $f(x) = x^2 - 6x$

**4.** $f(x) = -4x^2 + 4x + 1$

In Exercises 5–12, find all relative extrema of the function.

**5.** $g(x) = 6x^3 - 15x^2 + 12x$

**6.** $g(x) = \frac{1}{5}x^5 - x$

**7.** $h(x) = -(x + 4)^3$

**8.** $h(x) = 2(x - 3)^3$

**9.** $f(x) = x^3 - 6x^2 + 15$

**10.** $f(x) = x^4 - 32x + 4$

**11.** $f(x) = x^4 - 2x^3 + x + 1$

**12.** $f(x) = x^4 - 12x^3$

(T) In Exercises 13–18, use a graphing utility to graph the function. Then find all relative extrema of the function.

**13.** $f(x) = (x - 1)^{2/3}$

**14.** $f(t) = (t - 1)^{1/3}$

**15.** $g(t) = t - \dfrac{1}{2t^2}$

**16.** $f(x) = x + \dfrac{1}{x}$

**17.** $f(x) = \dfrac{x}{x + 1}$

**18.** $h(x) = \dfrac{4}{x^2 + 1}$

In Exercises 19–30, find the absolute extrema of the function on the closed interval. Use a graphing utility to verify your results.

**19.** $f(x) = 2(3 - x),$ $[-1, 2]$

**20.** $f(x) = \frac{1}{3}(2x + 5),$ $[0, 5]$

**21.** $f(x) = 5 - 2x^2,$ $[0, 3]$

**22.** $f(x) = x^2 + 2x - 4,$ $[-1, 1]$

**23.** $f(x) = x^3 - 3x^2,$ $[-1, 3]$

**24.** $f(x) = x^3 - 12x,$ $[0, 4]$

**25.** $h(s) = \dfrac{1}{3 - s},$ $[0, 2]$

**26.** $h(t) = \dfrac{t}{t - 2},$ $[3, 5]$
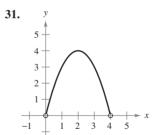
**27.** $f(x) = 3x^{2/3} - 2x,$ $[-1, 2]$
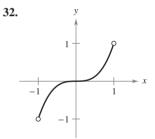
**28.** $g(t) = \dfrac{t^2}{t^2 + 3},$ $[-1, 1]$

**29.** $h(t) = (t - 1)^{2/3},$ $[-7, 2]$

**30.** $g(x) = 4\left(1 + \dfrac{1}{x} + \dfrac{1}{x^2}\right),$ $[-4, 5]$

In Exercises 31 and 32, approximate the critical numbers of the function shown in the graph. Determine whether the function has a relative maximum, a relative minimum, an absolute maximum, an absolute minimum, or none of these at each critical number on the interval shown.

**31.**



**32.**

(T) In Exercises 33–36, use a graphing utility to find graphically the absolute extrema of the function on the closed interval.

**33.** $f(x) = 0.4x^3 - 1.8x^2 + x - 3, \quad [0, 5]$

**34.** $f(x) = 3.2x^5 + 5x^3 - 3.5x, \quad [0, 1]$

**35.** $f(x) = \frac{4}{3}x\sqrt{3 - x}, \quad [0, 3]$

**36.** $f(x) = 4\sqrt{x} - 2x + 1, \quad [0, 6]$

In Exercises 37–40, find the absolute extrema of the function on the interval $[0, \infty)$.

**37.** $f(x) = \dfrac{4x}{x^2 + 1}$ 　　　　　**38.** $f(x) = \dfrac{8}{x + 1}$

**39.** $f(x) = \dfrac{2x}{x^2 + 4}$ 　　　　　**40.** $f(x) = 8 - \dfrac{4x}{x^2 + 1}$

In Exercises 41 and 42, find the maximum value of $|f''(x)|$ on the closed interval. (You will use this skill in Section 6.4 to estimate the error in the Trapezoidal Rule.)

**41.** $f(x) = \sqrt{1 + x^3}, \quad [0, 2]$ 　　**42.** $f(x) = \dfrac{1}{x^2 + 1}, \quad [0, 3]$

In Exercises 43 and 44, find the maximum value of $|f^{(4)}(x)|$ on the closed interval. (You will use this skill in Section 6.4 to estimate the error in Simpson's Rule. )

**43.** $f(x) = (x + 1)^{2/3}, \quad [0, 2]$

**44.** $f(x) = \dfrac{1}{x^2 + 1}, \quad [-1, 1]$

In Exercises 45 and 46, graph a function on the interval $[-2, 5]$ having the given characteristics.

**45.** Absolute maximum at $x = -2$

　　Absolute minimum at $x = 1$

　　Relative maximum at $x = 3$

**46.** Relative minimum at $x = -1$

　　Critical number at $x = 0$, but no extrema

　　Absolute maximum at $x = 2$
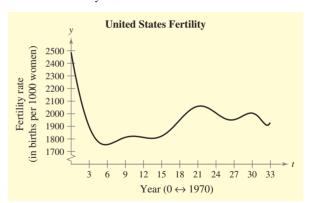
　　Absolute minimum at $x = 5$

**47. Cost**　A retailer has determined the cost $C$ for ordering and storing $x$ units of a product to be modeled by

$$C = 3x + \frac{20,000}{x}, \quad 0 < x \le 200.$$

The delivery truck can bring at most 200 units per order. Find the order size that will minimize the cost. Use a graphing utility to verify your result.

**48. Profit**　The quantity demanded $x$ for a product is inversely proportional to the cube of the price $p$ for $p > 1$. When the price is $10 per unit, the quantity demanded is eight units. The initial cost is $100 and the cost per unit is $4. What price will yield a maximum profit?

(T) **49. Profit**　When soft drinks were sold for $1.00 per can at football games, approximately 6000 cans were sold. When the price was raised to $1.20 per can, the quantity demanded dropped to 5600. The initial cost is $5000 and the cost per unit is $0.50. Assuming that the demand function is linear, use the *table* feature of a graphing utility to determine the price that will yield a maximum profit.

**50. Medical Science**　Coughing forces the trachea (windpipe) to contract, which in turn affects the velocity of the air through the trachea. The velocity of the air during coughing can be modeled by $v = k(R - r)r^2$, $0 \le r < R$, where $k$ is a constant, $R$ is the normal radius of the trachea, and $r$ is the radius during coughing. What radius $r$ will produce the maximum air velocity?

**51. Population**　The resident population $P$ (in millions) of the United States from 1790 through 2000 can be modeled by $P = 0.00000583t^3 + 0.005003t^2 + 0.13776t + 4.658$, $-10 \le t \le 200$, where $t = 0$ corresponds to 1800. *(Source: U.S. Census Bureau)*

(a) Make a conjecture about the maximum and minimum populations in the U.S. from 1790 to 2000.

(b) Analytically find the maximum and minimum populations over the interval.

(c) Write a brief paragraph comparing your conjecture with your results in part (b).

**52. Biology: Fertility Rates**　The graph of the United States fertility rate shows the number of births per 1000 women in their lifetime according to the birth rate in that particular year.　*(Source: U.S. National Center for Health Statistics)*

(a) Around what year was the fertility rate the highest, and to how many births per 1000 women did this rate correspond?

(b) During which time periods was the fertility rate increasing most rapidly? Most slowly?

(c) During which time periods was the fertility rate decreasing most rapidly? Most slowly?

(d) Give some possible real-life reasons for fluctuations in the fertility rate.



United States Fertility

# Section 3.3

# Concavity and the Second-Derivative Test

■ Determine the intervals on which the graphs of functions are concave upward or concave downward.
■ Find the points of inflection of the graphs of functions.
■ Use the Second-Derivative Test to find the relative extrema of functions.
■ Find the points of diminishing returns of input-output models.

## Concavity

You already know that locating the intervals over which a function $f$ increases or decreases is helpful in determining its graph. In this section, you will see that locating the intervals on which $f'$ increases or decreases can determine where the graph of $f$ is curving upward or curving downward. This property of curving upward or downward is defined formally as the **concavity** of the graph of the function.

Concave upward,
$f'$ is increasing.

### Definition of Concavity
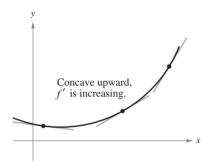
Let $f$ be differentiable on an open interval $I$. The graph of $f$ is

1. **concave upward** on $I$ if $f'$ is increasing on the interval.

2. **concave downward** on $I$ if $f'$ is decreasing on the interval.

Concave downward,
$f'$ is decreasing.

From Figure 3.20, you can observe the following graphical interpretation of concavity.

1. A curve that is concave upward lies *above* its tangent line.

2. A curve that is concave downward lies *below* its tangent line.

This visual test for concavity is useful when the graph of a function is given. To determine concavity without seeing a graph, you need an analytic test. It turns out that you can use the second derivative to determine these intervals in much the same way that you use the first derivative to determine the intervals on which $f$ is increasing or decreasing.

**FIGURE 3.20**

### Test for Concavity

Let $f$ be a function whose second derivative exists on an open interval $I$.

1. If $f''(x) > 0$ for all $x$ in $I$, then $f$ is concave upward on $I$.

2. If $f''(x) < 0$ for all $x$ in $I$, then $f$ is concave downward on $I$.

For a *continuous* function $f$, you can find the open intervals on which the graph of $f$ is concave upward and concave downward as follows. [For a function that is not continuous, the test intervals should be formed using points of discontinuity, along with the points at which $f''(x)$ is zero or undefined.]

**DISCOVERY**

Use a graphing utility to graph the function $f(x) = x^3 - x$ and its second derivative $f''(x) = 6x$ in the same viewing window. On what interval is $f$ concave upward? On what interval is $f''$ positive? Describe how the second derivative can be used to determine where a function is concave upward and concave downward. Repeat this analysis for the functions $g(x) = x^4 - 6x^2$ and $g''(x) = 12x^2 - 12$.

---

**Guidelines for Applying Concavity Test**

1. Locate the $x$-values at which $f''(x) = 0$ or $f''(x)$ is undefined.

2. Use these $x$-values to determine the test intervals.

3. Test the sign of $f''(x)$ in each test interval.

---

**Example 1**   **Applying the Test for Concavity**

**a.** The graph of the function

$$f(x) = x^2 \qquad \text{Original function}$$

is concave upward on the entire real line because its second derivative

$$f''(x) = 2 \qquad \text{Second derivative}$$

is positive for all $x$. (See Figure 3.21.)

**b.** The graph of the function
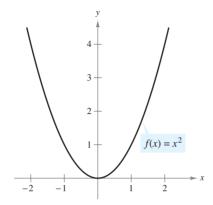
$$f(x) = \sqrt{x} \qquad \text{Original function}$$

is concave downward for $x > 0$ because its second derivative

$$f''(x) = -\frac{1}{4}x^{-3/2} \qquad \text{Second derivative}$$

is negative for all $x > 0$. (See Figure 3.22.)

✓**CHECKPOINT 1**

**a.** Find the second derivative of $f(x) = -2x^2$ and discuss the concavity of the graph.

**b.** Find the second derivative of $f(x) = -2\sqrt{x}$ and discuss the concavity of the graph. ■



**FIGURE 3.21**   Concave Upward
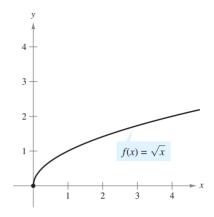


**FIGURE 3.22**   Concave Downward

*Example 2*   **Determining Concavity**

Determine the open intervals on which the graph of the function is concave upward or concave downward.

$$f(x) = \frac{6}{x^2 + 3}$$

**SOLUTION**   Begin by finding the second derivative of $f$.

$$f(x) = 6(x^2 + 3)^{-1} \qquad \text{Rewrite original function.}$$

$$f'(x) = (-6)(2x)(x^2 + 3)^{-2} \qquad \text{Chain Rule}$$

$$= \frac{-12x}{(x^2 + 3)^2} \qquad \text{Simplify.}$$

$$f''(x) = \frac{(x^2 + 3)^2(-12) - (-12x)(2)(2x)(x^2 + 3)}{(x^2 + 3)^4} \qquad \text{Quotient Rule}$$

$$= \frac{-12(x^2 + 3) + (48x^2)}{(x^2 + 3)^3} \qquad \text{Simplify.}$$

$$= \frac{36(x^2 - 1)}{(x^2 + 3)^3} \qquad \text{Simplify.}$$

From this, you can see that $f''(x)$ is defined for all real numbers and $f''(x) = 0$ when $x = \pm 1$. So, you can test the concavity of $f$ by testing the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, as shown in the table. The graph of $f$ is shown in Figure 3.23.

| Interval | $-\infty < x < -1$ | $-1 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Test value | $x = -2$ | $x = 0$ | $x = 2$ |
| Sign of $f''(x)$ | $f''(-2) > 0$ | $f''(0) < 0$ | $f''(2) > 0$ |
| Conclusion | Concave upward | Concave downward | Concave upward |



$$f(x) = \frac{6}{x^2 + 3}$$

Concave upward, $f''(x) > 0$

Concave downward, $f''(x) < 0$

Concave upward, $f''(x) > 0$

**FIGURE 3.23**

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**Algebra Review**

For help on the algebra in Example 2, see Example 1(a) in the *Chapter 3 Algebra Review*, on page 283.

---

**STUDY TIP**

In Example 2, $f'$ is increasing on the interval $(1, \infty)$ even though $f$ is decreasing there. Be sure you see that the increasing or decreasing of $f'$ does not necessarily correspond to the increasing or decreasing of $f$.

---

✓**CHECKPOINT 2**

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x) = \frac{12}{x^2 + 4} \ \blacksquare$$

## Points of Inflection

If the tangent line to a graph exists at a point at which the concavity changes, then the point is a **point of inflection.** Three examples of inflection points are shown in Figure 3.24. (Note that the third graph has a vertical tangent line at its point of inflection.)

**FIGURE 3.24**    The graph *crosses* its tangent line at a point of inflection.

**Definition of Point of Inflection**

If the graph of a continuous function has a tangent line at a point where its concavity changes from upward to downward (or downward to upward), then the point is a **point of inflection.**

### *DISCOVERY*

Use a graphing utility to graph

$$f(x) = x^3 - 6x^2 + 12x - 6 \quad \text{and} \quad f''(x) = 6x - 12$$

in the same viewing window. At what $x$-value does $f''(x) = 0$? At what $x$-value does the point of inflection occur? Repeat this analysis for
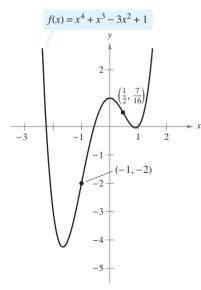
$$g(x) = x^4 - 5x^2 + 7 \quad \text{and} \quad g''(x) = 12x^2 - 10.$$

Make a general statement about the relationship of the point of inflection of a function and the second derivative of the function.

Because a point of inflection occurs where the concavity of a graph changes, it must be true that at such points the sign of $f''$ changes. So, to locate possible points of inflection, you only need to determine the values of $x$ for which $f''(x) = 0$ or for which $f''(x)$ does not exist. This parallels the procedure for locating the relative extrema of $f$ by determining the critical numbers of $f$.

**Property of Points of Inflection**

If $(c, f(c))$ is a point of inflection of the graph of $f$, then either $f''(c) = 0$ or $f''(c)$ is undefined.

$f(x) = x^4 + x^3 - 3x^2 + 1$

**FIGURE 3.25**  Two Points of Inflection

*Example 3*  **Finding Points of Inflection**

Discuss the concavity of the graph of $f$ and find its points of inflection.

$$f(x) = x^4 + x^3 - 3x^2 + 1$$

**SOLUTION**  Begin by finding the second derivative of $f$.

| | |
|---|---|
| $f(x) = x^4 + x^3 - 3x^2 + 1$ | Write original function. |
| $f'(x) = 4x^3 + 3x^2 - 6x$ | Find first derivative. |
| $f''(x) = 12x^2 + 6x - 6$ | Find second derivative. |
| $\quad = 6(2x - 1)(x + 1)$ | Factor. |
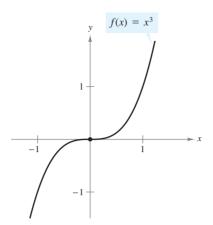
From this, you can see that the possible points of inflection occur at $x = \frac{1}{2}$ and $x = -1$. After testing the intervals $(-\infty, -1)$, $\left(-1, \frac{1}{2}\right)$, and $\left(\frac{1}{2}, \infty\right)$, you can determine that the graph is concave upward on $(-\infty, -1)$, concave downward on $\left(-1, \frac{1}{2}\right)$, and concave upward on $\left(\frac{1}{2}, \infty\right)$. Because the concavity changes at $x = -1$ and $x = \frac{1}{2}$, you can conclude that the graph has points of inflection at these $x$-values, as shown in Figure 3.25.

✓**CHECKPOINT 3**
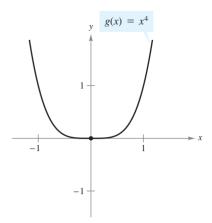
Discuss the concavity of the graph of $f$ and find its points of inflection.

$$f(x) = x^4 - 2x^3 + 1 \quad \blacksquare$$

It is possible for the second derivative to be zero at a point that is *not* a point of inflection. For example, compare the graphs of $f(x) = x^3$ and $g(x) = x^4$, as shown in Figure 3.26. Both second derivatives are zero when $x = 0$, but only the graph of $f$ has a point of inflection at $x = 0$. This shows that before concluding that a point of inflection exists at a value of $x$ for which $f''(x) = 0$, you must test to be certain that the concavity actually changes at that point.

$f(x) = x^3$

$g(x) = x^4$
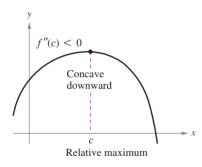
$f''(0) = 0$, and $(0, 0)$ is a point of inflection.

$g''(0) = 0$, but $(0, 0)$ is not a point of inflection.

**FIGURE 3.26**
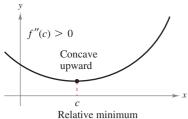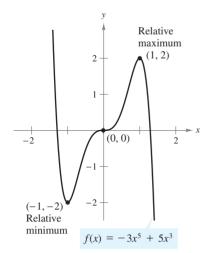
## The Second-Derivative Test

The second derivative can be used to perform a simple test for relative minima and relative maxima. If $f$ is a function such that $f'(c) = 0$ and the graph of $f$ is concave upward at $x = c$, then $f(c)$ is a relative minimum of $f$. Similarly, if $f$ is a function such that $f'(c) = 0$ and the graph of $f$ is concave downward at $x = c$, then $f(c)$ is a relative maximum of $f$, as shown in Figure 3.27.



**FIGURE 3.27**

---

**Second-Derivative Test**

Let $f'(c) = 0$, and let $f''$ exist on an open interval containing $c$.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.

2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

3. If $f''(c) = 0$, then the test fails. In such cases, you can use the First-Derivative Test to determine whether $f(c)$ is a relative minimum, a relative maximum, or neither.

---

**Example 4**    **Using the Second-Derivative Test**

Find the relative extrema of

$$f(x) = -3x^5 + 5x^3.$$

**SOLUTION**    Begin by finding the first derivative of $f$.

$$f'(x) = -15x^4 + 15x^2$$
$$= 15x^2(1 - x^2)$$

From this derivative, you can see that $x = 0$, $x = -1$, and $x = 1$ are the only critical numbers of $f$. Using the second derivative

$$f''(x) = -60x^3 + 30x$$

you can apply the Second-Derivative Test, as shown.

| Point | Sign of $f''(x)$ | Conclusion |
|-------|------------------|------------|
| $(-1, -2)$ | $f''(-1) = 30 > 0$ | Relative minimum |
| $(0, 0)$ | $f''(0) = 0$ | Test fails. |
| $(1, 2)$ | $f''(1) = -30 < 0$ | Relative maximum |

Because the test fails at $(0, 0)$, you can apply the First-Derivative Test to conclude that the point $(0, 0)$ is neither a relative minimum nor a relative maximum—a test for concavity would show that this point is a point of inflection. The graph of $f$ is shown in Figure 3.28.



**FIGURE 3.28**

✓ **CHECKPOINT 4**

Find all relative extrema of $f(x) = x^4 - 4x^3 + 1$. ■

## Extended Application: Diminishing Returns

In economics, the notion of concavity is related to the concept of **diminishing returns.** Consider a function

Output⟶          ⟵Input

$$y = f(x)$$

where $x$ measures input (in dollars) and $y$ measures output (in dollars). In Figure 3.29, notice that the graph of this function is concave upward on the interval $(a, c)$ and is concave downward on the interval $(c, b)$. On the interval $(a, c)$, each additional dollar of input returns more than the previous input dollar. By contrast, on the interval $(c, b)$, each additional dollar of input returns less than the previous input dollar. The point $(c, f(c))$ is called the **point of diminishing returns.** An increased investment beyond this point is usually considered a poor use of capital.

**FIGURE 3.29**

## Example 5 Exploring Diminishing Returns ®

By increasing its advertising cost $x$ (in thousands of dollars) for a product, a company discovers that it can increase the sales $y$ (in thousands of dollars) according to the model

$$y = -\frac{1}{10}x^3 + 6x^2 + 400, \quad 0 \le x \le 40.$$

Find the point of diminishing returns for this product.

**SOLUTION** Begin by finding the first and second derivatives.

$$y' = 12x - \frac{3x^2}{10} \qquad \text{First derivative}$$

$$y'' = 12 - \frac{3x}{5} \qquad \text{Second derivative}$$

The second derivative is zero only when $x = 20$. By testing the intervals $(0, 20)$ and $(20, 40)$, you can conclude that the graph has a point of diminishing returns when $x = 20$, as shown in Figure 3.30. So, the point of diminishing returns for this product occurs when $20,000$ is spent on advertising.

**FIGURE 3.30**

### ✓ CHECKPOINT 5

Find the point of diminishing returns for the model below, where $R$ is the revenue (in thousands of dollars) and $x$ is the advertising cost (in thousands of dollars).

$$R = \frac{1}{20,000}(450x^2 - x^3),$$

$$0 \le x \le 300 \quad ◾$$

### CONCEPT CHECK

1. Let $f$ be differentiable on an open interval $I$. If the graph of $f$ is concave upward on $I$, what can you conclude about the behavior of $f'$ on the interval?

2. Let $f$ be a function whose second derivative exists on an open interval $I$ and $f''(x) > 0$ for all $x$ in $I$. Is $f$ concave upward or concave downward on $I$?

3. Let $f'(c) = 0$, and let $f''$ exist on an open interval containing $c$. According to the Second-Derivative Test, what are the possible classifications for $f(c)$?

4. A newspaper headline states that "The rate of growth of the national deficit is decreasing." What does this mean? What does it imply about the graph of the deficit as a function of time?

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**Skills Review 3.3**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.4, 2.6, and 3.1.

In Exercises 1–6, find the second derivative of the function.

**1.** $f(x) = 4x^4 - 9x^3 + 5x - 1$

**2.** $g(s) = (s^2 - 1)(s^2 - 3s + 2)$

**3.** $g(x) = (x^2 + 1)^4$

**4.** $f(x) = (x - 3)^{4/3}$

**5.** $h(x) = \dfrac{4x + 3}{5x - 1}$

**6.** $f(x) = \dfrac{2x - 1}{3x + 2}$

In Exercises 7–10, find the critical numbers of the function.

**7.** $f(x) = 5x^3 - 5x + 11$

**8.** $f(x) = x^4 - 4x^3 - 10$

**9.** $g(t) = \dfrac{16 + t^2}{t}$

**10.** $h(x) = \dfrac{x^4 - 50x^2}{8}$

---

## Exercises 3.3

In Exercises 1–8, analytically find the open intervals on which the graph is concave upward and those on which it is concave downward.

**1.** $y = x^2 - x - 2$

**2.** $y = -x^3 + 3x^2 - 2$

**3.** $f(x) = \dfrac{x^2 - 1}{2x + 1}$

**4.** $f(x) = \dfrac{x^2 + 4}{4 - x^2}$

**5.** $f(x) = \dfrac{24}{x^2 + 12}$

**6.** $f(x) = \dfrac{x^2}{x^2 + 1}$

**7.** $y = -x^3 + 6x^2 - 9x - 1$

**8.** $y = x^5 + 5x^4 - 40x^2$

In Exercises 9–22, find all relative extrema of the function. Use the Second-Derivative Test when applicable.

**9.** $f(x) = 6x - x^2$

**10.** $f(x) = (x - 5)^2$

**11.** $f(x) = x^3 - 5x^2 + 7x$

**12.** $f(x) = x^4 - 4x^3 + 2$

**13.** $f(x) = x^{2/3} - 3$

**14.** $f(x) = x + \dfrac{4}{x}$

**15.** $f(x) = \sqrt{x^2 + 1}$

**16.** $f(x) = \sqrt{2x^2 + 6}$

**17.** $f(x) = \sqrt{9 - x^2}$

**18.** $f(x) = \sqrt{4 - x^2}$

**19.** $f(x) = \dfrac{8}{x^2 + 2}$

**20.** $f(x) = \dfrac{18}{x^2 + 3}$

**21.** $f(x) = \dfrac{x}{x - 1}$

**22.** $f(x) = \dfrac{x}{x^2 - 1}$

(T) In Exercises 23–26, use a graphing utility to estimate graphically all relative extrema of the function.

**23.** $f(x) = \frac{1}{2}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2$

**24.** $f(x) = -\frac{1}{3}x^5 - \frac{1}{2}x^4 + x$
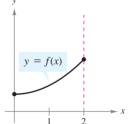
**25.** $f(x) = 5 + 3x^2 - x^3$

**26.** $f(x) = 3x^3 + 5x^2 - 2$
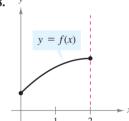
In Exercises 27–30, state the signs of $f'(x)$ and $f''(x)$ on the interval (0, 2).

**27.**
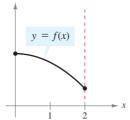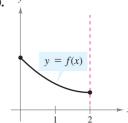


**28.**



**29.**



**30.**



In Exercises 31–38, find the point(s) of inflection of the graph of the function.

**31.** $f(x) = x^3 - 9x^2 + 24x - 18$

**32.** $f(x) = x(6 - x)^2$

**33.** $f(x) = (x - 1)^3(x - 5)$

**34.** $f(x) = x^4 - 18x^2 + 5$

**35.** $g(x) = 2x^4 - 8x^3 + 12x^2 + 12x$

**36.** $f(x) = -4x^3 - 8x^2 + 32$

**37.** $h(x) = (x - 2)^3(x - 1)$

**38.** $f(t) = (1 - t)(t - 4)(t^2 - 4)$

(T) In Exercises 39–50, use a graphing utility to graph the function and identify all relative extrema and points of inflection.

**39.** $f(x) = x^3 - 12x$

**40.** $f(x) = x^3 - 3x$

**41.** $f(x) = x^3 - 6x^2 + 12x$

**42.** $f(x) = x^3 - \frac{3}{2}x^2 - 6x$

**43.** $f(x) = \frac{1}{4}x^4 - 2x^2$

**44.** $f(x) = 2x^4 - 8x + 3$

**45.** $g(x) = (x - 2)(x + 1)^2$

**46.** $g(x) = (x - 6)(x + 2)^3$

**47.** $g(x) = x\sqrt{x + 3}$

**48.** $g(x) = x\sqrt{9 - x}$

**49.** $f(x) = \dfrac{4}{1 + x^2}$

**50.** $f(x) = \dfrac{2}{x^2 - 1}$

In Exercises 51–54, sketch a graph of a function $f$ having the given characteristics.

**51.** $f(2) = f(4) = 0$
  $f'(x) < 0$ if $x < 3$
  $f'(3) = 0$
  $f'(x) > 0$ if $x > 3$
  $f''(x) > 0$

**52.** $f(2) = f(4) = 0$
  $f'(x) > 0$ if $x < 3$
  $f'(3)$ is undefined.
  $f'(x) < 0$ if $x > 3$
  $f''(x) > 0, x \neq 3$

**53.** $f(0) = f(2) = 0$
  $f'(x) > 0$ if $x < 1$
  $f'(1) = 0$
  $f'(x) < 0$ if $x > 1$
  $f''(x) < 0$

**54.** $f(0) = f(2) = 0$
  $f'(x) < 0$ if $x < 1$
  $f'(1) = 0$
  $f'(x) > 0$ if $x > 1$
  $f''(x) > 0$
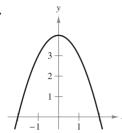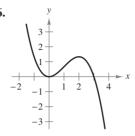
In Exercises 55 and 56, use the graph to sketch the graph of $f'$. Find the intervals on which (a) $f'(x)$ is positive, (b) $f'(x)$ is negative, (c) $f'$ is increasing, and (d) $f'$ is decreasing. For each of these intervals, describe the corresponding behavior of $f$.

**55.**



**56.**



In Exercises 57–60, you are given $f'$. Find the intervals on which (a) $f'(x)$ is increasing or decreasing and (b) the graph of $f$ is concave upward or concave downward. (c) Find the relative extrema and inflection points of $f$. (d) Then sketch a graph of $f$.

**57.** $f'(x) = 2x + 5$

**58.** $f'(x) = 3x^2 - 2$

**59.** $f'(x) = -x^2 + 2x - 1$

**60.** $f'(x) = x^2 + x - 6$

**Point of Diminishing Returns**   In Exercises 61 and 62, identify the point of diminishing returns for the input-output function. For each function, $R$ is the revenue and $x$ is the amount spent on advertising. Use a graphing utility to verify your results.

**61.** $R = \dfrac{1}{50,000}(600x^2 - x^3), \quad 0 \leq x \leq 400$

**62.** $R = -\frac{4}{9}(x^3 - 9x^2 - 27), \quad 0 \leq x \leq 5$

**Average Cost**   In Exercises 63 and 64, you are given the total cost of producing $x$ units. Find the production level that minimizes the average cost per unit. Use a graphing utility to verify your results.

**63.** $C = 0.5x^2 + 15x + 5000$

**64.** $C = 0.002x^3 + 20x + 500$

**Productivity**   In Exercises 65 and 66, consider a college student who works from 7 P.M. to 11 P.M. assembling mechanical components. The number $N$ of components assembled after $t$ hours is given by the function. At what time is the student assembling components at the greatest rate?

**65.** $N = -0.12t^3 + 0.54t^2 + 8.22t, \quad 0 \leq t \leq 4$

**66.** $N = \dfrac{20t^2}{4 + t^2}, \quad 0 \leq t \leq 4$

**Sales Growth**   In Exercises 67 and 68, find the time $t$ in years when the annual sales $x$ of a new product are increasing at the greatest rate. Use a graphing utility to verify your results.

**67.** $x = \dfrac{10,000t^2}{9 + t^2}$

**68.** $x = \dfrac{500,000t^2}{36 + t^2}$

(T) In Exercises 69–72, use a graphing utility to graph $f$, $f'$, and $f''$ in the same viewing window. Graphically locate the relative extrema and points of inflection of the graph of $f$. State the relationship between the behavior of $f$ and the signs of $f'$ and $f''$.

**69.** $f(x) = \frac{1}{2}x^3 - x^2 + 3x - 5, \quad [0, 3]$

**70.** $f(x) = -\frac{1}{20}x^5 - \frac{1}{12}x^2 - \frac{1}{3}x + 1, \quad [-2, 2]$

**71.** $f(x) = \dfrac{2}{x^2 + 1}, \quad [-3, 3]$   **72.** $f(x) = \dfrac{x^2}{x^2 + 1}, \quad [-3, 3]$
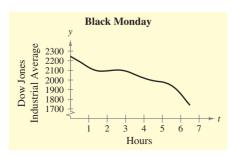
**73. Average Cost**   A manufacturer has determined that the total cost $C$ of operating a factory is $C = 0.5x^2 + 10x + 7200$, where $x$ is the number of units produced. At what level of production will the average cost per unit be minimized? (The average cost per unit is $C/x$.)

**74. Inventory Cost**   The cost $C$ for ordering and storing $x$ units is $C = 2x + 300,000/x$. What order size will produce a minimum cost?

**75. Phishing** Phishing is a criminal activity used by an individual or group to fraudulently acquire information by masquerading as a trustworthy person or business in an electronic communication. Criminals create spoof sites on the Internet to trick victims into giving them information. The sites are designed to copy the exact look and feel of a "real" site. A model for the number of reported spoof sites from November 2005 through October 2006 is

$$f(t) = 88.253t^3 - 1116.16t^2 + 4541.4t + 4161, \ 0 \le t \le 11$$

where $t$ represents the number of months since November 2005. *(Source: Anti-Phishing Working Group)*

(T) (a) Use a graphing utility to graph the model on the interval $[0, 11]$.

(b) Use the graph in part (a) to estimate the month corresponding to the absolute minimum number of spoof sites.

(c) Use the graph in part (a) to estimate the month corresponding to the absolute maximum number of spoof sites.

(d) During approximately which month was the rate of increase of the number of spoof sites the greatest? the least?

**76. Dow Jones Industrial Average** The graph shows the Dow Jones Industrial Average $y$ on Black Monday, October 19, 1987, where $t = 0$ corresponds to 9:30 A.M., when the market opens, and $t = 6.5$ corresponds to 4 P.M., the closing time. *(Source: Wall Street Journal)*



Black Monday

(a) Estimate the relative extrema and absolute extrema of the graph. Interpret your results in the context of the problem.

(b) Estimate the point of inflection of the graph on the interval $[1, 3]$. Interpret your result in the context of the problem.

**77. Think About It** Let $S$ represent monthly sales of a new digital audio player. Write a statement describing $S'$ and $S''$ for each of the following.

(a) The rate of change of sales is increasing.

(b) Sales are increasing, but at a greater rate.

(c) The rate of change of sales is steady.

(d) Sales are steady.

(e) Sales are declining, but at a lower rate.

(f) Sales have bottomed out and have begun to rise.

**78. Medicine** The spread of a virus can be modeled by

$$N = -t^3 + 12t^2, \quad 0 \le t \le 12$$

where $N$ is the number of people infected (in hundreds), and $t$ is the time (in weeks).

(a) What is the maximum number of people projected to be infected?

(b) When will the virus be spreading most rapidly?

(T) (c) Use a graphing utility to graph the model and to verify your results.

**Business Capsule**



Photo courtesy of Pat Alexander Sanford

In 1985, Pat Alexander Sanford started Alexander Perry, Inc., in Philadelphia, Pennsylvania. The company specializes in providing interior architecture and space planning to corporations, educational institutions, and private residences. Sanford started the company using about $5000 from her personal savings and a grant from the Women's Enterprise Center in Philadelphia. The company was incorporated in 1992. Revenues for the company topped $714,000 in 2004 and contracts for 2006 totaled about $6 million. Projected sales are currently expected to approach $10 million.

**79. Research Project** Use your school's library, the Internet, or some other reference source to research the financial history of a small company like the one above. Gather the data on the company's costs and revenues over a period of time, and use a graphing utility to graph a scatter plot of the data. Fit models to the data. Do the models appear to be concave upward or downward? Do they appear to be increasing or decreasing? Discuss the implications of your answers.

# Section 3.4

# Optimization Problems

■ Solve real-life optimization problems.

## Solving Optimization Problems

One of the most common applications of calculus is the determination of optimum (minimum or maximum) values. Before learning a general method for solving optimization problems, consider the next example.

### Example 1 Finding the Maximum Volume ®

A manufacturer wants to design an open box that has a square base and a surface area of 108 square inches, as shown in Figure 3.31. What dimensions will produce a box with a maximum volume?

**SOLUTION** Because the base of the box is square, the volume is

$$V = x^2 h. \qquad \text{Primary equation}$$

This equation is called the **primary equation** because it gives a formula for the quantity to be optimized. The surface area of the box is

$$S = (\text{area of base}) + (\text{area of four sides})$$
$$108 = x^2 + 4xh. \qquad \text{Secondary equation}$$

Because $V$ is to be optimized, it helps to express $V$ as a function of just one variable. To do this, solve the secondary equation for $h$ in terms of $x$ to obtain

$$h = \frac{108 - x^2}{4x}$$

and substitute into the primary equation.

$$V = x^2 h = x^2 \left( \frac{108 - x^2}{4x} \right) = 27x - \frac{1}{4}x^3 \qquad \text{Function of one variable}$$

Before finding which $x$-value yields a maximum value of $V$, you need to determine the *feasible domain* of the function. That is, what values of $x$ make sense in the problem? Because $x$ must be nonnegative and the area of the base $(A = x^2)$ is at most 108, you can conclude that the feasible domain is

$$0 \leq x \leq \sqrt{108}. \qquad \text{Feasible domain}$$

Using the techniques described in the first three sections of this chapter, you can determine that $\left( \text{on the interval } 0 \leq x \leq \sqrt{108} \right)$ this function has an absolute maximum when $x = 6$ inches and $h = 3$ inches.



**FIGURE 3.31** Open Box with Square Base: $S = x^2 + 4xh = 108$

### ✓ CHECKPOINT 1

Use a graphing utility to graph the volume function $V = 27x - \frac{1}{4}x^3$ on $0 \leq x \leq \sqrt{108}$ from Example 1. Verify that the function has an absolute maximum when $x = 6$. What is the maximum volume? ■

In studying Example 1, be sure that you understand the basic question that it asks. Some students have trouble with optimization problems because they are too eager to start solving the problem by using a standard formula. For instance, in Example 1, you should realize that there are infinitely many open boxes having 108 square inches of surface area. You might begin to solve this problem by asking yourself which basic shape would seem to yield a maximum volume. Should the box be tall, squat, or nearly cubical? You might even try calculating a few volumes, as shown in Figure 3.32, to see if you can get a good feeling for what the optimum dimensions should be.

Volume $= 74\frac{1}{4}$

Volume $= 92$

Volume $= 103\frac{3}{4}$

$3 \times 3 \times 8\frac{1}{4}$        $4 \times 4 \times 5\frac{3}{4}$        $5 \times 5 \times 4\frac{3}{20}$

Volume $= 108$

Volume $= 88$

$6 \times 6 \times 3$        $8 \times 8 \times 1\frac{3}{8}$

**FIGURE 3.32**    Which box has the greatest volume?

There are several steps in the solution of Example 1. The first step is to sketch a diagram and identify all *known* quantities and all quantities *to be determined*. The second step is to write a primary equation for the quantity to be optimized. Then, a secondary equation is used to rewrite the primary equation as a function of one variable. Finally, calculus is used to determine the optimum value. These steps are summarized below.

**Guidelines for Solving Optimization Problems**

1. Identify all given quantities and all quantities to be determined. If possible, make a sketch.

2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A summary of several common formulas is given in Appendix D.)

3. Reduce the primary equation to one having a single independent variable. This may involve the use of a **secondary equation** that relates the independent variables of the primary equation.

4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.

5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 3.1 through 3.3.

*Example 2*  **Finding a Minimum Sum**

The product of two positive numbers is 288. Minimize the sum of the second number and twice the first number.

**SOLUTION**

1. Let $x$ be the first number, $y$ the second, and $S$ the sum to be minimized.

2. Because you want to minimize $S$, the primary equation is

$$S = 2x + y. \qquad \text{Primary equation}$$

3. Because the product of the two numbers is 288, you can write the secondary equation as

$$xy = 288 \qquad \text{Secondary equation}$$

$$y = \frac{288}{x}.$$

Using this result, you can rewrite the primary equation as a function of one variable.

$$S = 2x + \frac{288}{x} \qquad \text{Function of one variable}$$

4. Because the numbers are positive, the feasible domain is

$$x > 0. \qquad \text{Feasible domain}$$

5. To find the minimum value of $S$, begin by finding its critical numbers.

$$\frac{dS}{dx} = 2 - \frac{288}{x^2} \qquad \text{Find derivative of } S.$$

$$0 = 2 - \frac{288}{x^2} \qquad \text{Set derivative equal to 0.}$$

$$x^2 = 144 \qquad \text{Simplify.}$$

$$x = \pm 12 \qquad \text{Critical numbers}$$

Choosing the positive $x$-value, you can use the First-Derivative Test to conclude that $S$ is decreasing on the interval $(0, 12)$ and increasing on the interval $(12, \infty)$, as shown in the table. So, $x = 12$ yields a minimum, and the two numbers are

$$x = 12 \quad \text{and} \quad y = \frac{288}{12} = 24.$$

| Interval | $0 < x < 12$ | $12 < x < \infty$ |
|---|---|---|
| Test value | $x = 1$ | $x = 13$ |
| Sign of $\dfrac{dS}{dx}$ | $\dfrac{dS}{dx} < 0$ | $\dfrac{dS}{dx} > 0$ |
| Conclusion | $S$ is decreasing. | $S$ is increasing. |

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**Algebra Review**

For help on the algebra in Example 2, see Example 1(b) in the *Chapter 3 Algebra Review*, on page 283.

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**TECHNOLOGY**

(T) After you have written the primary equation as a function of a single variable, you can estimate the optimum value by graphing the function. For instance, the graph of

$$S = 2x + \frac{288}{x}$$

shown below indicates that the minimum value of $S$ occurs when $x$ is about 12.



Relative minimum when $x \approx 12$

---

✓**CHECKPOINT 2**

The product of two numbers is 72. Minimize the sum of the second number and twice the first number. ■

$y = 4 - x^2$

$d$

$(x, y)$

$(0, 2)$

$d = \sqrt{(x-0)^2 + (y-2)^2}$

**FIGURE 3.33**

*Example 3*   **Finding a Minimum Distance**

Find the points on the graph of

$$y = 4 - x^2$$

that are closest to $(0, 2)$.

**SOLUTION**

1. Figure 3.33 indicates that there are two points at a minimum distance from the point $(0, 2)$.

2. You are asked to minimize the distance $d$. So, you can use the Distance Formula to obtain a primary equation.

$$d = \sqrt{(x - 0)^2 + (y - 2)^2} \qquad \text{Primary equation}$$

3. Using the secondary equation $y = 4 - x^2$, you can rewrite the primary equation as a function of a single variable.

$$d = \sqrt{x^2 + (4 - x^2 - 2)^2} \qquad \text{Substitute } 4 - x^2 \text{ for } y.$$
$$= \sqrt{x^4 - 3x^2 + 4} \qquad \text{Simplify.}$$

Because $d$ is smallest when the expression under the radical is smallest, you simplify the problem by finding the minimum value of $f(x) = x^4 - 3x^2 + 4$.

4. The domain of $f$ is the entire real line.

5. To find the minimum value of $f(x)$, first find the critical numbers of $f$.

$$f'(x) = 4x^3 - 6x \qquad \text{Find derivative of } f.$$
$$0 = 4x^3 - 6x \qquad \text{Set derivative equal to 0.}$$
$$0 = 2x(2x^2 - 3) \qquad \text{Factor.}$$
$$x = 0, x = \sqrt{\tfrac{3}{2}}, x = -\sqrt{\tfrac{3}{2}} \qquad \text{Critical numbers}$$

By the First-Derivative Test, you can conclude that $x = 0$ yields a relative maximum, whereas both $\sqrt{3/2}$ and $-\sqrt{3/2}$ yield a minimum. So, on the graph of $y = 4 - x^2$, the points that are closest to the point $(0, 2)$ are

$$\left(\sqrt{\tfrac{3}{2}}, \tfrac{5}{2}\right) \quad \text{and} \quad \left(-\sqrt{\tfrac{3}{2}}, \tfrac{5}{2}\right).$$

✓**CHECKPOINT 3**

Find the points on the graph of $y = 4 - x^2$ that are closest to $(0, 3)$. ∎

**Algebra Review**

For help on the algebra in Example 3, see Example 1(c) in the *Chapter 3 Algebra Review*, on page 283.

**STUDY TIP**

To confirm the result in Example 3, try computing the distances between several points on the graph of $y = 4 - x^2$ and the point $(0, 2)$. For instance, the distance between $(1, 3)$ and $(0, 2)$ is

$$d = \sqrt{(0 - 1)^2 + (2 - 3)^2} = \sqrt{2} \approx 1.414.$$

Note that this is greater than the distance between $\left(\sqrt{3/2}, 5/2\right)$ and $(0, 2)$, which is

$$d = \sqrt{\left(0 - \sqrt{\tfrac{3}{2}}\right)^2 + \left(2 - \tfrac{5}{2}\right)^2} = \sqrt{\tfrac{7}{4}} \approx 1.323.$$

*A = (x + 3)(y + 2)*

**FIGURE 3.34**

*Example 4*  **Finding a Minimum Area**  Ⓡ

A rectangular page will contain 24 square inches of print. The margins at the top and bottom of the page are $1\frac{1}{2}$ inches wide. The margins on each side are 1 inch wide. What should the dimensions of the page be to minimize the amount of paper used?

**SOLUTION**

1.  A diagram of the page is shown in Figure 3.34.

2.  Letting $A$ be the area to be minimized, the primary equation is

$$A = (x + 3)(y + 2).$$    Primary equation

3.  The printed area inside the margins is given by

$$24 = xy.$$    Secondary equation

Solving this equation for $y$ produces

$$y = \frac{24}{x}.$$

By substituting this into the primary equation, you obtain

$$A = (x + 3)\left(\frac{24}{x} + 2\right)$$    Write as a function of one variable.

$$= (x + 3)\left(\frac{24 + 2x}{x}\right)$$    Rewrite second factor as a single fraction.

$$= \frac{2x^2}{x} + \frac{30x}{x} + \frac{72}{x}$$    Multiply and separate into terms.

$$= 2x + 30 + \frac{72}{x}.$$    Simplify.

4.  Because $x$ must be positive, the feasible domain is $x > 0$.

5.  To find the minimum area, begin by finding the critical numbers of $A$.

$$\frac{dA}{dx} = 2 - \frac{72}{x^2}$$    Find derivative of $A$.

$$0 = 2 - \frac{72}{x^2}$$    Set derivative equal to 0.

$$-2 = -\frac{72}{x^2}$$    Subtract 2 from each side.

$$x^2 = 36$$    Simplify.

$$x = \pm 6$$    Critical numbers

Because $x = -6$ is not in the feasible domain, you only need to consider the critical number $x = 6$. Using the First-Derivative Test, it follows that $A$ is a minimum when $x = 6$. So, the dimensions of the page should be

$$x + 3 = 6 + 3 = 9 \text{ inches by } y + 2 = \frac{24}{6} + 2 = 6 \text{ inches.}$$
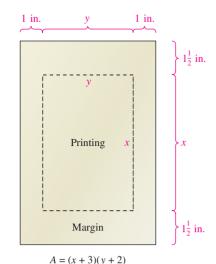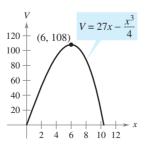
✓ **CHECKPOINT 4**
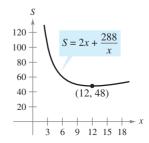
A rectangular page will contain 54 square inches of print. The margins at the top and bottom of the page are $1\frac{1}{2}$ inches wide. The margins on each side are 1 inch wide. What should the dimensions of the page be to minimize the amount of paper used? ▪

As applications go, the four examples described in this section are fairly simple, and yet the resulting primary equations are quite complicated. Real-life applications often involve equations that are at least as complex as these four. Remember that one of the main goals of this course is to enable you to use the power of calculus to analyze equations that at first glance seem formidable.
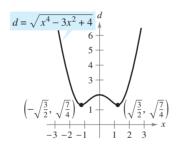
Also remember that once you have found the primary equation, you can use the graph of the equation to help solve the problem. For instance, the graphs of the primary equations in Examples 1 through 4 are shown in Figure 3.35.



Example 1
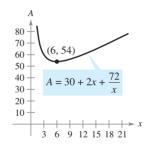


Example 2



Example 3



Example 4

**FIGURE 3.35**

## CONCEPT CHECK

1. Complete the following: In an optimization problem, the formula that represents the quantity to be optimized is called the _____ _____.

2. Explain what is meant by the term *feasible domain*.

3. Explain the difference between a primary equation and a secondary equation.

4. In your own words, state the guidelines for solving an optimization problem.

| **Skills Review 3.4** | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 3.1. |

In Exercises 1–4, write a formula for the written statement.

**1.** The sum of one number and half a second number is 12
**2.** The product of one number and twice another is 24.
**3.** The area of a rectangle is 24 square units.
**4.** The distance between two points is 10 units.

In Exercises 5–10, find the critical numbers of the function.

**5.** $y = x^2 + 6x - 9$

**6.** $y = 2x^3 - x^2 - 4x$

**7.** $y = 5x + \dfrac{125}{x}$

**8.** $y = 3x + \dfrac{96}{x^2}$

**9.** $y = \dfrac{x^2 + 1}{x}$

**10.** $y = \dfrac{x}{x^2 + 9}$

# Exercises 3.4

In Exercises 1–6, find two positive numbers satisfying the given requirements.

**1.** The sum is 120 and the product is a maximum.

**2.** The sum is $S$ and the product is a maximum.

**3.** The sum of the first and twice the second is 36 and the product is a maximum.

**4.** The sum of the first and twice the second is 100 and the product is a maximum.

**5.** The product is 192 and the sum is a minimum.

**6.** The product is 192 and the sum of the first plus three times the second is a minimum.
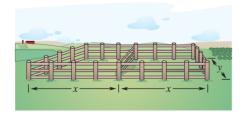
In Exercises 7 and 8, find the length and width of a rectangle that has the given perimeter and a maximum area.

**7.** Perimeter: 100 meters

**8.** Perimeter: $P$ units

In Exercises 9 and 10, find the length and width of a rectangle that has the given area and a minimum perimeter.
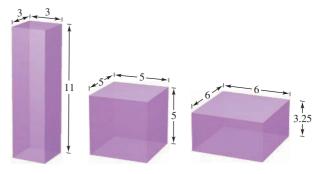
**9.** Area: 64 square feet

**10.** Area: $A$ square centimeters

**11. Maximum Area**   A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?



**12. Area**   A dairy farmer plans to enclose a rectangular pasture adjacent to a river. To provide enough grass for the herd, the pasture must contain 180,000 square meters. No fencing is required along the river. What dimensions will use the least amount of fencing?
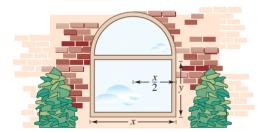
**13. Maximum Volume**
   (a) Verify that each of the rectangular solids shown in the figure has a surface area of 150 square inches.
   (b) Find the volume of each solid.
   (c) Determine the dimensions of a rectangular solid (with a square base) of maximum volume if its surface area is 150 square inches.



**14. Maximum Volume**   Determine the dimensions of a rectangular solid (with a square base) with maximum volume if its surface area is 337.5 square centimeters.

**15. Minimum Cost**   A storage box with a square base must have a volume of 80 cubic centimeters. The top and bottom cost $0.20 per square centimeter and the sides cost $0.10 per square centimeter. Find the dimensions that will minimize cost.

**16. Maximum Area**   A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). Find the dimensions of a Norman window of maximum area if the total perimeter is 16 feet.



**17. Minimum Surface Area**   A net enclosure for golf practice is open at one end (see figure). The volume of the enclosure is $83\frac{1}{3}$ cubic meters. Find the dimensions that require the least amount of netting.
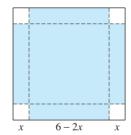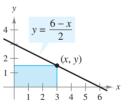


Figure for 17          Figure for 18

**18. Volume**   An open box is to be made from a six-inch by six-inch square piece of material by cutting equal squares from the corners and turning up the sides (see figure). Find the volume of the largest box that can be made.

**19. Volume**   An open box is to be made from a two-foot by three-foot rectangular piece of material by cutting equal squares from the corners and turning up the sides. Find the volume of the largest box that can be made in this manner.

**20. Maximum Yield**   A home gardener estimates that 16 apple trees will have an average yield of 80 apples per tree. But because of the size of the garden, for each additional tree planted the yield will decrease by four apples per tree. How many trees should be planted to maximize the total yield of apples? What is the maximum yield?

**21. Area**   A rectangular page is to contain 36 square inches of print. The margins at the top and bottom and on each side are to be $1\frac{1}{2}$ inches. Find the dimensions of the page that will minimize the amount of paper used.

**22. Area**   A rectangular page is to contain 30 square inches of print. The margins at the top and bottom of the page are to be 2 inches wide. The margins on each side are to be 1 inch wide. Find the dimensions of the page such that the least amount of paper is used.

**23. Maximum Area**   A rectangle is bounded by the $x$- and $y$-axes and the graph of $y = (6 - x)/2$ (see figure). What length and width should the rectangle have so that its area is a maximum?
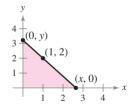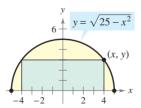


Figure for 23          Figure for 24

**24. Minimum Length**   A right triangle is formed in the first quadrant by the $x$- and $y$-axes and a line through the point $(1, 2)$ (see figure).

(a) Write the length $L$ of the hypotenuse as a function of $x$.

(T) (b) Use a graphing utility to approximate $x$ graphically such that the length of the hypotenuse is a minimum.

(c) Find the vertices of the triangle such that its area is a minimum.

**25. Maximum Area**   A rectangle is bounded by the $x$-axis and the semicircle

$$y = \sqrt{25 - x^2}$$

(see figure). What length and width should the rectangle have so that its area is a maximum?



**26. Area**   Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius $r$. (See Exercise 25.)

**27. Volume**   You are designing a soft drink container that has the shape of a right circular cylinder. The container is supposed to hold 12 fluid ounces (1 fluid ounce is approximately 1.80469 cubic inches). Find the dimensions that will use a minimum amount of construction material.

**28. Minimum Cost**   An energy drink container of the shape described in Exercise 27 must have a volume of 16 fluid ounces. The cost per square inch of constructing the top and bottom is twice the cost of constructing the sides. Find the dimensions that will minimize cost.

In Exercises 29–32, find the points on the graph of the function that are closest to the given point.
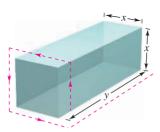
**29.** $f(x) = x^2$,   $\left(2, \dfrac{1}{2}\right)$

**30.** $f(x) = (x + 1)^2$,   $(5, 3)$
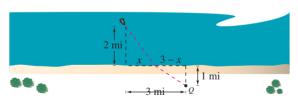
**31.** $f(x) = \sqrt{x}$,   $(4, 0)$

**32.** $f(x) = \sqrt{x - 8}$,   $(2, 0)$
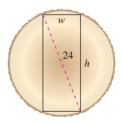
**33. Maximum Volume**   A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches. Find the dimensions of the package with maximum volume. Assume that the package's dimensions are $x$ by $x$ by $y$ (see figure).



**34. Minimum Surface Area**   A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid is 12 cubic inches. Find the radius of the cylinder that produces the minimum surface area.

**35. Minimum Cost**   An industrial tank of the shape described in Exercise 34 must have a volume of 3000 cubic feet. The hemispherical ends cost twice as much per square foot of surface area as the sides. Find the dimensions that will minimize cost.

**36. Minimum Area**   The sum of the perimeters of a circle and a square is 16. Find the dimensions of the circle and square that produce a minimum total area.

**37. Minimum Area**   The sum of the perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and square that produce a minimum total area.

**38. Minimum Time**   You are in a boat 2 miles from the nearest point on the coast. You are to go to point $Q$, located 3 miles down the coast and 1 mile inland (see figure). You can row at a rate of 2 miles per hour and you can walk at a rate of 4 miles per hour. Toward what point on the coast should you row in order to reach point $Q$ in the least time?



**39. Maximum Area**   An indoor physical fitness room consists of a rectangular region with a semicircle on each end. The perimeter of the room is to be a 200-meter running track. Find the dimensions that will make the area of the rectangular region as large as possible.

**40. Farming**   A strawberry farmer will receive $30 per bushel of strawberries during the first week of harvesting. Each week after that, the value will drop $0.80 per bushel. The farmer estimates that there are approximately 120 bushels of strawberries in the fields, and that the crop is increasing at a rate of four bushels per week. When should the farmer harvest the strawberries to maximize their value? How many bushels of strawberries will yield the maximum value? What is the maximum value of the strawberries?

**41. Beam Strength**   A wooden beam has a rectangular cross section of height $h$ and width $w$ (see figure). The strength $S$ of the beam is directly proportional to its width and the square of its height. What are the dimensions of the strongest beam that can be cut from a round log of diameter 24 inches?   (*Hint:* $S = kh^2w$, where $k > 0$ is the proportionality constant.)



**42. Area**   Four feet of wire is to be used to form a square and a circle.

(a) Express the sum of the areas of the square and the circle as a function $A$ of the side of the square $x$.

(b) What is the domain of $A$?

(T) (c) Use a graphing utility to graph $A$ on its domain.

(d) How much wire should be used for the square and how much for the circle in order to enclose the least total area? the greatest total area?

**43. Profit**   The profit $P$ (in thousands of dollars) for a company spending an amount $s$ (in thousands of dollars) on advertising is

$$P = -\frac{1}{10}s^3 + 6s^2 + 400.$$

(a) Find the amount of money the company should spend on advertising in order to yield a maximum profit.

(b) Find the point of diminishing returns.

## Mid-Chapter Quiz

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

Ⓣ In Exercises 1–3, find the critical numbers of the function and the open intervals on which the function is increasing or decreasing. Then use a graphing utility to graph the function.

**1.** $f(x) = x^2 - 6x + 1$     **2.** $f(x) = 2x^3 + 12x^2$     **3.** $f(x) = \dfrac{1}{x^2 + 2}$

Ⓣ In Exercises 4–6, use a graphing utility to graph the function. Then find all relative extrema of the function.

**4.** $f(x) = x^3 + 3x^2 - 5$     **5.** $f(x) = x^4 - 8x^2 + 3$     **6.** $f(x) = 2x^{2/3}$

In Exercises 7–9, find the absolute extrema of the function on the closed interval.

**7.** $f(x) = x^2 + 2x - 8, \quad [-2, 1]$     **8.** $f(x) = x^3 - 27x, \quad [-4, 4]$

**9.** $f(x) = \dfrac{x}{x^2 + 1}, \quad [0, 2]$

In Exercises 10 and 11, find the point(s) of inflection of the graph of the function. Then determine the open intervals on which the graph of the function is concave upward or concave downward.

**10.** $f(x) = x^3 - 6x^2 + 7x$     **11.** $f(x) = x^4 - 24x^2$

In Exercises 12 and 13, Use the Second-Derivative Test to find all relative extrema of the function.

**12.** $f(x) = 2x^3 + 3x^2 - 12x + 16$     **13.** $f(x) = \dfrac{x^2 + 1}{x}$

**14.** By increasing its advertising cost $x$ for a product, a company discovers that it can increase the sales $S$ according to the model

$$S = \frac{1}{3600}(360x^2 - x^3), \quad 0 \le x \le 240$$

where $x$ and $S$ are in thousands of dollars. Find the point of diminishing returns for this product.

**15.** A gardener has 200 feet of fencing to enclose a rectangular garden adjacent to a river (see figure). No fencing is needed along the river. What dimensions should be used so that the area of the garden will be a maximum?

Figure for 15

**16.** The resident population $P$ (in thousands) of the District of Columbia from 1999 through 2005 can be modeled by

$$P = 0.2694t^3 - 2.048t^2 - 0.73t + 571.9$$

where $-1 \le t \le 5$ and $t = 0$ corresponds to 2000.   *(Source: U.S. Census Bureau)*

(a) During which year, from 1999 through 2005, was the population the greatest? the least?

(b) During which year(s) was the population increasing? decreasing?
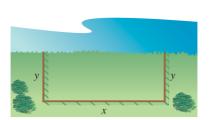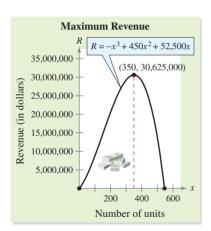
## Section 3.5

# Business and Economics Applications

- Solve business and economics optimization problems.
- Find the price elasticity of demand for demand functions.
- Recognize basic business terms and formulas.

## Optimization in Business and Economics

The problems in this section are primarily optimization problems. So, the five-step procedure used in Section 3.4 is an appropriate strategy to follow.

### Example 1   Finding the Maximum Revenue ®

A company has determined that its total revenue (in dollars) for a product can be modeled by

$$R = -x^3 + 450x^2 + 52{,}500x$$

where $x$ is the number of units produced (and sold). What production level will yield a maximum revenue?

**SOLUTION**

1. A sketch of the revenue function is shown in Figure 3.36.

2. The primary equation is the given revenue function.

$$R = -x^3 + 450x^2 + 52{,}500x \qquad \text{Primary equation}$$

3. Because $R$ is already given as a function of one variable, you do not need a secondary equation.

4. The feasible domain of the primary equation is

$$0 \le x \le 546. \qquad \text{Feasible domain}$$

This is determined by finding the $x$-intercepts of the revenue function, as shown in Figure 3.36.

5. To maximize the revenue, find the critical numbers.

$$\frac{dR}{dx} = -3x^2 + 900x + 52{,}500 = 0 \qquad \text{Set derivative equal to 0.}$$

$$-3(x - 350)(x + 50) = 0 \qquad \text{Factor.}$$

$$x = 350,\ x = -50 \qquad \text{Critical numbers}$$

The only critical number in the feasible domain is $x = 350$. From the graph of the function, you can see that the production level of 350 units corresponds to a maximum revenue.

**Maximum Revenue**

**FIGURE 3.36**   Maximum revenue occurs when $dR/dx = 0$.

### ✓ CHECKPOINT 1

Find the number of units that must be produced to maximize the revenue function $R = -x^3 + 150x^2 + 9375x$. What is the maximum revenue? ■

To study the effects of production levels on cost, economists use the **average cost function** $\overline{C}$, which is defined as

$$\overline{C} = \frac{C}{x} \qquad \text{Average cost function}$$

where $C = f(x)$ is the total cost function and $x$ is the number of units produced.

**Example 2**  **Finding the Minimum Average Cost** ®

A company estimates that the cost (in dollars) of producing $x$ units of a product can be modeled by $C = 800 + 0.04x + 0.0002x^2$. Find the production level that minimizes the average cost per unit.

**SOLUTION**

1. $C$ represents the total cost, $x$ represents the number of units produced, and $\overline{C}$ represents the average cost per unit.

2. The primary equation is

$$\overline{C} = \frac{C}{x}. \qquad \text{Primary equation}$$

3. Substituting the given equation for $C$ produces

$$\overline{C} = \frac{800 + 0.04x + 0.0002x^2}{x} \qquad \text{Substitute for } C.$$

$$= \frac{800}{x} + 0.04 + 0.0002x. \qquad \text{Function of one variable}$$

4. The feasible domain for this function is

$$x > 0. \qquad \text{Feasible domain}$$

5. You can find the critical numbers as shown.

$$\frac{d\overline{C}}{dx} = -\frac{800}{x^2} + 0.0002 = 0 \qquad \text{Set derivative equal to 0.}$$

$$0.0002 = \frac{800}{x^2}$$

$$x^2 = \frac{800}{0.0002} \qquad \begin{array}{l}\text{Multiply each side by } x^2 \text{ and divide}\\ \text{each side by 0.0002.}\end{array}$$

$$x^2 = 4{,}000{,}000$$

$$x = \pm 2000 \qquad \text{Critical numbers}$$
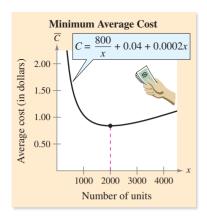
By choosing the positive value of $x$ and sketching the graph of $\overline{C}$, as shown in Figure 3.37, you can see that a production level of $x = 2000$ minimizes the average cost per unit.

**STUDY TIP**

To see that $x = 2000$ corresponds to a minimum average cost in Example 2, try evaluating $\overline{C}$ for several values of $x$. For instance, when $x = 400$, the average cost per unit is $\overline{C} = \$2.12$, but when $x = 2000$, the average cost per unit is $\overline{C} = \$0.84$.

**FIGURE 3.37**  Minimum average cost occurs when $d\overline{C}/dx = 0$.

✓ **CHECKPOINT 2**

Find the production level that minimizes the average cost per unit for the cost function $C = 400 + 0.05x + 0.0025x^2$. ■

*Example 3*   **Finding the Maximum Revenue**   ®

A business sells 2000 units of a product per month at a price of $10 each. It can sell 250 more items per month for each $0.25 reduction in price. What price per unit will maximize the monthly revenue?

**SOLUTION**

1. Let $x$ represent the number of units sold in a month, let $p$ represent the price per unit, and let $R$ represent the monthly revenue.

2. Because the revenue is to be maximized, the primary equation is

$$R = xp.$$  <span style="color:magenta">Primary equation</span>

3. A price of $p = \$10$ corresponds to $x = 2000$, and a price of $p = \$9.75$ corresponds to $x = 2250$. Using this information, you can use the point-slope form to create the demand equation.

$$p - 10 = \frac{10 - 9.75}{2000 - 2250}(x - 2000)$$   <span style="color:magenta">Point-slope form</span>

$$p - 10 = -0.001(x - 2000)$$   <span style="color:magenta">Simplify.</span>

$$p = -0.001x + 12$$   <span style="color:magenta">Secondary equation</span>

Substituting this value into the revenue equation produces

$$R = x(-0.001x + 12)$$   <span style="color:magenta">Substitute for $p$.</span>

$$= -0.001x^2 + 12x.$$   <span style="color:magenta">Function of one variable</span>

4. The feasible domain of the revenue function is

$$0 \le x \le 12{,}000.$$   <span style="color:magenta">Feasible domain</span>

5. To maximize the revenue, find the critical numbers.

$$\frac{dR}{dx} = 12 - 0.002x = 0$$   <span style="color:magenta">Set derivative equal to 0.</span>

$$-0.002x = -12$$

$$x = 6000$$   <span style="color:magenta">Critical number</span>

From the graph of $R$ in Figure 3.38, you can see that this production level yields a maximum revenue. The price that corresponds to this production level is

$$p = 12 - 0.001x$$   <span style="color:magenta">Demand function</span>

$$= 12 - 0.001(6000)$$   <span style="color:magenta">Substitute 6000 for $x$.</span>

$$= \$6.$$   <span style="color:magenta">Price per unit</span>



**Maximum Revenue**

(6000, 36,000)

$R = 12x - 0.001x^2$

Revenue (in dollars)

Number of units
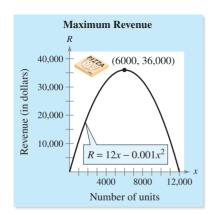
**FIGURE 3.38**

**STUDY TIP**
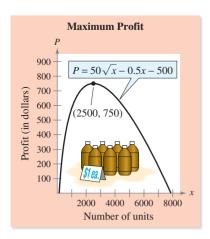
In Example 3, the revenue function was written as a function of $x$. It could also have been written as a function of $p$. That is, $R = 1000(12p - p^2)$. By finding the critical numbers of this function, you can determine that the maximum revenue occurs when $p = 6$.

✓**CHECKPOINT 3**

Find the price per unit that will maximize the monthly revenue for the business in Example 3 if it can sell only 200 more items per month for each $0.25 reduction in price. ■

**Maximum Profit**



**FIGURE 3.39**

✓ **CHECKPOINT 4**

Find the price that will maximize profit for the demand and cost functions.

$$p = \frac{40}{\sqrt{x}} \text{ and } C = 2x + 50 \text{ ■}$$

**Marginal Revenue and Marginal Cost**
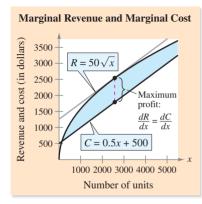


**FIGURE 3.40**

*Example 4* **Finding the Maximum Profit** ®

The marketing department of a business has determined that the demand for a product can be modeled by

$$p = \frac{50}{\sqrt{x}}.$$

The cost of producing $x$ units is given by $C = 0.5x + 500$. What price will yield a maximum profit?

**SOLUTION**

1. Let $R$ represent the revenue, $P$ the profit, $p$ the price per unit, $x$ the number of units, and $C$ the total cost of producing $x$ units.

2. Because you are maximizing the profit, the primary equation is

$$P = R - C. \qquad \text{Primary equation}$$

3. Because the revenue is $R = xp$, you can write the profit function as

$$\begin{aligned} P &= R - C \\ &= xp - (0.5x + 500) & \text{Substitute for } R \text{ and } C. \\ &= x\left(\frac{50}{\sqrt{x}}\right) - 0.5x - 500 & \text{Substitute for } p. \\ &= 50\sqrt{x} - 0.5x - 500. & \text{Function of one variable} \end{aligned}$$

4. The feasible domain of the function is $127 < x \le 7872$. (When $x$ is less than 127 or greater than 7872, the profit is negative.)

5. To maximize the profit, find the critical numbers.

$$\begin{aligned} \frac{dP}{dx} &= \frac{25}{\sqrt{x}} - 0.5 = 0 & \text{Set derivative equal to 0.} \\ \sqrt{x} &= 50 & \text{Isolate } x\text{-term on one side.} \\ x &= 2500 & \text{Critical number} \end{aligned}$$

From the graph of the profit function shown in Figure 3.39, you can see that a maximum profit occurs when $x = 2500$. The price that corresponds to $x = 2500$ is

$$p = \frac{50}{\sqrt{x}} = \frac{50}{\sqrt{2500}} = \frac{50}{50} = \$1.00. \qquad \text{Price per unit}$$

**STUDY TIP**

To find the maximum profit in Example 4, the equation $P = R - C$ was differentiated and set equal to zero. From the equation

$$\frac{dP}{dx} = \frac{dR}{dx} - \frac{dC}{dx} = 0$$

it follows that the maximum profit occurs when the marginal revenue is equal to the marginal cost, as shown in Figure 3.40.

## Price Elasticity of Demand

One way economists measure the responsiveness of consumers to a change in the price of a product is with **price elasticity of demand.** For example, a drop in the price of vegetables might result in a much greater demand for vegetables; such a demand is called **elastic.** On the other hand, the demand for items such as milk and water is relatively unresponsive to changes in price; the demand for such items is called **inelastic.**

More formally, the elasticity of demand is the percent change of a quantity demanded $x$, divided by the percent change in its price $p$. You can develop a formula for price elasticity of demand using the approximation

$$\frac{\Delta p}{\Delta x} \approx \frac{dp}{dx}$$

which is based on the definition of the derivative. Using this approximation, you can write

$$\text{Price elasticity of demand} = \frac{\text{rate of change in demand}}{\text{rate of change in price}}$$
$$= \frac{\Delta x/x}{\Delta p/p}$$
$$= \frac{p/x}{\Delta p/\Delta x}$$
$$\approx \frac{p/x}{dp/dx}.$$

**Definition of Price Elasticity of Demand**

If $p = f(x)$ is a differentiable function, then the **price elasticity of demand** is given by

$$\eta = \frac{p/x}{dp/dx}$$

where $\eta$ is the lowercase Greek letter eta. For a given price, the demand is **elastic** if $|\eta| > 1$, the demand is **inelastic** if $|\eta| < 1$, and the demand has **unit elasticity** if $|\eta| = 1$.

Price elasticity of demand is related to the total revenue function, as indicated in Figure 3.41 and the list below.

**1.** If the demand is *elastic*, then a decrease in price is accompanied by an increase in unit sales sufficient to increase the total revenue.

**2.** If the demand is *inelastic*, then a decrease in price is not accompanied by an increase in unit sales sufficient to increase the total revenue.

**FIGURE 3.41** Revenue Curve

**Example 5** **Comparing Elasticity and Revenue** ®

The demand function for a product is modeled by $p = 18 - 1.5\sqrt{x}$, $0 \le x \le 144$, as shown in Figure 3.42(a).

**a.** Find the intervals on which the demand is elastic, inelastic, and of unit elasticity.

**b.** Use the result of part (a) to describe the behavior of the revenue function.

**SOLUTION**

**a.** The price elasticity of demand is given by

$$\eta = \frac{p/x}{dp/dx} \qquad \text{Formula for price elasticity of demand}$$

$$= \frac{\dfrac{18 - 1.5\sqrt{x}}{x}}{\dfrac{-3}{4\sqrt{x}}} \qquad \text{Substitute for } p/x \text{ and } dp/dx.$$

$$= \frac{-24\sqrt{x} + 2x}{x} \qquad \text{Multiply numerator and denominator by } -\dfrac{4\sqrt{x}}{3}.$$

$$= -\frac{24\sqrt{x}}{x} + 2. \qquad \text{Rewrite as two fractions and simplify.}$$

The demand is of unit elasticity when $|\eta| = 1$. In the interval $[0, 144]$, the only solution of the equation

$$|\eta| = \left| -\frac{24\sqrt{x}}{x} + 2 \right| = 1 \qquad \text{Unit elasticity}$$

is $x = 64$. So, the demand is of unit elasticity when $x = 64$. For $x$-values in the interval $(0, 64)$,

$$|\eta| = \left| -\frac{24\sqrt{x}}{x} + 2 \right| > 1, \quad 0 < x < 64 \qquad \text{Elastic}$$

which implies that the demand is elastic when $0 < x < 64$. For $x$-values in the interval $(64, 144)$,
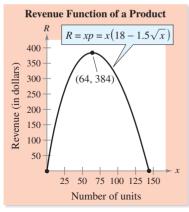
$$|\eta| = \left| -\frac{24\sqrt{x}}{x} + 2 \right| < 1, \quad 64 < x < 144 \qquad \text{Inelastic}$$

which implies that the demand is inelastic when $64 < x < 144$.

**b.** From part (a), you can conclude that the revenue function $R$ is increasing on the open interval $(0, 64)$, is decreasing on the open interval $(64, 144)$, and is a maximum when $x = 64$, as indicated in Figure 3.42(b).

**Demand Function of a Product**

$p = 18 - 1.5\sqrt{x}$

Number of units

(a)

**Revenue Function of a Product**

$R = xp = x(18 - 1.5\sqrt{x})$

(64, 384)

Number of units

(b)

**FIGURE 3.42**

✓ **CHECKPOINT 5**

Find the intervals on which the demand function $p = 36 - 2\sqrt{x}$, $0 \le x \le 324$, is elastic, inelastic, and of unit elasticity. ■

**STUDY TIP**

In the discussion of price elasticity of demand, the price is assumed to decrease as the quantity demanded increases. So, the demand function $p = f(x)$ is decreasing and $dp/dx$ is negative.

## Business Terms and Formulas

This section concludes with a summary of the basic business terms and formulas used in this section. A summary of the graphs of the demand, revenue, cost, and profit functions is shown in Figure 3.43.

*Summary of Business Terms and Formulas*

| | |
|---|---|
| $x$ = number of units produced (or sold) | $\eta$ = price elasticity of demand |
| $p$ = price per unit | $\quad = (p/x)/(dp/dx)$ |
| $R$ = total revenue from selling $x$ units $= xp$ | $dR/dx$ = marginal revenue |
| $C$ = total cost of producing $x$ units | $dC/dx$ = marginal cost |
| $P$ = total profit from selling $x$ units $= R - C$ | $dP/dx$ = marginal profit |
| $\overline{C}$ = average cost per unit $= \dfrac{C}{x}$ | |
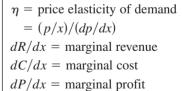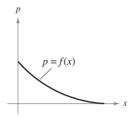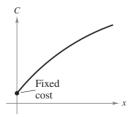


Demand function
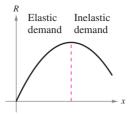
Quantity demanded increases as price decreases.



Revenue function

The low prices required to sell more units eventually result in a decreasing revenue.
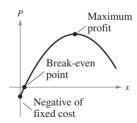


Cost function

The total cost to produce $x$ units includes the fixed cost.



Profit function

The break-even point occurs when $R = C$.

**FIGURE 3.43**

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**CONCEPT CHECK**

1. In the average cost function $\overline{C} = \dfrac{C}{x}$, what does $C$ represent? What does $x$ represent?

2. After a drop in the price of tomatoes, the demand for tomatoes increased. This is an example of what type of demand?

3. Even though the price of gasoline rose, the demand for gasoline was the same. This is an example of what type of demand?

4. Explain how price elasticity of demand is related to the total revenue function.

**Skills Review 3.5**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 0.3, 0.5, and 2.3.

In Exercises 1–4, evaluate the expression for $x = 150$.

**1.** $\left| -\dfrac{300}{x} + 3 \right|$

**2.** $\left| -\dfrac{600}{5x} + 2 \right|$

**3.** $\left| \dfrac{(20x^{-1/2})/x}{-10x^{-3/2}} \right|$

**4.** $\left| \dfrac{(4000/x^2)/x}{-8000x^{-3}} \right|$

In Exercises 5–10, find the marginal revenue, marginal cost, or marginal profit.

**5.** $C = 650 + 1.2x + 0.003x^2$

**6.** $P = 0.01x^2 + 11x$

**7.** $R = 14x - \dfrac{x^2}{2000}$

**8.** $R = 3.4x - \dfrac{x^2}{1500}$

**9.** $P = -0.7x^2 + 7x - 50$

**10.** $C = 1700 + 4.2x + 0.001x^3$

# Exercises 3.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find the number of units $x$ that produces a maximum revenue $R$.

**1.** $R = 800x - 0.2x^2$

**2.** $R = 48x^2 - 0.02x^3$

**3.** $R = 400x - x^2$

**4.** $R = 30x^{2/3} - 2x$

In Exercises 5–8, find the number of units $x$ that produces the minimum average cost per unit $\overline{C}$.

**5.** $C = 0.125x^2 + 20x + 5000$

**6.** $C = 0.001x^3 + 5x + 250$

**7.** $C = 2x^2 + 255x + 5000$

**8.** $C = 0.02x^3 + 55x^2 + 1380$

In Exercises 9–12, find the price per unit $p$ that produces the maximum profit $P$.

| Cost Function | Demand Function |
|---|---|
| **9.** $C = 100 + 30x$ | $p = 90 - x$ |
| **10.** $C = 0.5x + 500$ | $p = \dfrac{50}{\sqrt{x}}$ |
| **11.** $C = 8000 + 50x + 0.03x^2$ | $p = 70 - 0.01x$ |
| **12.** $C = 35x + 500$ | $p = 50 - 0.1\sqrt{x}$ |

Ⓣ **Average Cost** In Exercises 13 and 14, use the cost function to find the production level for which the average cost is a minimum. For this production level, show that the marginal cost and average cost are equal. Use a graphing utility to graph the average cost function and verify your results.

**13.** $C = 2x^2 + 5x + 18$

**14.** $C = x^3 - 6x^2 + 13x$

**15. Maximum Profit** A commodity has a demand function modeled by $p = 100 - 0.5x$, and a total cost function modeled by $C = 40x + 37.5$.

(a) What price yields a maximum profit?

(b) When the profit is maximized, what is the average cost per unit?

**16. Maximum Profit** How would the answer to Exercise 15 change if the marginal cost rose from $40 per unit to $50 per unit? In other words, rework Exercise 15 using the cost function $C = 50x + 37.5$.
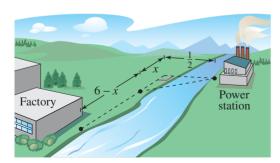
**Maximum Profit** In Exercises 17 and 18, find the amount $s$ of advertising that maximizes the profit $P$. ($s$ and $P$ are measured in thousands of dollars.) Find the point of diminishing returns.

**17.** $P = -2s^3 + 35s^2 - 100s + 200$

**18.** $P = -0.1s^3 + 6s^2 + 400$

**19. Maximum Profit** The cost per unit of producing a type of digital audio player is $60. The manufacturer charges $90 per unit for orders of 100 or less. To encourage large orders, however, the manufacturer reduces the charge by $0.10 per player for each order in excess of 100 units. For instance, an order of 101 players would be $89.90 per player, an order of 102 players would be $89.80 per player, and so on. Find the largest order the manufacturer should allow to obtain a maximum profit.

**20. Maximum Profit**   A real estate office handles a 50-unit apartment complex. When the rent is $580 per month, all units are occupied. For each $40 increase in rent, however, an average of one unit becomes vacant. Each occupied unit requires an average of $45 per month for service and repairs. What rent should be charged to obtain a maximum profit?

**21. Maximum Revenue**   When a wholesaler sold a product at $40 per unit, sales were 300 units per week. After a price increase of $5, however, the average number of units sold dropped to 275 per week. Assuming that the demand function is linear, what price per unit will yield a maximum total revenue?

**22. Maximum Profit**   Assume that the amount of money deposited in a bank is proportional to the square of the interest rate the bank pays on the money. Furthermore, the bank can reinvest the money at 12% simple interest. Find the interest rate the bank should pay to maximize its profit.

**(T) 23. Minimum Cost**   A power station is on one side of a river that is 0.5 mile wide, and a factory is 6 miles downstream on the other side of the river (see figure). It costs $18 per foot to run overland power lines and $25 per foot to run underwater power lines. Write a cost function for running the power lines from the power station to the factory. Use a graphing utility to graph your function. Estimate the value of $x$ that minimizes the cost. Explain your results.



**24. Minimum Cost**   An offshore oil well is 1 mile off the coast. The oil refinery is 2 miles down the coast. Laying pipe in the ocean is twice as expensive as laying it on land. Find the most economical path for the pipe from the well to the oil refinery.

**Minimum Cost**   In Exercises 25 and 26, find the speed $v$, in miles per hour, that will minimize costs on a 110-mile delivery trip. The cost per hour for fuel is $C$ dollars, and the driver is paid $W$ dollars per hour. (Assume there are no costs other than wages and fuel.)

**25.** Fuel cost: $C = \dfrac{v^2}{300}$

Driver: $W = \$12$

**26.** Fuel cost: $C = \dfrac{v^2}{500}$

Driver: $W = \$9.50$

**(T) Elasticity**   In Exercises 27–32, find the price elasticity of demand for the demand function at the indicated $x$-value. Is the demand elastic, inelastic, or of unit elasticity at the indicated $x$-value? Use a graphing utility to graph the revenue function, and identify the intervals of elasticity and inelasticity.

|  | Demand Function | Quantity Demanded |
|---|---|---|
| **27.** | $p = 600 - 5x$ | $x = 30$ |
| **28.** | $p = 400 - 3x$ | $x = 20$ |
| **29.** | $p = 5 - 0.03x$ | $x = 100$ |
| **30.** | $p = 20 - 0.0002x$ | $x = 30$ |
| **31.** | $p = \dfrac{500}{x + 2}$ | $x = 23$ |
| **32.** | $p = \dfrac{100}{x^2} + 2$ | $x = 10$ |

**33. Elasticity**   The demand function for a product is given by

$$p = 20 - 0.02x, \quad 0 < x < 1000.$$

(a) Find the price elasticity of demand when $x = 560$.

(b) Find the values of $x$ and $p$ that maximize the total revenue.

(c) For the value of $x$ found in part (b), show that the price elasticity of demand has unit elasticity.

**34. Elasticity**   The demand function for a product is given by

$$p = 800 - 4x, \quad 0 < x < 200.$$

(a) Find the price elasticity of demand when $x = 150$.

(b) Find the values of $x$ and $p$ that maximize the total revenue.

(c) For the value of $x$ found in part (b), show that the price elasticity of demand has unit elasticity.

**(T) 35. Minimum Cost**   The shipping and handling cost $C$ of a manufactured product is modeled by

$$C = 4\left(\frac{25}{x^2} - \frac{x}{x - 10}\right), \quad 0 < x < 10$$

where $C$ is measured in thousands of dollars and $x$ is the number of units shipped (in hundreds). Find the shipment size that minimizes the cost. (*Hint:* Use the *root* feature of a graphing utility.)

**(T) 36. Minimum Cost**   The ordering and transportation cost $C$ of the components used in manufacturing a product is modeled by

$$C = 8\left(\frac{2500}{x^2} - \frac{x}{x - 100}\right), \quad 0 < x < 100$$

where $C$ is measured in thousands of dollars and $x$ is the order size in hundreds. Find the order size that minimizes the cost. (*Hint:* Use the *root* feature of a graphing utility.)

**37.** *MAKE A DECISION: REVENUE*   The demand for a car wash is $x = 600 - 50p$, where the current price is \$5. Can revenue be increased by lowering the price and thus attracting more customers? Use price elasticity of demand to determine your answer.

**38. Revenue**   Repeat Exercise 37 for a demand function of

$$x = 800 - 40p.$$

**39. Sales**   The sales $S$ (in billions of dollars per year) for Procter & Gamble for the years 2001 through 2006 can be modeled by

$$S = 1.09312t^2 - 1.8682t + 39.831, \quad 1 \le t \le 6$$
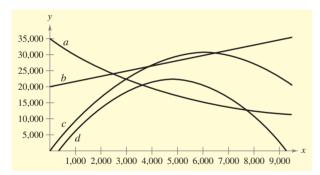
where $t$ represents the year, with $t = 1$ corresponding to 2001.   *(Source: Procter & Gamble Company)*

(a) During which year, from 2001 through 2006, were Procter & Gamble's sales increasing most rapidly?

(b) During which year were the sales increasing at the lowest rate?

(c) Find the rate of increase or decrease for each year in parts (a) and (b).

(T) (d) Use a graphing utility to graph the sales function. Then use the *zoom* and *trace* features to confirm the results in parts (a), (b), and (c).

**40. Revenue**   The revenue $R$ (in millions of dollars per year) for Papa John's from 1996 to 2005 can be modeled by

$$R = \frac{-485.0 + 116.68t}{1 - 0.12t + 0.0097t^2}, \quad 6 \le t \le 15$$

where $t$ represents the year, with $t = 6$ corresponding to 1996.   *(Source: Papa John's Int'l.)*

(a) During which year, from 1996 through 2005, was Papa John's revenue the greatest? the least?

(b) During which year was the revenue increasing at the greatest rate? decreasing at the greatest rate?

(T) (c) Use a graphing utility to graph the revenue function, and confirm your results in parts (a) and (b).

**41.** Match each graph with the function it best represents— a demand function, a revenue function, a cost function, or a profit function. Explain your reasoning. (The graphs are labeled *a–d*.)



**42. Demand**   A demand function is modeled by $x = a/p^m$, where $a$ is a constant and $m > 1$. Show that $\eta = -m$. In other words, show that a 1% increase in price results in an $m\%$ decrease in the quantity demanded.

**43. Think About It**   Throughout this text, it is assumed that demand functions are decreasing. Can you think of a product that has an increasing demand function? That is, can you think of a product that becomes more in demand as its price increases? Explain your reasoning, and sketch a graph of the function.

**44. Extended Application**   To work an extended application analyzing the sales per share for Lowe's from 1990 through 2005, visit this text's website at *college.hmco.com*.   *(Data Source: Lowe's Companies)*

**Business Capsule**



Photo courtesy of Jim Bell

Illinois native Jim Bell moved to California in 1996 to pursue his dream of working in the skateboarding industry. After a string of sales jobs with several skate companies, Bell started San Diego-based Jim Bell Skateboard Ramps in 2004 with an initial cash outlay of \$50. His custom-built skateboard ramp business brought in sales of \$250,000 the following year. His latest product, the U-Built-It Skateboard Ramp, is expected to nearly double his annual sales. Bell marketed his new product by featuring it at trade shows. He backed it up by showing pictures of the hundreds of ramps he has built. So, Bell was able to prove the demand existed, as well as the quality and customer satisfaction his work boasted.

**45. Research Project**   Choose an innovative product like the one described above. Use your school's library, the Internet, or some other reference source to research the history of the product or service. Collect data about the revenue that the product or service has generated, and find a mathematical model of the data. Summarize your findings.

# Asymptotes
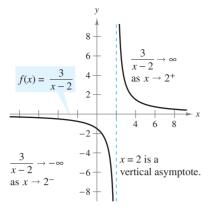
- Find the vertical asymptotes of functions and find infinite limits.
- Find the horizontal asymptotes of functions and find limits at infinity.
- Use asymptotes to answer questions about real-life situations.

## Vertical Asymptotes and Infinite Limits



$f(x) = \dfrac{3}{x-2}$

$\dfrac{3}{x-2} \to \infty$ as $x \to 2^+$

$\dfrac{3}{x-2} \to -\infty$ as $x \to 2^-$

$x = 2$ is a vertical asymptote.

**FIGURE 3.44**

In the first three sections of this chapter, you studied ways in which you can use calculus to help analyze the graph of a function. In this section, you will study another valuable aid to curve sketching: the determination of vertical and horizontal asymptotes.

Recall from Section 1.5, Example 10, that the function

$$f(x) = \frac{3}{x-2}$$

is unbounded as $x$ approaches 2 (see Figure 3.44). This type of behavior is described by saying that the line $x = 2$ is a **vertical asymptote** of the graph of $f$. The type of limit in which $f(x)$ approaches infinity (or negative infinity) as $x$ approaches $c$ from the left or from the right is an **infinite limit.** The infinite limits for the function $f(x) = 3/(x-2)$ can be written as

$$\lim_{x \to 2^-} \frac{3}{x-2} = -\infty$$

and

$$\lim_{x \to 2^+} \frac{3}{x-2} = \infty.$$

> **Definition of Vertical Asymptote**
>
> If $f(x)$ approaches infinity (or negative infinity) as $x$ approaches $c$ from the right or from the left, then the line $x = c$ is a **vertical asymptote** of the graph of $f$.
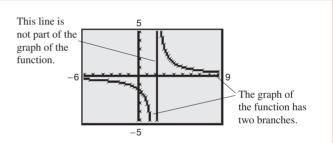
### TECHNOLOGY

When you use a graphing utility to graph a function that has a vertical asymptote, the utility may try to connect separate branches of the graph. For instance, the figure at the right shows the graph of

$$f(x) = \frac{3}{x-2}$$

on a graphing calculator.



This line is not part of the graph of the function.

The graph of the function has two branches.

**x Approaches 1 from the Left**

| $x$ | $f(x) = 1/(x - 1)$ |
|---|---|
| 0 | $-1$ |
| 0.9 | $-10$ |
| 0.99 | $-100$ |
| 0.999 | $-1000$ |
| 0.9999 | $-10,000$ |

**x Approaches 1 from the Right**

| $x$ | $f(x) = 1/(x - 1)$ |
|---|---|
| 2 | 1 |
| 1.1 | 10 |
| 1.01 | 100 |
| 1.001 | 1000 |
| 1.0001 | 10,000 |

✓ **CHECKPOINT 1**

Find each limit.

**a.** *Limit from the left*

$$\lim_{x \to 2^-} \frac{1}{x - 2}$$

*Limit from the right*

$$\lim_{x \to 2^+} \frac{1}{x - 2}$$

**b.** *Limit from the left*

$$\lim_{x \to -3^-} \frac{-1}{x + 3}$$

*Limit from the right*

$$\lim_{x \to -3^+} \frac{-1}{x + 3} \ \blacksquare$$

One of the most common instances of a vertical asymptote is the graph of a *rational function*—that is, a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials. If $c$ is a real number such that $q(c) = 0$ and $p(c) \neq 0$, the graph of $f$ has a vertical asymptote at $x = c$. Example 1 shows four cases.

### Example 1   Finding Infinite Limits

Find each limit.

*Limit from the left*          *Limit from the right*

**a.** $\lim\limits_{x \to 1^-} \dfrac{1}{x - 1} = -\infty$       $\lim\limits_{x \to 1^+} \dfrac{1}{x - 1} = \infty$      See Figure 3.45(a).

**b.** $\lim\limits_{x \to 1^-} \dfrac{-1}{x - 1} = \infty$       $\lim\limits_{x \to 1^+} \dfrac{-1}{x - 1} = -\infty$      See Figure 3.45(b).

**c.** $\lim\limits_{x \to 1^-} \dfrac{-1}{(x - 1)^2} = -\infty$       $\lim\limits_{x \to 1^+} \dfrac{-1}{(x - 1)^2} = -\infty$      See Figure 3.45(c).

**d.** $\lim\limits_{x \to 1^-} \dfrac{1}{(x - 1)^2} = \infty$       $\lim\limits_{x \to 1^+} \dfrac{1}{(x - 1)^2} = \infty$      See Figure 3.45(d).

$$\lim_{x \to 1^-} \frac{1}{x - 1} = -\infty \qquad \lim_{x \to 1^+} \frac{1}{x - 1} = \infty$$

(a)

$$\lim_{x \to 1^-} \frac{-1}{x - 1} = \infty \qquad \lim_{x \to 1^+} \frac{-1}{x - 1} = -\infty$$

(b)

$$\lim_{x \to 1} \frac{-1}{(x - 1)^2} = -\infty$$

(c)

$$\lim_{x \to 1} \frac{1}{(x - 1)^2} = \infty$$

(d)

**FIGURE 3.45**
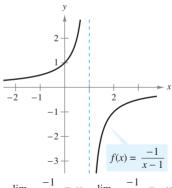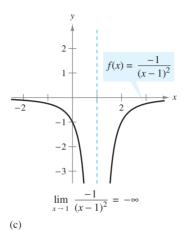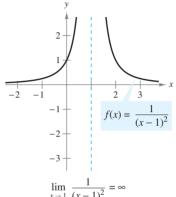
Each of the graphs in Example 1 has only one vertical asymptote. As shown in the next example, the graph of a rational function can have more than one vertical asymptote.



$f(x) = \dfrac{x + 2}{x^2 - 2x}$

**FIGURE 3.46** Vertical Asymptotes at $x = 0$ and $x = 2$

### Example 2 Finding Vertical Asymptotes

Find the vertical asymptotes of the graph of

$$f(x) = \frac{x + 2}{x^2 - 2x}.$$

**SOLUTION** The possible vertical asymptotes correspond to the $x$-values for which the denominator is zero.

| | |
|---|---|
| $x^2 - 2x = 0$ | Set denominator equal to 0. |
| $x(x - 2) = 0$ | Factor. |
| $x = 0, x = 2$ | Zeros of denominator |
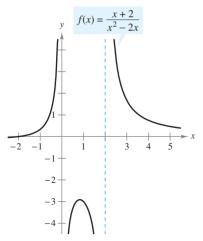
Because the numerator of $f$ is not zero at either of these $x$-values, you can conclude that the graph of $f$ has two vertical asymptotes—one at $x = 0$ and one at $x = 2$, as shown in Figure 3.46.

✓ **CHECKPOINT 2**

Find the vertical asymptote(s) of the graph of

$$f(x) = \frac{x + 4}{x^2 - 4x}. \ ■$$

### Example 3 Finding Vertical Asymptotes

Find the vertical asymptotes of the graph of

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$
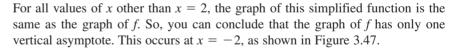
**SOLUTION** First factor the numerator and denominator. Then divide out like factors.
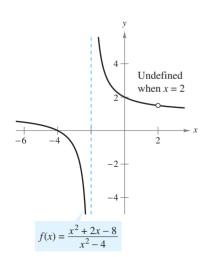
| | | |
|---|---|---|
| $f(x)$ | $= \dfrac{x^2 + 2x - 8}{x^2 - 4}$ | Write original function. |
| | $= \dfrac{(x + 4)(x - 2)}{(x + 2)(x - 2)}$ | Factor numerator and denominator. |
| | $= \dfrac{(x + 4)(x - 2)}{(x + 2)(x - 2)}$ | Divide out like factors. |
| | $= \dfrac{x + 4}{x + 2}, \quad x \neq 2$ | Simplify. |

For all values of $x$ other than $x = 2$, the graph of this simplified function is the same as the graph of $f$. So, you can conclude that the graph of $f$ has only one vertical asymptote. This occurs at $x = -2$, as shown in Figure 3.47.



Undefined when $x = 2$

$f(x) = \dfrac{x^2 + 2x - 8}{x^2 - 4}$

**FIGURE 3.47** Vertical Asymptote at $x = -2$

✓ **CHECKPOINT 3**

Find the vertical asymptotes of the graph of

$$f(x) = \frac{x^2 + 4x + 3}{x^2 - 9}. \ ■$$

From Example 3, you know that the graph of

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 4}$$

has a vertical asymptote at $x = -2$. This implies that the limit of $f(x)$ as $x \to -2$ from the right (or from the left) is either $\infty$ or $-\infty$. But without looking at the graph, how can you determine that the limit from the left is *negative* infinity and the limit from the right is *positive* infinity? That is, why is the limit from the left

$$\lim_{x \to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = -\infty \qquad \text{Limit from the left}$$

and why is the limit from the right

$$\lim_{x \to -2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \infty? \qquad \text{Limit from the right}$$

It is cumbersome to determine these limits analytically, and you may find the graphical method shown in Example 4 to be more efficient.

From the left, $f(x)$
approaches positive infinity.



From the right, $f(x)$
approaches negative infinity.
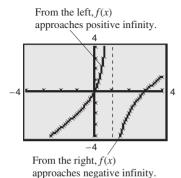
**FIGURE 3.48**

### Example 4    Determining Infinite Limits

Find the limits.

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} \quad \text{and} \quad \lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1}$$

**SOLUTION**    Begin by considering the function

$$f(x) = \frac{x^2 - 3x}{x - 1}.$$

Because the denominator is zero when $x = 1$ and the numerator is not zero when $x = 1$, it follows that the graph of the function has a vertical asymptote at $x = 1$. This implies that each of the given limits is either $\infty$ or $-\infty$. To determine which, use a graphing utility to graph the function, as shown in Figure 3.48. From the graph, you can see that the limit from the left is positive infinity and the limit from the right is negative infinity. That is,

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} = \infty \qquad \text{Limit from the left}$$

and

$$\lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1} = -\infty. \qquad \text{Limit from the right}$$

**STUDY TIP**

In Example 4, try evaluating $f(x)$ at $x$-values that are just barely to the left of 1. You will find that you can make the values of $f(x)$ arbitrarily large by choosing $x$ sufficiently close to 1. For instance, $f(0.99999) = 199,999$.

### ✓ CHECKPOINT 4

Find the limits.

$$\lim_{x \to 2^-} \frac{x^2 - 4x}{x - 2} \quad \text{and} \quad \lim_{x \to 2^+} \frac{x^2 - 4x}{x - 2}$$

Then verify your solution by graphing the function. ■

## Horizontal Asymptotes and Limits at Infinity
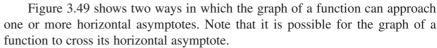
Another type of limit, called a **limit at infinity,** specifies a finite value approached by a function as $x$ increases (or decreases) without bound.



> **Definition of Horizontal Asymptote**
>
> If $f$ is a function and $L_1$ and $L_2$ are real numbers, the statements
>
> $$\lim_{x \to \infty} f(x) = L_1 \quad \text{and} \quad \lim_{x \to -\infty} f(x) = L_2$$
>
> denote **limits at infinity.** The lines $y = L_1$ and $y = L_2$ are **horizontal asymptotes** of the graph of $f$.

Figure 3.49 shows two ways in which the graph of a function can approach one or more horizontal asymptotes. Note that it is possible for the graph of a function to cross its horizontal asymptote.

Limits at infinity share many of the properties of limits discussed in Section 1.5. When finding horizontal asymptotes, you can use the property that

$$\lim_{x \to \infty} \frac{1}{x^r} = 0, \quad r > 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^r} = 0, \quad r > 0.$$

**FIGURE 3.49**

(The second limit assumes that $x^r$ is defined when $x < 0$.)

*Example 5* **Finding Limits at Infinity**

Find the limit: $\lim_{x \to \infty} \left( 5 - \frac{2}{x^2} \right)$.

**SOLUTION**

$$\lim_{x \to \infty} \left( 5 - \frac{2}{x^2} \right) = \lim_{x \to \infty} 5 - \lim_{x \to \infty} \frac{2}{x^2} \qquad \textcolor{red}{\lim_{x \to \infty} [f(x) - g(x)] = \lim_{x \to \infty} f(x) - \lim_{x \to \infty} g(x)}$$

$$= \lim_{x \to \infty} 5 - 2 \left( \lim_{x \to \infty} \frac{1}{x^2} \right) \qquad \textcolor{red}{\lim_{x \to \infty} cf(x) = c \lim_{x \to \infty} f(x)}$$

$$= 5 - 2(0)$$

$$= 5$$
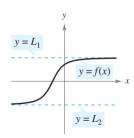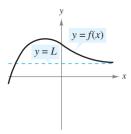
You can verify this limit by sketching the graph of
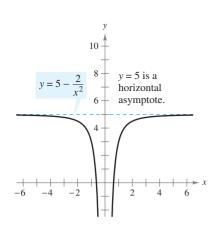
$$f(x) = 5 - \frac{2}{x^2}$$



as shown in Figure 3.50. Note that the graph has $y = 5$ as a horizontal asymptote to the right. By evaluating the limit of $f(x)$ as $x \to -\infty$, you can show that this line is also a horizontal asymptote to the left.

**FIGURE 3.50**

✓ **CHECKPOINT 5**

Find the limit: $\lim_{x \to \infty} \left( 2 + \frac{5}{x^2} \right)$. ■

There is an easy way to determine whether the graph of a *rational* function has a horizontal asymptote. This shortcut is based on a comparison of the degrees of the numerator and denominator of the rational function.

**Horizontal Asymptotes of Rational Functions**

Let $f(x) = p(x)/q(x)$ be a rational function.

1. If the degree of the numerator is less than the degree of the denominator, then $y = 0$ is a horizontal asymptote of the graph of $f$ (to the left and to the right).

2. If the degree of the numerator is equal to the degree of the denominator, then $y = a/b$ is a horizontal asymptote of the graph of $f$ (to the left and to the right), where $a$ and $b$ are the leading coefficients of $p(x)$ and $q(x)$, respectively.

3. If the degree of the numerator is greater than the degree of the denominator, then the graph of $f$ has no horizontal asymptote.

✓**CHECKPOINT 6**

Find the horizontal asymptote of the graph of each function.

**a.** $y = \dfrac{2x + 1}{4x^2 + 5}$

**b.** $y = \dfrac{2x^2 + 1}{4x^2 + 5}$

**c.** $y = \dfrac{2x^3 + 1}{4x^2 + 5}$ ■

**Example 6**  **Finding Horizontal Asymptotes**

Find the horizontal asymptote of the graph of each function.
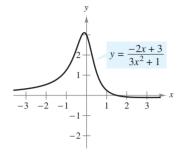
**a.** $y = \dfrac{-2x + 3}{3x^2 + 1}$    **b.** $y = \dfrac{-2x^2 + 3}{3x^2 + 1}$    **c.** $y = \dfrac{-2x^3 + 3}{3x^2 + 1}$
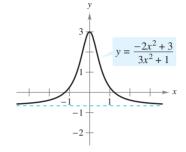
**SOLUTION**
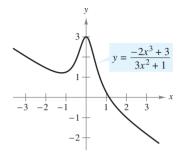
**a.** Because the degree of the numerator is less than the degree of the denominator, $y = 0$ is a horizontal asymptote. [See Figure 3.51(a).]

**b.** Because the degree of the numerator is equal to the degree of the denominator, the line $y = -\frac{2}{3}$ is a horizontal asymptote. [See Figure 3.51(b).]

**c.** Because the degree of the numerator is greater than the degree of the denominator, the graph has no horizontal asymptote. [See Figure 3.51(c).]



(a) $y = 0$ is a horizontal asymptote.    (b) $y = -\frac{2}{3}$ is a horizontal asymptote.    (c) No horizontal asymptote

**FIGURE 3.51**

## Applications of Asymptotes

There are many examples of asymptotic behavior in real life. For instance, Example 7 describes the asymptotic behavior of an average cost function.

### Example 7  Modeling Average Cost ®

A small business invests $5000 in a new product. In addition to this initial investment, the product will cost $0.50 per unit to produce. Find the average cost per unit if 1000 units are produced, if 10,000 units are produced, and if 100,000 units are produced. What is the limit of the average cost as the number of units produced increases?

**SOLUTION**    From the given information, you can model the total cost $C$ (in dollars) by

$$C = 0.5x + 5000 \qquad \text{Total cost function}$$

where $x$ is the number of units produced. This implies that the average cost function is

$$\overline{C} = \frac{C}{x} = 0.5 + \frac{5000}{x}. \qquad \text{Average cost function}$$

If only 1000 units are produced, then the average cost per unit is

$$\overline{C} = 0.5 + \frac{5000}{1000} = \$5.50. \qquad \text{Average cost for 1000 units}$$

If 10,000 units are produced, then the average cost per unit is

$$\overline{C} = 0.5 + \frac{5000}{10,000} = \$1.00. \qquad \text{Average cost for 10,000 units}$$
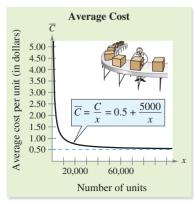
If 100,000 units are produced, then the average cost per unit is

$$\overline{C} = 0.5 + \frac{5000}{100,000} = \$0.55. \qquad \text{Average cost for 100,000 units}$$

As $x$ approaches infinity, the limiting average cost per unit is

$$\lim_{x \to \infty} \left( 0.5 + \frac{5000}{x} \right) = \$0.50.$$
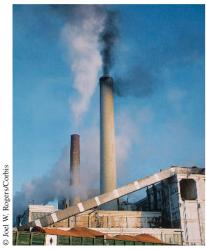
As shown in Figure 3.52, this example points out one of the major problems of small businesses. That is, it is difficult to have competitively low prices when the production level is low.

**Average Cost**

**FIGURE 3.52**    As $x \to \infty$, the average cost per unit approaches $0.50.

### ✓ CHECKPOINT 7

A small business invests $25,000 in a new product. In addition, the product will cost $0.75 per unit to produce. Find the cost function and the average cost function. What is the limit of the average cost function as production increases?

Since the 1980s, industries in the United States have spent billions of dollars to reduce air pollution.

© Joel W. Rogers/Corbis

*Example 8* **Modeling Smokestack Emission** ®

A manufacturing plant has determined that the cost $C$ (in dollars) of removing $p\%$ of the smokestack pollutants of its main smokestack is modeled by

$$C = \frac{80,000p}{100 - p}, \quad 0 \le p < 100.$$

What is the vertical asymptote of this function? What does the vertical asymptote mean to the plant owners?

**SOLUTION** The graph of the cost function is shown in Figure 3.53. From the graph, you can see that $p = 100$ is the vertical asymptote. This means that as the plant attempts to remove higher and higher percents of the pollutants, the cost increases dramatically. For instance, the cost of removing 85% of the pollutants is

$$C = \frac{80,000(85)}{100 - 85} \approx \$453,333 \qquad \text{Cost for 85\% removal}$$

but the cost of removing 90% is

$$C = \frac{80,000(90)}{100 - 90} = \$720,000. \qquad \text{Cost for 90\% removal}$$



**Smokestack Emission**

$$C = \frac{80,000p}{100 - p}$$

(90, 720,000)

(85, 453,333)

Cost (in dollars)

Percent of pollutants removed
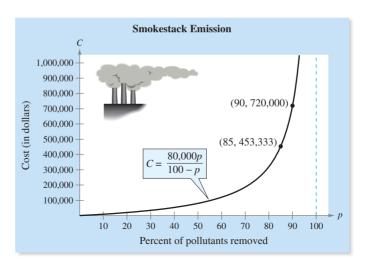
**FIGURE 3.53**

✓ **CHECKPOINT 8**

According to the cost function in Example 8, is it possible to remove 100% of the smokestack pollutants? Why or why not? ■

**CONCEPT CHECK**

1. Complete the following: If $f(x) \to \pm\infty$ as $x \to c$ from the right or the left, then the line $x = c$ is a _____ _____ of the graph of $f$.

2. Describe in your own words what is meant by $\lim\limits_{x \to \infty} f(x) = 4$.

3. Describe in your own words what is meant by $\lim\limits_{x \to -\infty} f(x) = 2$.

4. Complete the following: Given a rational function $f$, if the degree of the numerator is less than the degree of the denominator, then _____ is a horizontal asymptote of the graph of $f$ (to the left and to the right).

**Skills Review 3.6**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.5, 2.3, and 3.5.

In Exercises 1–8, find the limit.

**1.** $\lim\limits_{x \to 2} (x + 1)$

**2.** $\lim\limits_{x \to -1} (3x + 4)$

**3.** $\lim\limits_{x \to -3} \dfrac{2x^2 + x - 15}{x + 3}$

**4.** $\lim\limits_{x \to 2} \dfrac{3x^2 - 8x + 4}{x - 2}$

**5.** $\lim\limits_{x \to 2^+} \dfrac{x^2 - 5x + 6}{x^2 - 4}$

**6.** $\lim\limits_{x \to 1^-} \dfrac{x^2 - 6x + 5}{x^2 - 1}$

**7.** $\lim\limits_{x \to 0^+} \sqrt{x}$

**8.** $\lim\limits_{x \to 1^+} \left( x + \sqrt{x - 1} \right)$

In Exercises 9–12, find the average cost and the marginal cost.

**9.** $C = 150 + 3x$

**10.** $C = 1900 + 1.7x + 0.002x^2$

**11.** $C = 0.005x^2 + 0.5x + 1375$
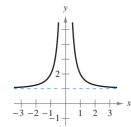
**12.** $C = 760 + 0.05x$

---

# Exercises 3.6
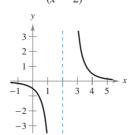
In Exercises 1–8, find the vertical and horizontal asymptotes. Write the asymptotes as equations of lines.
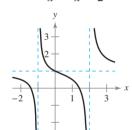
**1.** $f(x) = \dfrac{x^2 + 1}{x^2}$
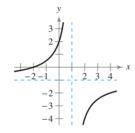


**2.** $f(x) = \dfrac{4}{(x - 2)^3}$
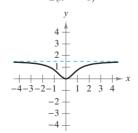


**3.** $f(x) = \dfrac{x^2 - 2}{x^2 - x - 2}$
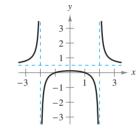


**4.** $f(x) = \dfrac{2 + x}{1 - x}$



**5.** $f(x) = \dfrac{3x^2}{2(x^2 + 1)}$



**6.** $f(x) = \dfrac{-4x}{x^2 + 4}$



**7.** $f(x) = \dfrac{x^2 - 1}{2x^2 - 8}$



**8.** $f(x) = \dfrac{x^2 + 1}{x^3 - 8}$
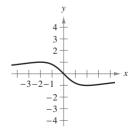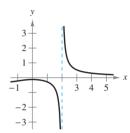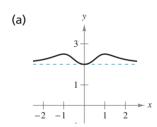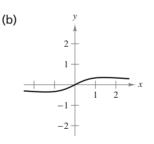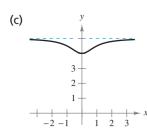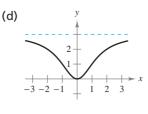
In Exercises 9–12, match the function with its graph. Use horizontal asymptotes as an aid. [The graphs are labeled (a)–(d).]

(a)

(b)

(c)

(d)

**9.** $f(x) = \dfrac{3x^2}{x^2 + 2}$

**10.** $f(x) = \dfrac{x}{x^2 + 2}$

**11.** $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

**12.** $f(x) = 5 - \dfrac{1}{x^2 + 1}$

In Exercises 13–20, find the limit.

**13.** $\lim\limits_{x \to -2^-} \dfrac{1}{(x + 2)^2}$

**14.** $\lim\limits_{x \to -2^-} \dfrac{1}{x + 2}$

**15.** $\lim\limits_{x \to 3^+} \dfrac{x - 4}{x - 3}$

**16.** $\lim\limits_{x \to 1^+} \dfrac{2 + x}{1 - x}$

**17.** $\lim\limits_{x \to 4^-} \dfrac{x^2}{x^2 - 16}$

**18.** $\lim\limits_{x \to 4} \dfrac{x^2}{x^2 + 16}$

**19.** $\lim\limits_{x \to 0^-} \left(1 + \dfrac{1}{x}\right)$

**20.** $\lim\limits_{x \to 0^-} \left(x^2 - \dfrac{1}{x}\right)$

Ⓣ Ⓢ In Exercises 21–24, use a graphing utility or spreadsheet software program to complete the table. Then use the result to estimate the limit of $f(x)$ as $x$ approaches infinity.

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|-----|--------|--------|--------|--------|--------|--------|--------|
| $f(x)$ | | | | | | | |

**21.** $f(x) = \dfrac{x + 1}{x\sqrt{x}}$

**22.** $f(x) = \dfrac{2x^2}{x + 1}$

**23.** $f(x) = \dfrac{x^2 - 1}{0.02x^2}$

**24.** $f(x) = \dfrac{3x^2}{0.1x^2 + 1}$

Ⓣ Ⓢ In Exercises 25 and 26, use a graphing utility or a spreadsheet software program to complete the table and use the result to estimate the limit of $f(x)$ as $x$ approaches infinity and as $x$ approaches negative infinity.

| $x$ | $-10^6$ | $-10^4$ | $-10^2$ | $10^0$ | $10^2$ | $10^4$ | $10^6$ |
|-----|---------|---------|---------|--------|--------|--------|--------|
| $f(x)$ | | | | | | | |

**25.** $f(x) = \dfrac{2x}{\sqrt{x^2 + 4}}$

**26.** $f(x) = x - \sqrt{x(x - 1)}$

In Exercises 27 and 28, find $\lim\limits_{x \to \infty} h(x)$, if possible.

**27.** $f(x) = 5x^3 - 3$

(a) $h(x) = \dfrac{f(x)}{x^2}$    (b) $h(x) = \dfrac{f(x)}{x^3}$    (c) $h(x) = \dfrac{f(x)}{x^4}$

**28.** $f(x) = 3x^2 + 7$

(a) $h(x) = \dfrac{f(x)}{x}$    (b) $h(x) = \dfrac{f(x)}{x^2}$    (c) $h(x) = \dfrac{f(x)}{x^3}$

In Exercises 29 and 30, find each limit, if possible.

**29.** (a) $\lim\limits_{x \to \infty} \dfrac{x^2 + 2}{x^3 - 1}$

(b) $\lim\limits_{x \to \infty} \dfrac{x^2 + 2}{x^2 - 1}$

(c) $\lim\limits_{x \to \infty} \dfrac{x^2 + 2}{x - 1}$

**30.** (a) $\lim\limits_{x \to \infty} \dfrac{3 - 2x}{3x^3 - 1}$

(b) $\lim\limits_{x \to \infty} \dfrac{3 - 2x}{3x - 1}$

(c) $\lim\limits_{x \to \infty} \dfrac{3 - 2x^2}{3x - 1}$

In Exercises 31–40, find the limit.

**31.** $\lim\limits_{x \to \infty} \dfrac{4x - 3}{2x + 1}$

**32.** $\lim\limits_{x \to \infty} \dfrac{5x^3 + 1}{10x^3 - 3x^2 + 7}$

**33.** $\lim\limits_{x \to \infty} \dfrac{3x}{4x^2 - 1}$

**34.** $\lim\limits_{x \to -\infty} \dfrac{2x^2 - 5x - 12}{1 - 6x - 8x^2}$

**35.** $\lim\limits_{x \to -\infty} \dfrac{5x^2}{x + 3}$

**36.** $\lim\limits_{x \to \infty} \dfrac{x^3 - 2x^2 + 3x + 1}{x^2 - 3x + 2}$

**37.** $\lim\limits_{x \to \infty} (2x - x^{-2})$

**38.** $\lim\limits_{x \to \infty} (2 - x^{-3})$

**39.** $\lim\limits_{x \to -\infty} \left(\dfrac{2x}{x - 1} + \dfrac{3x}{x + 1}\right)$

**40.** $\lim\limits_{x \to \infty} \left(\dfrac{2x^2}{x - 1} + \dfrac{3x}{x + 1}\right)$

In Exercises 41–58, sketch the graph of the equation. Use intercepts, extrema, and asymptotes as sketching aids.

**41.** $y = \dfrac{3x}{1 - x}$

**42.** $y = \dfrac{x - 3}{x - 2}$

**43.** $f(x) = \dfrac{x^2}{x^2 + 9}$

**44.** $f(x) = \dfrac{x}{x^2 + 4}$

**45.** $g(x) = \dfrac{x^2}{x^2 - 16}$

**46.** $g(x) = \dfrac{x}{x^2 - 4}$

**47.** $xy^2 = 4$

**48.** $x^2y = 4$

**49.** $y = \dfrac{2x}{1 - x}$

**50.** $y = \dfrac{2x}{1 - x^2}$

**51.** $y = 1 - 3x^{-2}$

**52.** $y = 1 + x^{-1}$

**53.** $f(x) = \dfrac{1}{x^2 - x - 2}$

**54.** $f(x) = \dfrac{x - 2}{x^2 - 4x + 3}$

**55.** $g(x) = \dfrac{x^2 - x - 2}{x - 2}$

**56.** $g(x) = \dfrac{x^2 - 9}{x + 3}$

**57.** $y = \dfrac{2x^2 - 6}{(x - 1)^2}$

**58.** $y = \dfrac{x}{(x + 1)^2}$

**59. Cost** The cost $C$ (in dollars) of producing $x$ units of a product is $C = 1.35x + 4570$.

(a) Find the average cost function $\overline{C}$.

(b) Find $\overline{C}$ when $x = 100$ and when $x = 1000$.

(c) What is the limit of $\overline{C}$ as $x$ approaches infinity?

**60. Average Cost** A business has a cost (in dollars) of $C = 0.5x + 500$ for producing $x$ units.

(a) Find the average cost function $\overline{C}$.

(b) Find $\overline{C}$ when $x = 250$ and when $x = 1250$.

(c) What is the limit of $\overline{C}$ as $x$ approaches infinity?

**61. Average Cost** The cost function for a certain model of personal digital assistant (PDA) is given by $C = 13.50x + 45,750$, where $C$ is measured in dollars and $x$ is the number of PDAs produced.

(a) Find the average cost function $\overline{C}$.

(b) Find $\overline{C}$ when $x = 100$ and $x = 1000$.

(c) Determine the limit of the average cost function as $x$ approaches infinity. Interpret the limit in the context of the problem.

**62. Average Cost** The cost function for a company to recycle $x$ tons of material is given by $C = 1.25x + 10,500$, where $C$ is measured in dollars.

(a) Find the average cost function $\overline{C}$.

(b) Find the average costs of recycling 100 tons of material and 1000 tons of material.

(c) Determine the limit of the average cost function as $x$ approaches infinity. Interpret the limit in the context of the problem.

**63. Seizing Drugs** The cost $C$ (in millions of dollars) for the federal government to seize $p\%$ of a type of illegal drug as it enters the country is modeled by

$C = 528p/(100 - p), \quad 0 \le p < 100.$

(a) Find the costs of seizing 25%, 50%, and 75%.

(b) Find the limit of $C$ as $p \to 100^-$. Interpret the limit in the context of the problem. Use a graphing utility to verify your result.

**64. Removing Pollutants** The cost $C$ (in dollars) of removing $p\%$ of the air pollutants in the stack emission of a utility company that burns coal is modeled by

$C = 80,000p/(100 - p), \quad 0 \le p < 100.$

(a) Find the costs of removing 15%, 50%, and 90%.

(b) Find the limit of $C$ as $p \to 100^-$. Interpret the limit in the context of the problem. Use a graphing utility to verify your result.

**65. Learning Curve** Psychologists have developed mathematical models to predict performance $P$ (the percent of correct responses) as a function of $n$, the number of times a task is performed. One such model is

$P = \dfrac{0.5 + 0.9(n - 1)}{1 + 0.9(n - 1)}, \quad 0 < n.$

(S) (a) Use a spreadsheet software program to complete the table for the model.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $P$ | | | | | | | | | | |

(b) Find the limit as $n$ approaches infinity.

(T) (c) Use a graphing utility to graph this learning curve, and interpret the graph in the context of the problem.

**66. Biology: Wildlife Management** The state game commission introduces 30 elk into a new state park. The population $N$ of the herd is modeled by
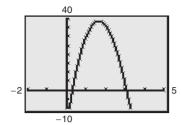
$N = [10(3 + 4t)]/(1 + 0.1t)$

where $t$ is the time in years.

(a) Find the size of the herd after 5, 10, and 25 years.

(b) According to this model, what is the limiting size of the herd as time progresses?

**67. Average Profit** The cost and revenue functions for a product are $C = 34.5x + 15,000$ and $R = 69.9x$.

(a) Find the average profit function $\overline{P} = (R - C)/x$.

(b) Find the average profits when $x$ is 1000, 10,000, and 100,000.

(c) What is the limit of the average profit function as $x$ approaches infinity? Explain your reasoning.

**68. Average Profit** The cost and revenue functions for a product are $C = 25.5x + 1000$ and $R = 75.5x$.

(a) Find the average profit function $\overline{P} = \dfrac{R - C}{x}$.

(b) Find the average profits when $x$ is 100, 500, and 1000.

(c) What is the limit of the average profit function as $x$ approaches infinity? Explain your reasoning.

# Curve Sketching: A Summary
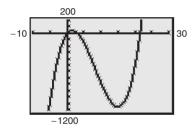
■ Analyze the graphs of functions.
■ Recognize the graphs of simple polynomial functions.

## Summary of Curve-Sketching Techniques

It would be difficult to overstate the importance of using graphs in mathematics. Descartes's introduction of analytic geometry contributed significantly to the rapid advances in calculus that began during the mid-seventeenth century.

So far, you have studied several concepts that are useful in analyzing the graph of a function.

- $x$-intercepts and $y$-intercepts            (Section 1.2)
- Domain and range                              (Section 1.4)
- Continuity                                     (Section 1.6)
- Differentiability                              (Section 2.1)
- Relative extrema                               (Section 3.2)
- Concavity                                      (Section 3.3)
- Points of inflection                           (Section 3.3)
- Vertical asymptotes                            (Section 3.6)
- Horizontal asymptotes                          (Section 3.6)



**FIGURE 3.54**

When you are sketching the graph of a function, either by hand or with a graphing utility, remember that you cannot normally show the *entire* graph. The decision as to which part of the graph to show is crucial. For instance, which of the viewing windows in Figure 3.54 better represents the graph of

$$f(x) = x^3 - 25x^2 + 74x - 20?$$

The lower viewing window gives a more complete view of the graph, but the context of the problem might indicate that the upper view is better. Here are some guidelines for analyzing the graph of a function.

**Guidelines for Analyzing the Graph of a Function**

1. Determine the domain and range of the function. If the function models a real-life situation, consider the context.

2. Determine the intercepts and asymptotes of the graph.

3. Locate the $x$-values where $f'(x)$ and $f''(x)$ are zero or undefined. Use the results to determine where the relative extrema and the points of inflection occur.

In these guidelines, note the importance of *algebra* (as well as calculus) for solving the equations $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$.

**TECHNOLOGY**

(T) Which of the viewing windows best represents the graph of the function

$$f(x) = \frac{x^3 + 8x^2 - 33x}{5}?$$

**a.** Xmin $= -15$, Xmax $= 1$,
    Ymin $= -10$, Ymax $= 60$

**b.** Xmin $= -10$, Xmax $= 10$,
    Ymin $= -10$, Ymax $= 10$

**c.** Xmin $= -13$, Xmax $= 5$,
    Ymin $= -10$, Ymax $= 60$

*Example 1* **Analyzing a Graph**

Analyze the graph of

$$f(x) = x^3 + 3x^2 - 9x + 5. \qquad \text{Original function}$$

**SOLUTION** Begin by finding the intercepts of the graph. This function factors as

$$f(x) = (x - 1)^2(x + 5). \qquad \text{Factored form}$$
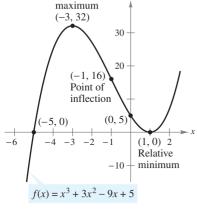
So, the $x$-intercepts occur when $x = 1$ and $x = -5$. The derivative is

$$f'(x) = 3x^2 + 6x - 9 \qquad \text{First derivative}$$
$$= 3(x - 1)(x + 3). \qquad \text{Factored form}$$

So, the critical numbers of $f$ are $x = 1$ and $x = -3$. The second derivative of $f$ is

$$f''(x) = 6x + 6 \qquad \text{Second derivative}$$
$$= 6(x + 1). \qquad \text{Factored form}$$

which implies that the second derivative is zero when $x = -1$. By testing the values of $f'(x)$ and $f''(x)$, as shown in the table, you can see that $f$ has one relative minimum, one relative maximum, and one point of inflection. The graph of $f$ is shown in Figure 3.55.



Relative maximum (−3, 32)

(−1, 16) Point of inflection

(−5, 0)  (0, 5)

(1, 0) Relative minimum

$f(x) = x^3 + 3x^2 - 9x + 5$

**FIGURE 3.55**

|  | $f(x)$ | $f'(x)$ | $f''(x)$ | Characteristics of graph |
|---|---|---|---|---|
| $x$ in $(-\infty, -3)$ |  | $+$ | $-$ | Increasing, concave downward |
| $x = -3$ | 32 | 0 | $-$ | Relative maximum |
| $x$ in $(-3, -1)$ |  | $-$ | $-$ | Decreasing, concave downward |
| $x = -1$ | 16 | $-$ | 0 | Point of inflection |
| $x$ in $(-1, 1)$ |  | $-$ | $+$ | Decreasing, concave upward |
| $x = 1$ | 0 | 0 | $+$ | Relative minimum |
| $x$ in $(1, \infty)$ |  | $+$ | $+$ | Increasing, concave upward |

✓**CHECKPOINT 1**

Analyze the graph of $f(x) = -x^3 + 3x^2 + 9x - 27$. ■

---

**TECHNOLOGY**

Ⓣ  In Example 1, you are able to find the zeros of $f, f'$, and $f''$ algebraically (by factoring). When this is not feasible, you can use a graphing utility to find the zeros. For instance, the function

$$g(x) = x^3 + 3x^2 - 9x + 6$$

is similar to the function in the example, but it does not factor with integer coefficients. Using a graphing utility, you can determine that the function has only one $x$-intercept, $x \approx -5.0275$.

Example 2 **Analyzing a Graph**

Analyze the graph of

$$f(x) = x^4 - 12x^3 + 48x^2 - 64x. \qquad \text{Original function}$$

**SOLUTION** Begin by finding the intercepts of the graph. This function factors as

$$f(x) = x(x^3 - 12x^2 + 48x - 64)$$
$$= x(x - 4)^3. \qquad \text{Factored form}$$

So, the $x$-intercepts occur when $x = 0$ and $x = 4$. The derivative is

$$f'(x) = 4x^3 - 36x^2 + 96x - 64 \qquad \text{First derivative}$$
$$= 4(x - 1)(x - 4)^2. \qquad \text{Factored form}$$

So, the critical numbers of $f$ are $x = 1$ and $x = 4$. The second derivative of $f$ is

$$f''(x) = 12x^2 - 72x + 96 \qquad \text{Second derivative}$$
$$= 12(x - 4)(x - 2). \qquad \text{Factored form}$$

which implies that the second derivative is zero when $x = 2$ and $x = 4$. By testing the values of $f'(x)$ and $f''(x)$, as shown in the table, you can see that $f$ has one relative minimum and two points of inflection. The graph is shown in Figure 3.56.

**FIGURE 3.56**

|  | $f(x)$ | $f'(x)$ | $f''(x)$ | Characteristics of graph |
|---|---|---|---|---|
| $x$ in $(-\infty, 1)$ |  | $-$ | $+$ | Decreasing, concave upward |
| $x = 1$ | $-27$ | $0$ | $+$ | Relative minimum |
| $x$ in $(1, 2)$ |  | $+$ | $+$ | Increasing, concave upward |
| $x = 2$ | $-16$ | $+$ | $0$ | Point of inflection |
| $x$ in $(2, 4)$ |  | $+$ | $-$ | Increasing, concave downward |
| $x = 4$ | $0$ | $0$ | $0$ | Point of inflection |
| $x$ in $(4, \infty)$ |  | $+$ | $+$ | Increasing, concave upward |
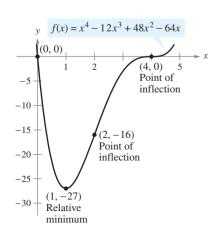
✓**CHECKPOINT 2**

Analyze the graph of $f(x) = x^4 - 4x^3 + 5$. ∎

---

*DISCOVERY*

A polynomial function of degree $n$ can have at most $n - 1$ relative extrema and at most $n - 2$ points of inflection. For instance, the third-degree polynomial in Example 1 has two relative extrema and one point of inflection. Similarly, the fourth-degree polynomial function in Example 2 has one relative extremum and two points of inflection. Is it possible for a third-degree function to have no relative extrema? Is it possible for a fourth-degree function to have no relative extrema?

$$f(x) = \frac{x^2 - 2x + 4}{x - 2}$$
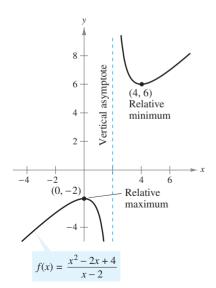
**FIGURE 3.57**

## Example 3   Analyzing a Graph

Analyze the graph of

$$f(x) = \frac{x^2 - 2x + 4}{x - 2}.$$      Original function

**SOLUTION**   The $y$-intercept occurs at $(0, -2)$. Using the Quadratic Formula on the numerator, you can see that there are no $x$-intercepts. Because the denominator is zero when $x = 2$ (and the numerator is not zero when $x = 2$), it follows that $x = 2$ is a vertical asymptote of the graph. There are no horizontal asymptotes because the degree of the numerator is greater than the degree of the denominator. The derivative is

$$f'(x) = \frac{(x - 2)(2x - 2) - (x^2 - 2x + 4)}{(x - 2)^2}$$      First derivative

$$= \frac{x(x - 4)}{(x - 2)^2}.$$      Factored form

So, the critical numbers of $f$ are $x = 0$ and $x = 4$. The second derivative is

$$f''(x) = \frac{(x - 2)^2(2x - 4) - (x^2 - 4x)(2)(x - 2)}{(x - 2)^4}$$      Second derivative

$$= \frac{(x - 2)(2x^2 - 8x + 8 - 2x^2 + 8x)}{(x - 2)^4}$$

$$= \frac{8}{(x - 2)^3}.$$      Factored form

Because the second derivative has no zeros and because $x = 2$ is not in the domain of the function, you can conclude that the graph has no points of inflection. By testing the values of $f'(x)$ and $f''(x)$, as shown in the table, you can see that $f$ has one relative minimum and one relative maximum. The graph of $f$ is shown in Figure 3.57.

|  | $f(x)$ | $f'(x)$ | $f''(x)$ | Characteristics of graph |
|---|---|---|---|---|
| $x$ in $(-\infty, 0)$ |  | $+$ | $-$ | Increasing, concave downward |
| $x = 0$ | $-2$ | $0$ | $-$ | Relative maximum |
| $x$ in $(0, 2)$ |  | $-$ | $-$ | Decreasing, concave downward |
| $x = 2$ | Undef. | Undef. | Undef. | Vertical asymptote |
| $x$ in $(2, 4)$ |  | $-$ | $+$ | Decreasing, concave upward |
| $x = 4$ | $6$ | $0$ | $+$ | Relative minimum |
| $x$ in $(4, \infty)$ |  | $+$ | $+$ | Increasing, concave upward |

✓ **CHECKPOINT 3**

Analyze the graph of $f(x) = \dfrac{x^2}{x - 1}$. ∎

**Example 4** **Analyzing a Graph**

Analyze the graph of

$$f(x) = \frac{2(x^2 - 9)}{x^2 - 4}.$$   Original function

**SOLUTION** Begin by writing the function in factored form.

$$f(x) = \frac{2(x - 3)(x + 3)}{(x - 2)(x + 2)}$$   Factored form

The $y$-intercept is $\left(0, \frac{9}{2}\right)$, and the $x$-intercepts are $(-3, 0)$ and $(3, 0)$. The graph of $f$ has vertical asymptotes at $x = \pm 2$ and a horizontal asymptote at $y = 2$. The first derivative is
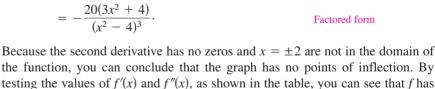
$$f'(x) = \frac{2[(x^2 - 4)(2x) - (x^2 - 9)(2x)]}{(x^2 - 4)^2}$$   First derivative

$$= \frac{20x}{(x^2 - 4)^2}.$$   Factored form

So, the critical number of $f$ is $x = 0$. The second derivative of $f$ is

$$f''(x) = \frac{(x^2 - 4)^2(20) - (20x)(2)(2x)(x^2 - 4)}{(x^2 - 4)^4}$$   Second derivative

$$= \frac{20(x^2 - 4)(x^2 - 4 - 4x^2)}{(x^2 - 4)^4}$$

$$= -\frac{20(3x^2 + 4)}{(x^2 - 4)^3}.$$   Factored form

Because the second derivative has no zeros and $x = \pm 2$ are not in the domain of the function, you can conclude that the graph has no points of inflection. By testing the values of $f'(x)$ and $f''(x)$, as shown in the table, you can see that $f$ has one relative minimum. The graph of $f$ is shown in Figure 3.58.



$f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$

$(-3, 0)$  $(3, 0)$

$\left(0, \frac{9}{2}\right)$ Relative minimum

**FIGURE 3.58**

|  | $f(x)$ | $f'(x)$ | $f''(x)$ | Characteristics of graph |
|---|---|---|---|---|
| $x$ in $(-\infty, -2)$ |  | $-$ | $-$ | Decreasing, concave downward |
| $x = -2$ | Undef. | Undef. | Undef. | Vertical asymptote |
| $x$ in $(-2, 0)$ |  | $-$ | $+$ | Decreasing, concave upward |
| $x = 0$ | $\frac{9}{2}$ | $0$ | $+$ | Relative minimum |
| $x$ in $(0, 2)$ |  | $+$ | $+$ | Increasing, concave upward |
| $x = 2$ | Undef. | Undef. | Undef. | Vertical asymptote |
| $x$ in $(2, \infty)$ |  | $+$ | $-$ | Increasing, concave downward |
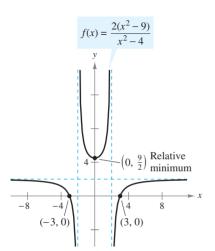
✓ **CHECKPOINT 4**

Analyze the graph of $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$. ◼

*Example 5* **Analyzing a Graph**

Analyze the graph of

$$f(x) = 2x^{5/3} - 5x^{4/3}.$$    Original function

**SOLUTION**  Begin by writing the function in factored form.

$$f(x) = x^{4/3}(2x^{1/3} - 5)$$    Factored form

One of the intercepts is $(0, 0)$. A second $x$-intercept occurs when $2x^{1/3} = 5$.

$$2x^{1/3} = 5$$
$$x^{1/3} = \frac{5}{2}$$
$$x = \left(\frac{5}{2}\right)^3$$
$$x = \frac{125}{8}$$

The first derivative is

$$f'(x) = \frac{10}{3}x^{2/3} - \frac{20}{3}x^{1/3}$$    First derivative
$$= \frac{10}{3}x^{1/3}(x^{1/3} - 2).$$    Factored form

So, the critical numbers of $f$ are $x = 0$ and $x = 8$. The second derivative is

$$f''(x) = \frac{20}{9}x^{-1/3} - \frac{20}{9}x^{-2/3}$$    Second derivative
$$= \frac{20}{9}x^{-2/3}(x^{1/3} - 1)$$
$$= \frac{20(x^{1/3} - 1)}{9x^{2/3}}.$$    Factored form

So, possible points of inflection occur when $x = 1$ and when $x = 0$. By testing the values of $f'(x)$ and $f''(x)$, as shown in the table, you can see that $f$ has one relative maximum, one relative minimum, and one point of inflection. The graph of $f$ is shown in Figure 3.59.

**Algebra Review**
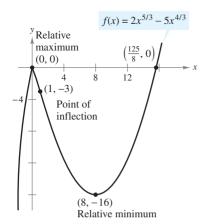
For help on the algebra in Example 5, see Example 2(a) in the *Chapter 3 Algebra Review*, on page 284.



**FIGURE 3.59**

| | $f(x)$ | $f'(x)$ | $f''(x)$ | Characteristics of graph |
|---|---|---|---|---|
| $x$ in $(-\infty, 0)$ | | $+$ | $-$ | Increasing, concave downward |
| $x = 0$ | $0$ | $0$ | Undef. | Relative maximum |
| $x$ in $(0, 1)$ | | $-$ | $-$ | Decreasing, concave downward |
| $x = 1$ | $-3$ | $-$ | $0$ | Point of inflection |
| $x$ in $(1, 8)$ | | $-$ | $+$ | Decreasing, concave upward |
| $x = 8$ | $-16$ | $0$ | $+$ | Relative minimum |
| $x$ in $(8, \infty)$ | | $+$ | $+$ | Increasing, concave upward |
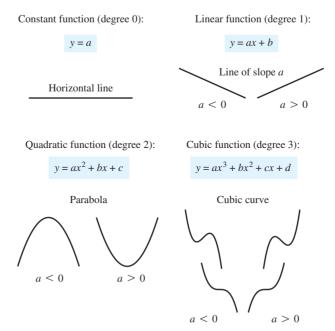
✓ **CHECKPOINT 5**

Analyze the graph of

$$f(x) = 2x^{3/2} - 6x^{1/2}.$$ ■

## Summary of Simple Polynomial Graphs

A summary of the graphs of polynomial functions of degrees 0, 1, 2, and 3 is shown in Figure 3.60. Because of their simplicity, lower-degree polynomial functions are commonly used as mathematical models.

Constant function (degree 0):

$$y = a$$

Horizontal line

Linear function (degree 1):

$$y = ax + b$$

Line of slope $a$

$a < 0$        $a > 0$

Quadratic function (degree 2):

$$y = ax^2 + bx + c$$

Parabola

$a < 0$        $a > 0$

Cubic function (degree 3):

$$y = ax^3 + bx^2 + cx + d$$

Cubic curve

$a < 0$        $a > 0$

**FIGURE 3.60**

### STUDY TIP

The graph of any cubic polynomial has one point of inflection. The slope of the graph at the point of inflection may be zero or nonzero.

### CONCEPT CHECK

1. A fourth-degree polynomial can have at most how many relative extrema?

2. A fourth-degree polynomial can have at most how many points of inflection?

3. Complete the following: A polynomial function of degree $n$ can have at most _____ relative extrema.

4. Complete the following: A polynomial function of degree $n$ can have at most _____ points of inflection.

| **Skills Review 3.7** | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 3.1 and 3.6. |

In Exercises 1–4, find the vertical and horizontal asymptotes of the graph.

**1.** $f(x) = \dfrac{1}{x^2}$ 　　**2.** $f(x) = \dfrac{8}{(x-2)^2}$ 　　**3.** $f(x) = \dfrac{40x}{x+3}$ 　　**4.** $f(x) = \dfrac{x^2 - 3}{x^2 - 4x + 3}$

In Exercises 5–10, determine the open intervals on which the function is increasing or decreasing.

**5.** $f(x) = x^2 + 4x + 2$ 　　**6.** $f(x) = -x^2 - 8x + 1$ 　　**7.** $f(x) = x^3 - 3x + 1$

**8.** $f(x) = \dfrac{-x^3 + x^2 - 1}{x^2}$ 　　**9.** $f(x) = \dfrac{x-2}{x-1}$ 　　**10.** $f(x) = -x^3 - 4x^2 + 3x + 2$

## Exercises 3.7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–22, sketch the graph of the function. Choose a scale that allows all relative extrema and points of inflection to be identified on the graph.

**1.** $y = -x^2 - 2x + 3$ 　　**2.** $y = 2x^2 - 4x + 1$

**3.** $y = x^3 - 4x^2 + 6$ 　　**4.** $y = -x^3 + x - 2$

**5.** $y = 2 - x - x^3$ 　　**6.** $y = x^3 + 3x^2 + 3x + 2$

**7.** $y = 3x^3 - 9x + 1$ 　　**8.** $y = -4x^3 + 6x^2$

**9.** $y = 3x^4 + 4x^3$ 　　**10.** $y = x^4 - 2x^2$

**11.** $y = x^3 - 6x^2 + 3x + 10$

**12.** $y = -x^3 + 3x^2 + 9x - 2$

**13.** $y = x^4 - 8x^3 + 18x^2 - 16x + 5$

**14.** $y = x^4 - 4x^3 + 16x - 16$

**15.** $y = x^4 - 4x^3 + 16x$ 　　**16.** $y = x^5 + 1$

**17.** $y = x^5 - 5x$ 　　**18.** $y = (x-1)^5$

**19.** $y = \dfrac{x^2 + 1}{x}$ 　　**20.** $y = \dfrac{x+2}{x}$

**21.** $y = \begin{cases} x^2 + 1, & x \le 0 \\ 1 - 2x, & x > 0 \end{cases}$ 　　**22.** $y = \begin{cases} x^2 + 4, & x < 0 \\ 4 - x, & x \ge 0 \end{cases}$

(T) In Exercises 23–34, use a graphing utility to graph the function. Choose a window that allows all relative extrema and points of inflection to be identified on the graph.

**23.** $y = \dfrac{x^2}{x^2 + 3}$ 　　**24.** $y = \dfrac{x}{x^2 + 1}$

**25.** $y = 3x^{2/3} - 2x$ 　　**26.** $y = 3x^{2/3} - x^2$

**27.** $y = 1 - x^{2/3}$ 　　**28.** $y = (1 - x)^{2/3}$

**29.** $y = x^{1/3} + 1$ 　　**30.** $y = x^{-1/3}$

**31.** $y = x^{5/3} - 5x^{2/3}$ 　　**32.** $y = x^{4/3}$

**33.** $y = x\sqrt{x^2 - 9}$ 　　**34.** $y = \dfrac{x}{\sqrt{x^2 - 4}}$

In Exercises 35–44, sketch the graph of the function. Label the intercepts, relative extrema, points of inflection, and asymptotes. Then state the domain of the function.

**35.** $y = \dfrac{5 - 3x}{x - 2}$ 　　**36.** $y = \dfrac{x^2 + 1}{x^2 - 9}$

**37.** $y = \dfrac{2x}{x^2 - 1}$ 　　**38.** $y = \dfrac{x^2 - 6x + 12}{x - 4}$

**39.** $y = x\sqrt{4 - x}$ 　　**40.** $y = x\sqrt{4 - x^2}$

**41.** $y = \dfrac{x - 3}{x}$ 　　**42.** $y = x + \dfrac{32}{x^2}$

**43.** $y = \dfrac{x^3}{x^3 - 1}$ 　　**44.** $y = \dfrac{x^4}{x^4 - 1}$

In Exercises 45 and 46, find values of $a$, $b$, $c$, and $d$ such that the graph of $f(x) = ax^3 + bx^2 + cx + d$ will resemble the given graph. Then use a graphing utility to verify your result. (There are many correct answers.)

**45.** 　　　　　　　　　　**46.**

In Exercises 47–50, use the graph of $f'$ or $f''$ to sketch the graph of $f$. (There are many correct answers.)

**47.**



**48.**



**49.**



**50.**



In Exercises 51 and 52, sketch a graph of a function $f$ having the given characteristics. (There are many correct answers.)

**51.** $f(-2) = f(0) = 0$

$f'(x) > 0$ if $x < -1$

$f'(x) < 0$ if $-1 < x < 0$

$f'(x) > 0$ if $x > 0$

$f'(-1) = f'(0) = 0$

**52.** $f(-1) = f(3) = 0$

$f'(1)$ is undefined.

$f'(x) < 0$ if $x < 1$

$f'(x) > 0$ if $x > 1$

$f''(x) < 0, x \neq 1$

$\lim_{x \to \infty} f(x) = 4$

In Exercises 53 and 54, create a function whose graph has the given characteristics. (There are many correct answers.)

**53.** Vertical asymptote: $x = 5$

Horizontal asymptote: $y = 0$

**54.** Vertical asymptote: $x = -3$

Horizontal asymptote: None

**55.** *MAKE A DECISION: SOCIAL SECURITY* The table lists the average monthly Social Security benefits $B$ (in dollars) for retired workers aged 62 and over from 1998 through 2005. A model for the data is

$$B = \frac{582.6 + 38.38t}{1 + 0.025t - 0.0009t^2}, \quad 8 \leq t \leq 15$$

where $t = 8$ corresponds to 1998. *(Source: U.S. Social Security Administration)*

| $t$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|-----|-----|-----|-----|-----|-----|-----|------|
| $B$ | 780 | 804 | 844 | 874 | 895 | 922 | 955 | 1002 |

(T) (a) Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window. How well does the model fit the data?

(b) Use the model to predict the average monthly benefit in 2008.

(c) Should this model be used to predict the average monthly Social Security benefits in future years? Why or why not?

**56.** **Cost** An employee of a delivery company earns $10 per hour driving a delivery van in an area where gasoline costs $2.80 per gallon. When the van is driven at a constant speed $s$ (in miles per hour, with $40 \leq s \leq 65$), the van gets $700/s$ miles per gallon.

(a) Find the cost $C$ as a function of $s$ for a 100-mile trip on an interstate highway.

(T) (b) Use a graphing utility to graph the function found in part (a) and determine the most economical speed.

**57.** *MAKE A DECISION: PROFIT* The management of a company is considering three possible models for predicting the company's profits from 2003 through 2008. Model I gives the expected annual profits if the current trends continue. Models II and III give the expected annual profits for various combinations of increased labor and energy costs. In each model, $p$ is the profit (in billions of dollars) and $t = 0$ corresponds to 2003.

Model I: $p = 0.03t^2 - 0.01t + 3.39$

Model II: $p = 0.08t + 3.36$

Model III: $p = -0.07t^2 + 0.05t + 3.38$

(T) (a) Use a graphing utility to graph all three models in the same viewing window.

(b) For which models are profits increasing during the interval from 2003 through 2008?

(c) Which model is the most optimistic? Which is the most pessimistic? Which model would you choose? Explain.

(T) **58.** **Meteorology** The monthly normal temperature $T$ (in degrees Fahrenheit) for Pittsburgh, Pennsylvania can be modeled by

$$T = \frac{22.329 - 0.7t + 0.029t^2}{1 - 0.203t + 0.014t^2}, \quad 1 \leq t \leq 12$$

where $t$ is the month, with $t = 1$ corresponding to January. Use a graphing utility to graph the model and find all absolute extrema. Interpret the meaning of these values in the context of the problem. *(Source: National Climatic Data Center)*

(T) **Writing** In Exercises 59 and 60, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.

**59.** $h(x) = \dfrac{6 - 2x}{3 - x}$

**60.** $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

## Section 3.8

# Differentials and Marginal Analysis

■ Find the differentials of functions.
■ Use differentials to approximate changes in functions.
■ Use differentials to approximate changes in real-life models.

## Differentials

When the derivative was defined in Section 2.1 as the limit of the ratio $\Delta y/\Delta x$, it seemed natural to retain the quotient symbolism for the limit itself. So, the derivative of $y$ with respect to $x$ was denoted by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

*even though* we did not interpret $dy/dx$ as the quotient of two separate quantities. In this section, you will see that the quantities $dy$ and $dx$ can be assigned meanings in such a way that their quotient, when $dx \neq 0$, is equal to the derivative of $y$ with respect to $x$.

**STUDY TIP**

In this definition, $dx$ can have any nonzero value. In most applications, however, $dx$ is chosen to be small and this choice is denoted by $dx = \Delta x$.

**Definition of Differentials**

Let $y = f(x)$ represent a differentiable function. The **differential of $x$** (denoted by $dx$) is any nonzero real number. The **differential of $y$** (denoted by $dy$) is $dy = f'(x)\, dx$.

One use of differentials is in approximating the change in $f(x)$ that corresponds to a change in $x$, as shown in Figure 3.61. This change is denoted by

$$\Delta y = f(x + \Delta x) - f(x). \qquad \text{Change in } y$$

**STUDY TIP**

Note in Figure 3.61 that near the point of tangency, the graph of $f$ is very close to the tangent line. This is the essence of the approximations used in this section. In other words, near the point of tangency, $dy \approx \Delta y$.

In Figure 3.61, notice that as $\Delta x$ gets smaller and smaller, the values of $dy$ and $\Delta y$ get closer and closer. That is, when $\Delta x$ is small, $dy \approx \Delta y$.
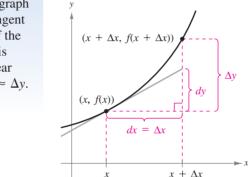


**FIGURE 3.61**

This **tangent line approximation** is the basis for most applications of differentials.

### Example 1   Interpreting Differentials Graphically

Consider the function given by

$$f(x) = x^2. \qquad\qquad \text{Original function}$$

Find the value of $dy$ when $x = 1$ and $dx = 0.01$. Compare this with the value of $\Delta y$ when $x = 1$ and $\Delta x = 0.01$. Interpret the results graphically.

**SOLUTION**   Begin by finding the derivative of $f$.

$$f'(x) = 2x \qquad\qquad \text{Derivative of } f$$

When $x = 1$ and $dx = 0.01$, the value of the differential $dy$ is

$$
\begin{aligned}
dy &= f'(x)\,dx & & \text{Differential of } y \\
&= f'(1)(0.01) & & \text{Substitute 1 for } x \text{ and 0.01 for } dx. \\
&= 2(1)(0.01) & & \text{Use } f'(x) = 2x. \\
&= 0.02. & & \text{Simplify.}
\end{aligned}
$$

When $x = 1$ and $\Delta x = 0.01$, the value of $\Delta y$ is

$$
\begin{aligned}
\Delta y &= f(x + \Delta x) - f(x) & & \text{Change in } y \\
&= f(1.01) - f(1) & & \text{Substitute 1 for } x \text{ and 0.01 for } \Delta x. \\
&= (1.01)^2 - (1)^2 & & \\
&= 0.0201. & & \text{Simplify.}
\end{aligned}
$$

Note that $dy \approx \Delta y$, as shown in Figure 3.62.

The figure shows the graph of $f(x) = x^2$ with the tangent line $y = 2x - 1$ at the point $(1, 1)$, illustrating $\Delta x = 0.01$, $dy$, $\Delta y$, $f(1)$, and $f(1.01)$.
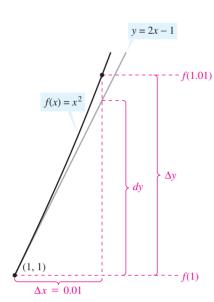
**FIGURE 3.62**

### ✓ CHECKPOINT 1

Find the value of $dy$ when $x = 2$ and $dx = 0.01$ for $f(x) = x^4$. Compare this with the value of $\Delta y$ when $x = 2$ and $\Delta x = 0.01$.  ∎

**STUDY TIP**

Find an equation of the tangent line $y = g(x)$ to the graph of $f(x) = x^2$ at the point $x = 1$. Evaluate $g(1.01)$ and $f(1.01)$.

The validity of the approximation

$$dy \approx \Delta y, \quad dx \neq 0$$

stems from the definition of the derivative. That is, the existence of the limit

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

implies that when $\Delta x$ is close to zero, then $f'(x)$ is close to the difference quotient. So, you can write

$$
\begin{aligned}
\frac{f(x + \Delta x) - f(x)}{\Delta x} &\approx f'(x) \\
f(x + \Delta x) - f(x) &\approx f'(x)\,\Delta x \\
\Delta y &\approx f'(x)\,\Delta x.
\end{aligned}
$$

Substituting $dx$ for $\Delta x$ and $dy$ for $f'(x)\,dx$ produces

$$\Delta y \approx dy.$$

## Marginal Analysis

Differentials are used in economics to approximate changes in revenue, cost, and profit. Suppose that $R = f(x)$ is the total revenue for selling $x$ units of a product. When the number of units increases by 1, the change in $x$ is $\Delta x = 1$, and the change in $R$ is

$$\Delta R = f(x + \Delta x) - f(x) \approx dR = \frac{dR}{dx}\,dx.$$

In other words, you can use the differential $dR$ to approximate the change in the revenue that accompanies the sale of one additional unit. Similarly, the differentials $dC$ and $dP$ can be used to approximate the changes in cost and profit that accompany the sale (or production) of one additional unit.

---

### Example 2 Using Marginal Analysis Ⓡ

The demand function for a product is modeled by

$$p = 400 - x, \quad 0 \le x \le 400.$$

Use differentials to approximate the change in revenue as sales increase from 149 units to 150 units. Compare this with the actual change in revenue.

**SOLUTION** Begin by finding the marginal revenue, $dR/dx$.

| | |
|---|---|
| $R = xp$ | Formula for revenue |
| $\quad = x(400 - x)$ | Use $p = 400 - x$ |
| $\quad = 400x - x^2$ | Multiply. |
| $\dfrac{dR}{dx} = 400 - 2x$ | Power Rule |

When $x = 149$ and $dx = \Delta x = 1$, you can approximate the change in the revenue to be

$$[400 - 2(149)](1) = \$102.$$

When $x$ increases from 149 to 150, the actual change in revenue is

$$\Delta R = \left[400(150) - 150^2\right] - \left[400(149) - 149^2\right]$$
$$= 37{,}500 - 37{,}399$$
$$= \$101$$

---

### ✓ CHECKPOINT 2

The demand function for a product is modeled by

$$p = 200 - x, \quad 0 \le x \le 200.$$

Use differentials to approximate the change in revenue as sales increase from 89 to 90 units. Compare this with the actual change in revenue. ■

*Example 3*
**MAKE A DECISION** **Using Marginal Analysis** Ⓡ

The profit derived from selling $x$ units of an item is modeled by

$$P = 0.0002x^3 + 10x.$$

Use the differential $dP$ to approximate the change in profit when the production level changes from 50 to 51 units. Compare this with the actual gain in profit obtained by increasing the production level from 50 to 51 units. Will the gain in profit exceed $11?

**SOLUTION** The marginal profit is

$$\frac{dP}{dx} = 0.0006x^2 + 10.$$

When $x = 50$ and $dx = 1$, the differential is

$$[0.0006(50)^2 + 10](1) = \$11.50.$$

When $x$ changes from 50 to 51 units, the actual change in profit is

$$\Delta P = [(0.0002)(51)^3 + 10(51)] - [(0.0002)(50)^3 + 10(50)]$$
$$\approx 536.53 - 525.00$$
$$= \$11.53.$$

These values are shown graphically in Figure 3.63. Note that the gain in profit will exceed $11.

**STUDY TIP**

Example 3 uses differentials to solve the same problem that was solved in Example 5 in Section 2.3. Look back at that solution. Which approach do you prefer?

**STUDY TIP**

Find an equation of the tangent line $y = f(x)$ to the graph of $P = 0.0002x^3 + 10x$ at the point $x = 50$. Evaluate $f(51)$ and $p(51)$.
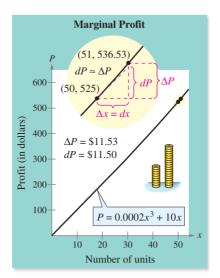


**FIGURE 3.63**

✓ **CHECKPOINT 3**

Use the differential $dP$ to approximate the change in profit for the profit function in Example 3 when the production level changes from 40 to 41 units. Compare this with the actual gain in profit obtained by increasing the production level from 40 to 41 units. ■

## Formulas for Differentials

You can use the definition of differentials to rewrite each differentiation rule in **differential form.** For example, if $u$ and $v$ are differentiable functions of $x$, then $du = (du/dx)\, dx$ and $dv = (dv/dx)\, dx$, which implies that you can write the Product Rule in the following differential form.

$$d[uv] = \frac{d}{dx}[uv]\, dx \qquad\qquad \text{Differential of } uv$$

$$= \left[ u\frac{dv}{dx} + v\frac{du}{dx} \right] dx \qquad\qquad \text{Product Rule}$$

$$= u\frac{dv}{dx}\, dx + v\frac{du}{dx}\, dx$$

$$= u\, dv + v\, du \qquad\qquad \text{Differential form of Product Rule}$$

The following summary gives the differential forms of the differentiation rules presented so far in the text.

---

### Differential Forms of Differentiation Rules

Constant Multiple Rule: $\qquad\qquad\qquad d[cu] = c\, du$

Sum or Difference Rule: $\qquad\qquad\qquad d[u \pm v] = du \pm dv$

Product Rule: $\qquad\qquad\qquad\qquad\quad d[uv] = u\, dv + v\, du$

Quotient Rule: $\qquad\qquad\qquad\qquad\quad d\left[\dfrac{u}{v}\right] = \dfrac{v\, du - u\, dv}{v^2}$

Constant Rule: $\qquad\qquad\qquad\qquad\quad d[c] = 0$

Power Rule: $\qquad\qquad\qquad\qquad\qquad d[x^n] = nx^{n-1}\, dx$

---

The next example compares the derivatives and differentials of several simple functions.

### Example 4 Finding Differentials

Find the differential $dy$ of each function.

| Function | Derivative | Differential |
|---|---|---|
| **a.** $y = x^2$ | $\dfrac{dy}{dx} = 2x$ | $dy = 2x\, dx$ |
| **b.** $y = \dfrac{3x + 2}{5}$ | $\dfrac{dy}{dx} = \dfrac{3}{5}$ | $dy = \dfrac{3}{5}\, dx$ |
| **c.** $y = 2x^2 - 3x$ | $\dfrac{dy}{dx} = 4x - 3$ | $dy = (4x - 3)\, dx$ |
| **d.** $y = \dfrac{1}{x}$ | $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$ | $dy = -\dfrac{1}{x^2}\, dx$ |

✓ **CHECKPOINT 4**

Find the differential $dy$ of each function.

**a.** $y = 4x^3$

**b.** $y = \dfrac{2x + 1}{3}$

**c.** $y = 3x^2 - 2x$

**d.** $y = \dfrac{1}{x^2}$ ∎

## Error Propagation

A common use of differentials is the estimation of errors that result from inaccuracies of physical measuring devices. This is shown in Example 5.

*Example 5*  **Estimating Measurement Errors**  Ⓡ

The radius of a ball bearing is measured to be 0.7 inch, as shown in Figure 3.64. This implies that the volume of the ball bearing is $\frac{4}{3}\pi(0.7)^3 \approx 1.4368$ cubic inches. You are told that the measurement of the radius is correct to within 0.01 inch. How far off could the calculation of the volume be?

**SOLUTION**   Because the value of $r$ can be off by 0.01 inch, it follows that

$$-0.01 \le \Delta r \le 0.01. \qquad \text{Possible error in measuring}$$

Using $\Delta r = dr$, you can estimate the possible error in the volume.

$$V = \tfrac{4}{3}\pi r^3 \qquad\qquad \text{Formula for volume}$$

$$dV = \frac{dV}{dr}\, dr = 4\pi r^2\, dr \qquad\qquad \text{Formula for differential of } V$$

The possible error in the volume is

$$4\pi r^2\, dr = 4\pi(0.7)^2(\pm 0.01) \qquad \text{Substitute for } r \text{ and } dr.$$

$$\approx \pm 0.0616 \text{ cubic inch.} \qquad \text{Possible error}$$

So, the volume of the ball bearing could range between

$$(1.4368 - 0.0616) = 1.3752 \text{ cubic inches}$$

and

$$(1.4368 + 0.0616) = 1.4984 \text{ cubic inches.}$$

In Example 5, the **relative error** in the volume is defined to be the ratio of $dV$ to $V$. This ratio is

$$\frac{dV}{V} \approx \frac{\pm 0.0616}{1.4368} \approx \pm 0.0429.$$

This corresponds to a **percentage error** of 4.29%.



**FIGURE 3.64**

✓ **CHECKPOINT 5**

Find the surface area of the ball bearing in Example 5. How far off could your calculation of the surface area be? The surface area of a sphere is given by $S = 4\pi r^2$. ∎

┌─ **CONCEPT CHECK** ─────────────────────

1.  Given a differentiable function $y = f(x)$, what is the differential of $x$?
2.  Given a differentiable function $y = f(x)$, write an expression for the differential of $y$.
3.  Write the differential form of the Quotient Rule.
4.  When using differentials, what is meant by the terms *relative error* and *percentage error*?

**Skills Review 3.8** The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.2 and 2.4.

In Exercises 1–12, find the derivative.

**1.** $C = 44 + 0.09x^2$

**2.** $C = 250 + 0.15x$

**3.** $R = x(1.25 + 0.02\sqrt{x})$

**4.** $R = x(15.5 - 1.55x)$

**5.** $P = -0.03x^{1/3} + 1.4x - 2250$

**6.** $P = -0.02x^2 + 25x - 1000$

**7.** $A = \frac{1}{4}\sqrt{3}x^2$

**8.** $A = 6x^2$

**9.** $C = 2\pi r$

**10.** $P = 4w$

**11.** $S = 4\pi r^2$

**12.** $P = 2x + \sqrt{2}x$

In Exercises 13–16, write a formula for the quantity.

**13.** Area $A$ of a circle of radius $r$

**14.** Area $A$ of a square of side $x$

**15.** Volume $V$ of a cube of edge $x$

**16.** Volume $V$ of a sphere of radius $r$

## Exercises 3.8

In Exercises 1–6, find the differential $dy$.

**1.** $y = 3x^2 - 4$

**2.** $y = 3x^{2/3}$

**3.** $y = (4x - 1)^3$

**4.** $y = \dfrac{x + 1}{2x - 1}$

**5.** $y = \sqrt{9 - x^2}$

**6.** $y = \sqrt[3]{6x^2}$

In Exercises 7–10, let $x = 1$ and $\Delta x = 0.01$. Find $\Delta y$.

**7.** $f(x) = 5x^2 - 1$

**8.** $f(x) = \sqrt{3x}$

**9.** $f(x) = \dfrac{4}{\sqrt[3]{x}}$

**10.** $f(x) = \dfrac{x}{x^2 + 1}$

In Exercises 11–14, compare the values of $dy$ and $\Delta y$.

**11.** $y = 0.5x^3$ $\quad x = 2$ $\quad \Delta x = dx = 0.1$

**12.** $y = 1 - 2x^2$ $\quad x = 0$ $\quad \Delta x = dx = -0.1$

**13.** $y = x^4 + 1$ $\quad x = -1$ $\quad \Delta x = dx = 0.01$

**14.** $y = 2x + 1$ $\quad x = 2$ $\quad \Delta x = dx = 0.01$

In Exercises 15–20, let $x = 2$ and complete the table for the function.

| $dx = \Delta x$ | $dy$ | $\Delta y$ | $\Delta y - dy$ | $dy/\Delta y$ |
|---|---|---|---|---|
| 1.000 | | | | |
| 0.500 | | | | |
| 0.100 | | | | |
| 0.010 | | | | |
| 0.001 | | | | |

**15.** $y = x^2$

**16.** $y = x^5$

**17.** $y = \dfrac{1}{x^2}$

**18.** $y = \dfrac{1}{x}$

**19.** $y = \sqrt[4]{x}$

**20.** $y = \sqrt{x}$

In Exercises 21–24, find an equation of the tangent line to the function at the given point. Then find the function values and the tangent line values at $f(x + \Delta x)$ and $y(x + \Delta x)$ for $\Delta x = -0.01$ and 0.01.

| Function | Point |
|---|---|
| **21.** $f(x) = 2x^3 - x^2 + 1$ | $(-2, -19)$ |
| **22.** $f(x) = 3x^2 - 1$ | $(2, 11)$ |
| **23.** $f(x) = \dfrac{x}{x^2 + 1}$ | $(0, 0)$ |
| **24.** $f(x) = \sqrt{25 - x^2}$ | $(3, 4)$ |

**25. Profit** The profit $P$ for a company producing $x$ units is

$$P = (500x - x^2) - \left(\frac{1}{2}x^2 - 77x + 3000\right).$$

Approximate the change and percent change in profit as production changes from $x = 115$ to $x = 120$ units.

**26. Revenue** The revenue $R$ for a company selling $x$ units is

$$R = 900x - 0.1x^2.$$

Use differentials to approximate the change in revenue if sales increase from $x = 3000$ to $x = 3100$ units.

**27. Demand**   The demand function for a product is modeled by

$$p = 75 - 0.25x.$$

(a) If $x$ changes from 7 to 8, what is the corresponding change in $p$? Compare the values of $\Delta p$ and $dp$.

(b) Repeat part (a) when $x$ changes from 70 to 71 units.

**28. Biology: Wildlife Management**   A state game commission introduces 50 deer into newly acquired state game lands. The population $N$ of the herd can be modeled by

$$N = \frac{10(5 + 3t)}{1 + 0.04t}$$

where $t$ is the time in years. Use differentials to approximate the change in the herd size from $t = 5$ to $t = 6$.

Ⓣ **Marginal Analysis**   In Exercises 29–34, use differentials to approximate the change in cost, revenue, or profit corresponding to an increase in sales of one unit. For instance, in Exercise 29, approximate the change in cost as $x$ increases from 12 units to 13 units. Then use a graphing utility to graph the function, and use the *trace* feature to verify your result.

| Function | x-Value |
|---|---|
| **29.** $C = 0.05x^2 + 4x + 10$ | $x = 12$ |
| **30.** $C = 0.025x^2 + 8x + 5$ | $x = 10$ |
| **31.** $R = 30x - 0.15x^2$ | $x = 75$ |
| **32.** $R = 50x - 1.5x^2$ | $x = 15$ |
| **33.** $P = -0.5x^3 + 2500x - 6000$ | $x = 50$ |
| **34.** $P = -x^2 + 60x - 100$ | $x = 25$ |

**35. Marginal Analysis**   A retailer has determined that the monthly sales $x$ of a watch are 150 units when the price is $50, but decrease to 120 units when the price is $60. Assume that the demand is a linear function of the price. Find the revenue $R$ as a function of $x$ and approximate the change in revenue for a one-unit increase in sales when $x = 141$. Make a sketch showing $dR$ and $\Delta R$.

**36. Marginal Analysis**   A manufacturer determines that the demand $x$ for a product is inversely proportional to the square of the price $p$. When the price is $10, the demand is 2500. Find the revenue $R$ as a function of $x$ and approximate the change in revenue for a one-unit increase in sales when $x = 3000$. Make a sketch showing $dR$ and $\Delta R$.

**37. Marginal Analysis**   The demand $x$ for a web camera is 30,000 units per month when the price is $25 and 40,000 units when the price is $20. The initial investment is $275,000 and the cost per unit is $17. Assume that the demand is a linear function of the price. Find the profit $P$ as a function of $x$ and approximate the change in profit for a one-unit increase in sales when $x = 28,000$. Make a sketch showing $dP$ and $\Delta P$.

**38. Marginal Analysis**   The variable cost for the production of a calculator is $14.25 and the initial investment is $110,000. Find the total cost $C$ as a function of $x$, the number of units produced. Then use differentials to approximate the change in the cost for a one-unit increase in production when $x = 50,000$. Make a sketch showing $dC$ and $\Delta C$. Explain why $dC = \Delta C$ in this problem.

**39. Area**   The side of a square is measured to be 12 inches, with a possible error of $\frac{1}{64}$ inch. Use differentials to approximate the possible error and the relative error in computing the area of the square.

**40. Volume**   The radius of a sphere is measured to be 6 inches, with a possible error of 0.02 inch. Use differentials to approximate the possible error and the relative error in computing the volume of the sphere.

**41. Economics: Gross Domestic Product**   The gross domestic product (GDP) of the United States for 2001 through 2005 is modeled by

$$G = 0.0026x^2 - 7.246x + 14{,}597.85$$

where $G$ is the GDP (in billions of dollars) and $x$ is the capital outlay (in billions of dollars). Use differentials to approximate the change in the GDP when the capital outlays change from $2100 billion to $2300 billion. *(Source: U.S. Bureau of Economic Analysis, U.S. Office of Management and Budget)*

**42. Medical Science**   The concentration $C$ (in milligrams per milliliter) of a drug in a patient's bloodstream $t$ hours after injection into muscle tissue is modeled by

$$C = \frac{3t}{27 + t^3}.$$

Use differentials to approximate the change in the concentration when $t$ changes from $t = 1$ to $t = 1.5$.

**43. Physiology: Body Surface Area**   The body surface area (BSA) of a 180-centimeter-tall (about six-feet-tall) person is modeled by

$$B = 0.1\sqrt{5w}$$

where $B$ is the BSA (in square meters) and $w$ is the weight (in kilograms). Use differentials to approximate the change in the person's BSA when the person's weight changes from 90 kilograms to 95 kilograms.

**True or False?**   In Exercises 44 and 45, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**44.** If $y = x + c$, then $dy = dx$.

**45.** If $y = ax + b$, then $\Delta y/\Delta x = dy/dx$.

# Algebra Review

## Solving Equations

Much of the algebra in Chapter 3 involves simplifying algebraic expressions (see pages 196 and 197) and solving algebraic equations (see page 106). The Algebra Review on page 106 illustrates some of the basic techniques for solving equations. On these two pages, you can review some of the more complicated techniques for solving equations.

When solving an equation, remember that your basic goal is to isolate the variable on one side of the equation. To do this, you use inverse operations. For instance, to get rid of the *subtract* 2 in

$$x - 2 = 0$$

you *add* 2 to each side of the equation. Similarly, to get rid of the *square root* in

$$\sqrt{x + 3} = 2$$

you *square* both sides of the equation.

### Example 1  Solving an Equation

Solve each equation.

**a.** $\dfrac{36(x^2 - 1)}{(x^2 + 3)^3} = 0$    **b.** $0 = 2 - \dfrac{288}{x^2}$    **c.** $0 = 2x(2x^2 - 3)$

**SOLUTION**

**a.** $\dfrac{36(x^2 - 1)}{(x^2 + 3)^3} = 0$      Example 2, page 227

$36(x^2 - 1) = 0$      A fraction is zero only if its numerator is zero.

$x^2 - 1 = 0$      Divide each side by 36.

$x^2 = 1$      Add 1 to each side.

$x = \pm 1$      Take the square root of each side.

**b.** $0 = 2 - \dfrac{288}{x^2}$      Example 2, page 237

$-2 = -\dfrac{288}{x^2}$      Subtract 2 from each side.

$1 = \dfrac{144}{x^2}$      Divide each side by $-2$.

$x^2 = 144$      Multiply each side by $x^2$.

$x = \pm 12$      Take the square root of each side.

**c.** $0 = 2x(2x^2 - 3)$      Example 3, page 238

$2x = 0 \implies x = 0$      Set first factor equal to zero.

$2x^2 - 3 = 0 \implies x = \pm\sqrt{\tfrac{3}{2}}$      Set second factor equal to zero.

*Example 2*  **Solve an Equation**

Solve each equation.

**a.** $\dfrac{20(x^{1/3} - 1)}{9x^{2/3}} = 0$     **b.** $\dfrac{25}{\sqrt{x}} - 0.5 = 0$

**c.** $x^2(4x - 3) = 0$     **d.** $\dfrac{4x}{3(x^2 - 4)^{1/3}} = 0$

**e.** $g'(x) = 0$, where $g(x) = (x - 2)(x + 1)^2$

**SOLUTION**

**a.** $\dfrac{20(x^{1/3} - 1)}{9x^{2/3}} = 0$      Example 5, page 271

     $20(x^{1/3} - 1) = 0$      A fraction is zero only if its numerator is zero.

         $x^{1/3} - 1 = 0$      Divide each side by 20.

            $x^{1/3} = 1$      Add 1 to each side.

               $x = 1$      Cube each side.

**b.** $\dfrac{25}{\sqrt{x}} - 0.5 = 0$      Example 4, page 248

       $\dfrac{25}{\sqrt{x}} = 0.5$      Add 0.5 to each side.

       $25 = 0.5\sqrt{x}$      Multiply each side by $\sqrt{x}$.

       $50 = \sqrt{x}$      Divide each side by 0.5.

       $2500 = x$      Square both sides.

**c.** $x^2(4x - 3) = 0$      Example 2, page 218

       $x^2 = 0 \implies x = 0$      Set first factor equal to zero.

     $4x - 3 = 0 \implies x = \frac{3}{4}$      Set second factor equal to zero.

**d.** $\dfrac{4x}{3(x^2 - 4)^{1/3}} = 0$      Example 4, page 210

       $4x = 0$      A fraction is zero only if its numerator is zero.

       $x = 0$      Divide each side by 4.

**e.** $g(x) = (x - 2)(x + 1)^2$      Exercise 45, page 233

     $(x - 2)(2)(x + 1) + (x + 1)^2(1) = 0$      Find derivative and set equal to zero.

     $(x + 1)[2(x - 2) + (x + 1)] = 0$      Factor.

     $(x + 1)(2x - 4 + x + 1) = 0$      Multiply factors.

     $(x + 1)(3x - 3) = 0$      Combine like terms.

       $x + 1 = 0 \implies x = -1$      Set first factor equal to zero.

     $3x - 3 = 0 \implies x = 1$      Set second factor equal to zero.

# Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 287.
Answers to odd-numbered Review Exercises are given in the back of the text.*

## Section 3.1                                                      Review Exercises

■ Find the critical numbers of a function.                              *1–4*

   $c$ is a critical number of $f$ if $f'(c) = 0$ or $f'(c)$ is undefined.

■ Find the open intervals on which a function is increasing or decreasing.    *5–8*

   Increasing if $f'(x) > 0$

   Decreasing if $f'(x) < 0$

■ Find intervals on which a real-life model is increasing or decreasing, and interpret    *9, 10, 95*
   the results in context.

## Section 3.2

■ Use the First-Derivative Test to find the relative extrema of a function.    *11–20*

■ Find the absolute extrema of a continuous function on a closed interval.     *21–30*

■ Find minimum and maximum values of a real-life model and interpret the results    *31, 32*
   in context.

## Section 3.3

■ Find the open intervals on which the graph of a function is concave upward or    *33–36*
   concave downward.

   Concave upward if $f''(x) > 0$

   Concave downward if $f''(x) < 0$

■ Find the points of inflection of the graph of a function.    *37–40*

■ Use the Second-Derivative Test to find the relative extrema of a function.    *41–44*

■ Find the point of diminishing returns of an input-output model.    *45, 46*

## Section 3.4

■ Solve real-life optimization problems.    *47–53, 96*

## Section 3.5

■ Solve business and economics optimization problems.    *54–58, 99*

■ Find the price elasticity of demand for a demand function.    *59–62*

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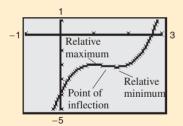| Section 3.6 | Review Exercises |
|---|---|
| ■ Find the vertical and horizontal asymptotes of a function and sketch its graph. | *63– 68* |
| ■ Find infinite limits and limits at infinity. | *69–76* |
| ■ Use asymptotes to answer questions about real life. | *77, 78* |

| Section 3.7 | |
|---|---|
| ■ Analyze the graph of a function. | *79–86* |

| Section 3.8 | |
|---|---|
| ■ Find the differential of a function. | *87–90* |
| ■ Use differentials to approximate changes in a function. | *91–94* |
| ■ Use differentials to approximate changes in real-life models. | *97, 98* |

## Study Strategies

■ **Solve Problems Graphically, Analytically, and Numerically**   When analyzing the graph of a function, use a variety of problem-solving strategies. For instance, if you were asked to analyze the graph of

$$f(x) = x^3 - 4x^2 + 5x - 4$$

you could begin *graphically*. That is, you could use a graphing utility to find a viewing window that appears to show the important characteristics of the graph. From the graph shown below, the function appears to have one relative minimum, one relative maximum, and one point of inflection.



Next, you could use calculus to *analyze* the graph. Because the derivative of $f$ is

$$f'(x) = 3x^2 - 8x + 5 = (3x - 5)(x - 1)$$

the critical numbers of $f$ are $x = \frac{5}{3}$ and $x = 1$. By the First-Derivative Test, you can conclude that $x = \frac{5}{3}$ yields a relative minimum and $x = 1$ yields a relative maximum. Because

$$f''(x) = 6x - 8$$

you can conclude that $x = \frac{4}{3}$ yields a point of inflection. Finally, you could analyze the graph *numerically*. For instance, you could construct a table of values and observe that $f$ is increasing on the interval $(-\infty, 1)$, decreasing on the interval $\left(1, \frac{5}{3}\right)$, and increasing on the interval $\left(\frac{5}{3}, \infty\right)$.

■ **Problem-Solving Strategies**   If you get stuck when trying to solve an optimization problem, consider the strategies below.

1. *Draw a Diagram.*   If feasible, draw a diagram that represents the problem. Label all known values and unknown values on the diagram.

2. *Solve a Simpler Problem.*   Simplify the problem, or write several simple examples of the problem. For instance, if you are asked to find the dimensions that will produce a maximum area, try calculating the areas of several examples.

3. *Rewrite the Problem in Your Own Words.*   Rewriting a problem can help you understand it better.

4. *Guess and Check.*   Try guessing the answer, then check your guess in the statement of the original problem. By refining your guesses, you may be able to think of a general strategy for solving the problem.

# Review Exercises

In Exercises 1–4, find the critical numbers of the function.

**1.** $f(x) = -x^2 + 2x + 4$

**2.** $g(x) = (x - 1)^2(x - 3)$

**3.** $h(x) = \sqrt{x}(x - 3)$

**4.** $f(x) = (x + 3)^2$

In Exercises 5–8, determine the open intervals on which the function is increasing or decreasing. Solve the problem analytically and graphically.

**5.** $f(x) = x^2 + x - 2$

**6.** $g(x) = (x + 2)^3$

**7.** $h(x) = \dfrac{x^2 - 3x - 4}{x - 3}$

**8.** $f(x) = -x^3 + 6x^2 - 2$

**9. Meteorology** The monthly normal temperature $T$ (in degrees Fahrenheit) for New York City can be modeled by

$T = 0.0380t^4 - 1.092t^3 + 9.23t^2 - 19.6t + 44$

where $1 \le t \le 12$ and $t = 1$ corresponds to January. *(Source: National Climatic Data Center)*

(a) Find the interval(s) on which the model is increasing.

(b) Find the interval(s) on which the model is decreasing.

(c) Interpret the results of parts (a) and (b).

(T) (d) Use a graphing utility to graph the model.

**10. CD Shipments** The number $S$ of manufacturer unit shipments (in millions) of CDs in the United States from 2000 through 2005 can be modeled by

$S = -4.17083t^4 + 40.3009t^3 - 110.524t^2$
$+ 19.40t + 941.6$

where $0 \le t \le 5$ and $t = 0$ corresponds to 2000. *(Source: Recording Industry Association of America)*

(a) Find the interval(s) on which the model is increasing.

(b) Find the interval(s) on which the model is decreasing.

(c) Interpret the results of parts (a) and (b).

(T) (d) Use a graphing utility to graph the model.

In Exercises 11–20, use the First-Derivative Test to find the relative extrema of the function. Then use a graphing utility to verify your result.

**11.** $f(x) = 4x^3 - 6x^2 - 2$    **12.** $f(x) = \frac{1}{4}x^4 - 8x$

**13.** $g(x) = x^2 - 16x + 12$

**14.** $h(x) = 4 + 10x - x^2$

**15.** $h(x) = 2x^2 - x^4$

**16.** $s(x) = x^4 - 8x^2 + 3$

**17.** $f(x) = \dfrac{6}{x^2 + 1}$

**18.** $f(x) = \dfrac{2}{x^2 - 1}$

**19.** $h(x) = \dfrac{x^2}{x - 2}$

**20.** $g(x) = x - 6\sqrt{x}, \quad x > 0$

In Exercises 21–30, find the absolute extrema of the function on the closed interval. Then use a graphing utility to confirm your result.

**21.** $f(x) = x^2 + 5x + 6; \quad [-3, 0]$

**22.** $f(x) = x^4 - 2x^3; \quad [0, 2]$

**23.** $f(x) = x^3 - 12x + 1; \quad [-4, 4]$

**24.** $f(x) = x^3 + 2x^2 - 3x + 4; \quad [-3, 2]$

**25.** $f(x) = 4\sqrt{x} - x^2; \quad [0, 3]$

**26.** $f(x) = 2\sqrt{x} - x; \quad [0, 9]$

**27.** $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}; \quad [0, 2]$

**28.** $f(x) = -x^4 + x^2 + 2; \quad [0, 2]$

**29.** $f(x) = \dfrac{2x}{x^2 + 1}; \quad [-1, 2]$

**30.** $f(x) = \dfrac{8}{x} + x; \quad [1, 4]$

(T) **31. Surface Area** A right circular cylinder of radius $r$ and height $h$ has a volume of 25 cubic inches. The total surface area of the cylinder in terms of $r$ is given by

$S = 2\pi r\left(r + \dfrac{25}{\pi r^2}\right).$

Use a graphing utility to graph $S$ and $S'$ and find the value of $r$ that yields the minimum surface area.

**32. Environment** When organic waste is dumped into a pond, the decomposition of the waste consumes oxygen. A model for the oxygen level $O$ (where 1 is the normal level) of a pond as waste material oxidizes is

$O = \dfrac{t^2 - t + 1}{t^2 + 1}, \quad 0 \le t$

where $t$ is the time in weeks.

(a) When is the oxygen level lowest? What is this level?

(b) When is the oxygen level highest? What is this level?

(c) Describe the oxygen level as $t$ increases.

In Exercises 33–36, determine the open intervals on which the graph of the function is concave upward or concave downward. Then use a graphing utility to confirm your result.

**33.** $f(x) = (x - 2)^3$       **34.** $h(x) = x^5 - 10x^2$

**35.** $g(x) = \frac{1}{4}(-x^4 + 8x^2 - 12)$

**36.** $h(x) = x^3 - 6x$

In Exercises 37–40, find the points of inflection of the graph of the function.

**37.** $f(x) = \frac{1}{2}x^4 - 4x^3$

**38.** $f(x) = \frac{1}{4}x^4 - 2x^2 - x$

**39.** $f(x) = x^3(x - 3)^2$

**40.** $f(x) = (x - 1)^2(x - 3)$

In Exercises 41–44, use the Second-Derivative Test to find the relative extrema of the function.

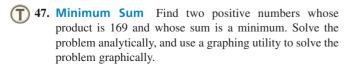**41.** $f(x) = x^5 - 5x^3$

**42.** $f(x) = x(x^2 - 3x - 9)$

**43.** $f(x) = 2x^2(1 - x^2)$

**44.** $f(x) = x - 4\sqrt{x + 1}$

**Point of Diminishing Returns**   In Exercises 45 and 46, identify the point of diminishing returns for the input-output function. For each function, $R$ is the revenue (in thousands of dollars) and $x$ is the amount spent on advertising (in thousands of dollars).

**45.** $R = \frac{1}{1500}(150x^2 - x^3), \quad 0 \le x \le 100$

**46.** $R = -\frac{2}{3}(x^3 - 12x^2 - 6), \quad 0 \le x \le 8$

(T) **47. Minimum Sum**   Find two positive numbers whose product is 169 and whose sum is a minimum. Solve the problem analytically, and use a graphing utility to solve the problem graphically.

**48. Length**   The wall of a building is to be braced by a beam that must pass over a five-foot fence that is parallel to the building and 4 feet from the building. Find the length of the shortest beam that can be used.

**49. Newspaper Circulation**   The total number $N$ of daily newspapers in circulation (in millions) in the United States from 1970 through 2005 can be modeled by

$N = 0.022t^3 - 1.27t^2 + 9.7t + 1746$

where $0 \le t \le 35$ and $t = 0$ corresponds to 1970. *(Source: Editor and Publisher Company)*

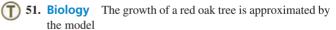(a) Find the absolute maximum and minimum over the time period.

(b) Find the year in which the circulation was changing at the greatest rate.

(c) Briefly explain your results for parts (a) and (b).

**50. Minimum Cost**   A fence is to be built to enclose a rectangular region of 4800 square feet. The fencing material along three sides costs $3 per foot. The fencing material along the fourth side costs $4 per foot.

(a) Find the most economical dimensions of the region.

(b) How would the result of part (a) change if the fencing material costs for all sides increased by $1 per foot?

(T) **51. Biology**   The growth of a red oak tree is approximated by the model

$y = -0.003x^3 + 0.137x^2 + 0.458x - 0.839,$

  $2 \le x \le 34$

where $y$ is the height of the tree in feet and $x$ is its age in years. Find the age of the tree when it is growing most rapidly. Then use a graphing utility to graph the function and to verify your result. (*Hint:* Use the viewing window $2 \le x \le 34$ and $-10 \le y \le 60$.)

**52. Consumer Trends**   The average number of hours $N$ (per person per year) of TV usage in the United States from 2000 through 2005 can be modeled by

$N = -0.382t^3 - 0.97t^2 + 30.5t + 1466, \quad 0 \le t \le 5$

where $t = 0$ corresponds to 2000. *(Source: Veronis Suhler Stevenson)*

(a) Find the intervals on which $dN/dt$ is increasing and decreasing.

(b) Find the limit of $N$ as $t \to \infty$.

(c) Briefly explain your results for parts (a) and (b).

**53. Medicine: Poiseuille's Law**   The speed of blood that is $r$ centimeters from the center of an artery is modeled by

$s(r) = c(R^2 - r^2), \quad c > 0$

where $c$ is a constant, $R$ is the radius of the artery, and $s$ is measured in centimeters per second. Show that the speed is a maximum at the center of an artery.

**54. Profit**   The demand and cost functions for a product are $p = 36 - 4x$ and $C = 2x^2 + 6$.

(a) What level of production will produce a maximum profit?

(b) What level of production will produce a minimum average cost per unit?

**55. Revenue**   For groups of 20 or more, a theater determines the ticket price $p$ according to the formula

$p = 15 - 0.1(n - 20), \quad 20 \le n \le N$

where $n$ is the number in the group. What should the value of $N$ be? Explain your reasoning.

**56. Minimum Cost**   The cost of fuel to run a locomotive is proportional to the $\frac{3}{2}$ power of the speed. At a speed of 25 miles per hour, the cost of fuel is $50 per hour. Other costs amount to $100 per hour. Find the speed that will minimize the cost per mile.

57. **Inventory Cost** The cost $C$ of inventory modeled by

$$C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r$$

depends on ordering and storage costs, where $Q$ is the number of units sold per year, $r$ is the cost of storing one unit for 1 year, $s$ is the cost of placing an order, and $x$ is the number of units in the order. Determine the order size that will minimize the cost when $Q = 10,000$, $s = 4.5$, and $r = 5.76$.

58. **Profit** The demand and cost functions for a product are given by

$$p = 600 - 3x$$

and

$$C = 0.3x^2 + 6x + 600$$

where $p$ is the price per unit, $x$ is the number of units, and $C$ is the total cost. The profit for producing $x$ units is given by

$$P = xp - C - xt$$

where $t$ is the excise tax per unit. Find the maximum profits for excise taxes of $t = \$5$, $t = \$10$, and $t = \$20$.

In Exercises 59–62, find the intervals on which the demand is elastic, inelastic, and of unit elasticity.

59. $p = 30 - 0.2x$, $0 \le x \le 150$

60. $p = 60 - 0.04x$, $0 \le x \le 1500$

61. $p = 300 - x$, $0 \le x \le 300$

62. $p = 960 - x$, $0 \le x \le 960$

(T) In Exercises 63–68, find the vertical and horizontal asymptotes of the graph. Then use a graphing utility to graph the function.

63. $h(x) = \dfrac{2x + 3}{x - 4}$

64. $g(x) = \dfrac{3}{x} - 2$

65. $f(x) = \dfrac{\sqrt{9x^2 + 1}}{x}$

66. $h(x) = \dfrac{3x}{\sqrt{x^2 + 2}}$

67. $f(x) = \dfrac{4}{x^2 + 1}$

68. $h(x) = \dfrac{2x^2 + 3x - 5}{x - 1}$

In Exercises 69–76, find the limit, if it exists.

69. $\displaystyle \lim_{x \to 0^+} \left(x - \frac{1}{x^3}\right)$

70. $\displaystyle \lim_{x \to 0^-} \left(3 + \frac{1}{x}\right)$

71. $\displaystyle \lim_{x \to -1^+} \frac{x^2 - 2x + 1}{x + 1}$

72. $\displaystyle \lim_{x \to 3^-} \frac{3x^2 + 1}{x^2 - 9}$

73. $\displaystyle \lim_{x \to \infty} \frac{2x^2}{3x^2 + 5}$

74. $\displaystyle \lim_{x \to \infty} \frac{3x^2 - 2x + 3}{x + 1}$

75. $\displaystyle \lim_{x \to -\infty} \frac{3x}{x^2 + 1}$

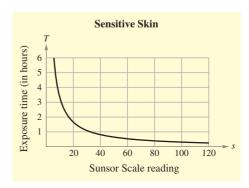76. $\displaystyle \lim_{x \to -\infty} \left(\frac{x}{x - 2} + \frac{2x}{x + 2}\right)$

77. **Health** For a person with sensitive skin, the maximum amount $T$ (in hours) of exposure to the sun that can be tolerated before skin damage occurs can be modeled by

$$T = \frac{-0.03s + 33.6}{s}, \quad 0 < s \le 120$$

where $s$ is the Sunsor Scale reading. *(Source: Sunsor, Inc.)*

(T) (a) Use a graphing utility to graph the model. Compare your result with the graph below.

(b) Describe the value of $T$ as $s$ increases.



**Sensitive Skin**

78. **Average Cost and Profit** The cost and revenue functions for a product are given by

$$C = 10,000 + 48.9x$$

and

$$R = 68.5x.$$

(a) Find the average cost function.

(b) What is the limit of the average cost as $x$ approaches infinity?

(c) Find the average profits when $x$ is 1 million, 2 million, and 10 million.

(d) What is the limit of the average profit as $x$ increases without bound?

Ⓣ In Exercises 79–86, use a graphing utility to graph the function. Use the graph to approximate any intercepts, relative extrema, points of inflection, and asymptotes. State the domain of the function.

**79.** $f(x) = 4x - x^2$

**80.** $f(x) = 4x^3 - x^4$

**81.** $f(x) = x\sqrt{16 - x^2}$

**82.** $f(x) = x^2\sqrt{9 - x^2}$

**83.** $f(x) = \dfrac{x + 1}{x - 1}$

**84.** $f(x) = \dfrac{x - 1}{3x^2 + 1}$

**85.** $f(x) = x^2 + \dfrac{2}{x}$

**86.** $f(x) = x^{4/5}$

In Exercises 87–90, find the differential $dy$.

**87.** $y = x(1 - x)$

**88.** $y = (3x^2 - 2)^3$

**89.** $y = \sqrt{36 - x^2}$

**90.** $y = \dfrac{2 - x}{x + 5}$

In Exercises 91–94, use differentials to approximate the change in cost, revenue, or profit corresponding to an increase in sales of one unit.

**91.** $C = 40x^2 + 1225, \quad x = 10$

**92.** $C = 1.5\sqrt[3]{x} + 500, \quad x = 125$

**93.** $R = 6.25x + 0.4x^{3/2}, \quad x = 225$

**94.** $P = 0.003x^2 + 0.019x - 1200, \quad x = 750$

Ⓣ **95. Revenue Per Share**  The revenues per share $R$ (in dollars) for the Walt Disney Company for the years 1994 through 2005 are shown in the table.  *(Source: The Walt Disney Company)*

| Year, $t$ | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|
| Revenue per share, $R$ | 6.40 | 7.70 | 10.50 | 11.10 | 11.21 | 11.34 |

| Year, $t$ | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|
| Revenue per share, $R$ | 12.09 | 12.52 | 12.40 | 13.23 | 15.05 | 15.91 |

(a) Use a graphing utility to create a scatter plot of the data, where $t$ is the time in years, with $t = 4$ corresponding to 1994.

(b) Describe any trends and/or patterns in the data.

(c) A model for the data is

$$R = \dfrac{5.75 - 2.043t + 0.1959t^2}{1 - 0.378t + 0.0438t^2 - 0.00117t^3},$$

$$4 \le t \le 15.$$

Graph the model and the data in the same viewing window.

(d) Find the years in which the revenue per share was increasing and decreasing.

(e) Find the years in which the rate of change of the revenue per share was increasing and decreasing.

(f) Briefly explain your results for parts (d) and (e).

**96. Medicine**  The effectiveness $E$ of a pain-killing drug $t$ hours after entering the bloodstream is modeled by

$$E = 22.5t + 7.5t^2 - 2.5t^3, \quad 0 \le t \le 4.5.$$

Ⓣ (a) Use a graphing utility to graph the equation. Choose an appropriate window.

(b) Find the maximum effectiveness the pain-killing drug attains over the interval $[0, 4.5]$.

**97. Surface Area and Volume**  The diameter of a sphere is measured to be 18 inches with a possible error of 0.05 inch. Use differentials to approximate the possible error in the surface area and the volume of the sphere.

**98. Demand**  A company finds that the demand for its product is modeled by $p = 85 - 0.125x$. If $x$ changes from 7 to 8, what is the corresponding change in $p$? Compare the values of $\Delta p$ and $dp$.

Ⓑ **99. Economics: Revenue**  Consider the following cost and demand information for a monopoly (in dollars). Complete the table, and then use the information to answer the questions.  *(Source: Adapted from Taylor, Economics, Fifth Edition)*

| Quantity of output | Price | Total revenue | Marginal revenue |
|---|---|---|---|
| 1 | 14.00 | | |
| 2 | 12.00 | | |
| 3 | 10.00 | | |
| 4 | 8.50 | | |
| 5 | 7.00 | | |
| 6 | 5.50 | | |

Ⓣ (a) Use the *regression* feature of a graphing utility to find a quadratic model for the total revenue data.

(b) From the total revenue model you found in part (a), use derivatives to find an equation for the marginal revenue. Now use the values for output in the table and compare the results with the values in the marginal revenue column of the table. How close was your model?

(c) What quantity maximizes total revenue for the monopoly?

## Chapter Test

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, find the critical numbers of the function and the open intervals on which the function is increasing or decreasing .

**1.** $f(x) = 3x^2 - 4$          **2.** $f(x) = x^3 - 12x$          **3.** $f(x) = (x - 5)^4$

In Exercises 4–6, use the First-Derivative Test to find all relative extrema of the function. Then use a graphing utility to verify your result.

**4.** $f(x) = \dfrac{1}{3}x^3 - 9x + 4$      **5.** $f(x) = 2x^4 - 4x^2 - 5$      **6.** $f(x) = \dfrac{5}{x^2 + 2}$

In Exercises 7–9, find the absolute extrema of the function on the closed interval.

**7.** $f(x) = x^2 + 6x + 8, \quad [-4, 0]$      **8.** $f(x) = 12\sqrt{x} - 4x, \quad [0, 5]$

**9.** $f(x) = \dfrac{6}{x} + \dfrac{x}{2}, \quad [1, 6]$

In Exercises 10 and 11, determine the open intervals on which the graph of the function is concave upward or concave downward.

**10.** $f(x) = x^5 - 4x^2$          **11.** $f(x) = \dfrac{20}{3x^2 + 8}$

In Exercises 12 and 13, find the point(s) of inflection of the graph of the function.

**12.** $f(x) = x^4 + 6$          **13.** $f(x) = \dfrac{1}{5}x^5 - 4x^2$

In Exercises 14 and 15, use the Second-Derivative Test to find all relative extrema of the function.

**14.** $f(x) = x^3 - 6x^2 - 24x + 12$      **15.** $f(x) = \dfrac{3}{5}x^5 - 9x^3$

(T) In Exercises 16–18, find the vertical and horizontal asymptotes of the graph. Then use a graphing utility to graph the function.

**16.** $f(x) = \dfrac{3x + 2}{x - 5}$      **17.** $f(x) = \dfrac{2x^2}{x^2 + 3}$      **18.** $f(x) = \dfrac{2x^2 - 5}{x - 1}$

In Exercises 19–21, find the limit, if it exists.

**19.** $\displaystyle\lim_{x \to \infty} \left( \dfrac{3}{x} + 1 \right)$      **20.** $\displaystyle\lim_{x \to \infty} \dfrac{3x^2 - 4x + 1}{x - 7}$      **21.** $\displaystyle\lim_{x \to -\infty} \dfrac{6x^2 + x - 5}{2x^2 - 5x}$

In Exercises 22–24, find the differential $dy$.

**22.** $y = 5x^2 - 3$      **23.** $y = \dfrac{1 - x}{x + 3}$      **24.** $y = (x + 4)^3$

**25.** The demand function for a product is modeled by $p = 250 - 0.4x, 0 \le x \le 625$, where $p$ is the price at which $x$ units of the product are demanded by the market. Find the interval of inelasticity for the function.

# 4

# Exponential and Logarithmic Functions



On May 26, 2006, Java, Indonesia experienced an earthquake measuring 6.3 on the Richter scale, a logarithmic function that serves as one way to calculate an earthquake's magnitude. (See Section 4.5, Exercise 87.)

## Applications

Exponential and logarithmic functions have many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Make a Decision:  Median Sales Prices, Exercise 37, page 298
- Learning Theory, Exercise 88, page 325
- Consumer Trends, Exercise 85, page 334
- Make a Decision:  Revenue, Exercise 41, page 343
- Make a Decision:  Modeling Data, Exercise 52, page 343

## Section 4.1

# Exponential Functions

- Use the properties of exponents to evaluate and simplify exponential expressions.
- Sketch the graphs of exponential functions.

### Exponential Functions

You are already familiar with the behavior of algebraic functions such as

$$f(x) = x^2, \quad g(x) = \sqrt{x} = x^{1/2}, \quad \text{and} \quad h(x) = \frac{1}{x} = x^{-1}$$

each of which involves a variable raised to a constant power. By interchanging roles and raising a constant to a variable power, you obtain another important class of functions called **exponential functions.** Some simple examples are

$$f(x) = 2^x, \quad g(x) = \left(\frac{1}{10}\right)^x = \frac{1}{10^x}, \quad \text{and} \quad h(x) = 3^{2x} = 9^x.$$

In general, you can use any positive base $a \neq 1$ as the base of an exponential function.

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**Definition of Exponential Function**

If $a > 0$ and $a \neq 1$, then the **exponential function** with base $a$ is given by

$$f(x) = a^x.$$

---

**STUDY TIP**

In the definition above, the base $a = 1$ is excluded because it yields $f(x) = 1^x = 1$. This is a constant function, not an exponential function.

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When working with exponential functions, the properties of exponents, shown below, are useful.

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**Properties of Exponents**

Let $a$ and $b$ be positive numbers.

**1.** $a^0 = 1$  **2.** $a^x a^y = a^{x+y}$  **3.** $\dfrac{a^x}{a^y} = a^{x-y}$

**4.** $(a^x)^y = a^{xy}$  **5.** $(ab)^x = a^x b^x$  **6.** $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

**7.** $a^{-x} = \dfrac{1}{a^x}$

**Example 1**   **Applying Properties of Exponents**

Simplify each expression using the properties of exponents.

**a.** $(2^2)(2^3)$     **b.** $(2^2)(2^{-3})$     **c.** $(3^2)^3$

**d.** $\left(\dfrac{1}{3}\right)^{-2}$     **e.** $\dfrac{3^2}{3^3}$     **f.** $(2^{1/2})(3^{1/2})$

**SOLUTION**

**a.** $(2^2)(2^3) = 2^{2+3} = 2^5 = 32$    <span style="color:red">Apply Property 2.</span>

**b.** $(2^2)(2^{-3}) = 2^{2-3} = 2^{-1} = \frac{1}{2}$    <span style="color:red">Apply Properties 2 and 7.</span>

**c.** $(3^2)^3 = 3^{2(3)} = 3^6 = 729$    <span style="color:red">Apply Property 4.</span>

**d.** $\left(\dfrac{1}{3}\right)^{-2} = \dfrac{1}{(1/3)^2} = \left(\dfrac{1}{1/3}\right)^2 = 3^2 = 9$    <span style="color:red">Apply Properties 6 and 7.</span>

**e.** $\dfrac{3^2}{3^3} = 3^{2-3} = 3^{-1} = \dfrac{1}{3}$    <span style="color:red">Apply Properties 3 and 7.</span>

**f.** $(2^{1/2})(3^{1/2}) = [(2)(3)]^{1/2} = 6^{1/2} = \sqrt{6}$    <span style="color:red">Apply Property 5.</span>

✓ **CHECKPOINT 1**

Simplify each expression using the properties of exponents.

**a.** $(3^2)(3^3)$     **b.** $(3^2)(3^{-1})$

**c.** $(2^3)^2$     **d.** $\left(\dfrac{1}{2}\right)^{-3}$

**e.** $\dfrac{2^2}{2^3}$     **f.** $(2^{1/2})(5^{1/2})$ ■

Although Example 1 demonstrates the properties of exponents with integer and rational exponents, it is important to realize that the properties hold for *all* real exponents. With a calculator, you can obtain approximations of $a^x$ for any base $a$ and any real exponent $x$. Here are some examples.

$$2^{-0.6} \approx 0.660, \qquad \pi^{0.75} \approx 2.360, \qquad (1.56)^{\sqrt{2}} \approx 1.876$$

**Example 2**   **Dating Organic Material**   ®

In living organic material, the ratio of radioactive carbon isotopes to the total number of carbon atoms is about 1 to $10^{12}$. When organic material dies, its radioactive carbon isotopes begin to decay, with a half-life of about 5715 years. This means that after 5715 years, the ratio of isotopes to atoms will have decreased to one-half the original ratio, after a second 5715 years the ratio will have decreased to one-fourth of the original, and so on. Figure 4.1 shows this decreasing ratio. The formula for the ratio $R$ of carbon isotopes to carbon atoms is

$$R = \left(\frac{1}{10^{12}}\right)\left(\frac{1}{2}\right)^{t/5715}$$

where $t$ is the time in years. Find the value of $R$ for each period of time.

**a.** 10,000 years     **b.** 20,000 years     **c.** 25,000 years

**SOLUTION**

**a.** $R = \left(\dfrac{1}{10^{12}}\right)\left(\dfrac{1}{2}\right)^{10,000/5715} \approx 2.973 \times 10^{-13}$    <span style="color:red">Ratio for 10,000 years</span>

**b.** $R = \left(\dfrac{1}{10^{12}}\right)\left(\dfrac{1}{2}\right)^{20,000/5715} \approx 8.842 \times 10^{-14}$    <span style="color:red">Ratio for 20,000 years</span>

**c.** $R = \left(\dfrac{1}{10^{12}}\right)\left(\dfrac{1}{2}\right)^{25,000/5715} \approx 4.821 \times 10^{-14}$    <span style="color:red">Ratio for 25,000 years</span>



**Organic Material**

**FIGURE 4.1**

✓ **CHECKPOINT 2**

Use the formula for the ratio of carbon isotopes to carbon atoms in Example 2 to find the value of $R$ for each period of time.

**a.** 5,000 years

**b.** 15,000 years

**c.** 30,000 years ■

## Graphs of Exponential Functions

The basic nature of the graph of an exponential function can be determined by the point-plotting method or by using a graphing utility.

### Example 3  Graphing Exponential Functions

Sketch the graph of each exponential function.

**a.** $f(x) = 2^x$    **b.** $g(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$    **c.** $h(x) = 3^x$

**SOLUTION**  To sketch these functions by hand, you can begin by constructing a table of values, as shown below.

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|
| $f(x) = 2^x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 | 16 |
| $g(x) = 2^{-x}$ | 8 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ |
| $h(x) = 3^x$ | $\frac{1}{27}$ | $\frac{1}{9}$ | $\frac{1}{3}$ | 1 | 3 | 9 | 27 | 81 |

The graphs of the three functions are shown in Figure 4.2. Note that the graphs of $f(x) = 2^x$ and $h(x) = 3^x$ are increasing, whereas the graph of $g(x) = 2^{-x}$ is decreasing.
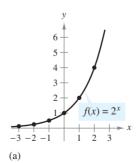


(a)    (b)    (c)

**FIGURE 4.2**
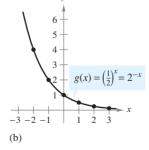
✓ **CHECKPOINT 3**

Complete the table of values for $f(x) = 5^x$. Sketch the graph of the exponential function.

| $x$ | $-3$ | $-2$ | $-1$ | 0 |
|---|---|---|---|---|
| $f(x)$ | | | | |

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(x)$ | | | |

### TECHNOLOGY
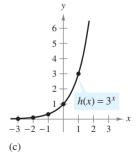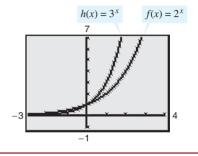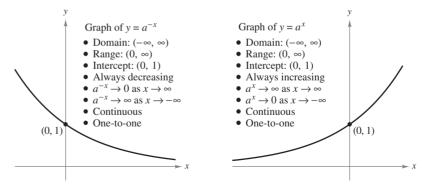
(T) Try graphing the functions $f(x) = 2^x$ and $h(x) = 3^x$ in the same viewing window, as shown at the right. From the display, you can see that the graph of $h$ is increasing more rapidly than the graph of $f$.*



*Specific calculator keystroke instructions for operations in this and other technology boxes can be found at *college.hmco.com/info/larsonapplied*.

The forms of the graphs in Figure 4.2 are typical of the graphs of the exponential functions $y = a^{-x}$ and $y = a^x$, where $a > 1$. The basic characteristics of such graphs are summarized in Figure 4.3.

Graph of $y = a^{-x}$
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Always decreasing
- $a^{-x} \to 0$ as $x \to \infty$
- $a^{-x} \to \infty$ as $x \to -\infty$
- Continuous
- One-to-one

$(0, 1)$

Graph of $y = a^x$
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Always increasing
- $a^x \to \infty$ as $x \to \infty$
- $a^x \to 0$ as $x \to -\infty$
- Continuous
- One-to-one

$(0, 1)$

**FIGURE 4.3**    Characteristics of the Exponential Functions $y = a^{-x}$ and $y = a^x$ $(a > 1)$

**Example 4**    **Graphing an Exponential Function**

Sketch the graph of $f(x) = 3^{-x} - 1$.

**SOLUTION**    Begin by creating a table of values, as shown below.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $3^2 - 1 = 8$ | $3^1 - 1 = 2$ | $3^0 - 1 = 0$ | $3^{-1} - 1 = -\frac{2}{3}$ | $3^{-2} - 1 = -\frac{8}{9}$ |

From the limit

$$\lim_{x \to \infty} (3^{-x} - 1) = \lim_{x \to \infty} 3^{-x} - \lim_{x \to \infty} 1$$

$$= \lim_{x \to \infty} \frac{1}{3^x} - \lim_{x \to \infty} 1$$

$$= 0 - 1$$

$$= -1$$

you can see that $y = -1$ is a horizontal asymptote of the graph. The graph is shown in Figure 4.4.



$(-2, 8)$

$f(x) = 3^{-x} - 1$

$(-1, 2)$

$(0, 0)$

$\left(1, -\frac{2}{3}\right)$ $\left(2, -\frac{8}{9}\right)$

**FIGURE 4.4**
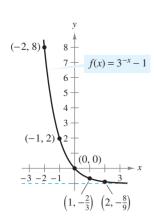
**✓ CHECKPOINT 4**

Complete the table of values for $f(x) = 2^{-x} + 1$. Sketch the graph of the function. Determine the horizontal asymptote of the graph.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|---|
| $f(x)$ | | | | |

| $x$ | $1$ | $2$ | $3$ |
|---|---|---|---|
| $f(x)$ | | | |

**CONCEPT CHECK**

1. Complete the following: If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a(n) _____ function.

2. Identify the domain and range of the exponential functions (a) $y = a^{-x}$ and (b) $y = a^x$. (Assume $a > 1$.)

3. As $x$ approaches $\infty$, what does $a^{-x}$ approach? (Assume $a > 1$.)

4. Explain why $1^x$ is *not* an exponential function.

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.4 and 1.6.

In Exercises 1–6, describe how the graph of g is related to the graph of f.

**1.** $g(x) = f(x + 2)$  **2.** $g(x) = -f(x)$  **3.** $g(x) = -1 + f(x)$

**4.** $g(x) = f(-x)$  **5.** $g(x) = f(x - 1)$  **6.** $g(x) = f(x) + 2$

In Exercises 7–10, discuss the continuity of the function.

**7.** $f(x) = \dfrac{x^2 + 2x - 1}{x + 4}$  **8.** $f(x) = \dfrac{x^2 - 3x + 1}{x^2 + 2}$

**9.** $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 1}$  **10.** $f(x) = \dfrac{x^2 - 5x + 4}{x^2 + 1}$

In Exercises 11–16, solve for x.

**11.** $2x - 6 = 4$  **12.** $3x + 1 = 5$  **13.** $(x + 4)^2 = 25$

**14.** $(x - 2)^2 = 8$  **15.** $x^2 + 4x - 5 = 0$  **16.** $2x^2 - 3x + 1 = 0$

## Exercises 4.1

In Exercises 1 and 2, evaluate each expression.

**1.** (a) $5(5^3)$  (b) $27^{2/3}$
    (c) $64^{3/4}$  (d) $81^{1/2}$
    (e) $25^{3/2}$  (f) $32^{2/5}$

**2.** (a) $\left(\frac{1}{5}\right)^3$  (b) $\left(\frac{1}{8}\right)^{1/3}$
    (c) $64^{2/3}$  (d) $\left(\frac{5}{8}\right)^2$
    (e) $100^{3/2}$  (f) $4^{5/2}$

In Exercises 3–6, use the properties of exponents to simplify the expression.

**3.** (a) $(5^2)(5^3)$  (b) $(5^2)(5^{-3})$
    (c) $(5^2)^2$  (d) $5^{-3}$

**4.** (a) $\dfrac{5^3}{5^6}$  (b) $\left(\dfrac{1}{5}\right)^{-2}$
    (c) $(8^{1/2})(2^{1/2})$  (d) $(32^{3/2})(\frac{1}{2})^{3/2}$

**5.** (a) $\dfrac{5^3}{25^2}$  (b) $(9^{2/3})(3)(3^{2/3})$
    (c) $[(25^{1/2})(5^2)]^{1/3}$  (d) $(8^2)(4^3)$

**6.** (a) $(4^3)(4^2)$  (b) $(\frac{1}{4})^2(4^2)$
    (c) $(4^6)^{1/2}$  (d) $[(8^{-1})(8^{2/3})]^3$

Ⓣ In Exercises 7–10, evaluate the function. If necessary, use a graphing utility, rounding your answers to three decimal places.

**7.** $f(x) = 2^{x-1}$
    (a) $f(3)$  (b) $f(\frac{1}{2})$  (c) $f(-2)$  (d) $f(-\frac{3}{2})$

**8.** $f(x) = 3^{x+2}$
    (a) $f(-4)$  (b) $f(-\frac{1}{2})$  (c) $f(2)$  (d) $f(-\frac{5}{2})$

**9.** $g(x) = 1.05^x$
    (a) $g(-2)$  (b) $g(120)$  (c) $g(12)$  (d) $g(5.5)$

**10.** $g(x) = 1.075^x$
    (a) $g(1.2)$  (b) $g(180)$  (c) $g(60)$  (d) $g(12.5)$

**11. Radioactive Decay**  After $t$ years, the remaining mass $y$ (in grams) of 16 grams of a radioactive element whose half-life is 30 years is given by

$$y = 16\left(\frac{1}{2}\right)^{t/30}, \quad t \geq 0.$$

How much of the initial mass remains after 90 years?

**12. Radioactive Decay**  After $t$ years, the remaining mass $y$ (in grams) of 23 grams of a radioactive element whose half-life is 45 years is given by
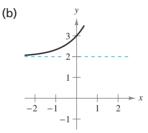
$$y = 23\left(\frac{1}{2}\right)^{t/45}, \quad t \geq 0.$$
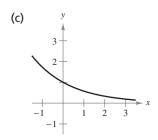
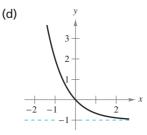How much of the initial mass remains after 150 years?

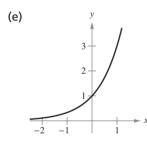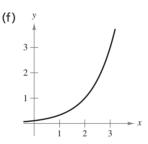In Exercises 13–18, match the function with its graph. [The graphs are labeled (a)–(f).]

(a)

(b)

(c)

(d)

(e)

(f)

13. $f(x) = 3^x$

14. $f(x) = 3^{-x/2}$

15. $f(x) = -3^x$

16. $f(x) = 3^{x-2}$

17. $f(x) = 3^{-x} - 1$

18. $f(x) = 3^x + 2$

(T) In Exercises 19–30, use a graphing utility to graph the function.

19. $f(x) = 6^x$

20. $f(x) = 4^x$

21. $f(x) = \left(\frac{1}{5}\right)^x = 5^{-x}$

22. $f(x) = \left(\frac{1}{4}\right)^x = 4^{-x}$

23. $y = 2^{x-1}$

24. $y = 4^x + 3$

25. $y = -2^x$

26. $y = -5^x$

27. $y = 3^{-x^2}$

28. $y = 2^{-x^2}$

29. $s(t) = \frac{1}{4}(3^{-t})$

30. $s(t) = 2^{-t} + 3$

31. **Population Growth**   The population $P$ (in millions) of the United States from 1992 through 2005 can be modeled by the exponential function $P(t) = 252.12(1.011)^t$, where $t$ is the time in years, with $t = 2$ corresponding to 1992. Use the model to estimate the population in the years (a) 2008 and (b) 2012.   *(Source: U.S. Census Bureau)*

32. **Sales**   The sales $S$ (in millions of dollars) for Starbucks from 1996 through 2005 can be modeled by the exponential function $S(t) = 182.34(1.272)^t$, where $t$ is the time in

years, with $t = 6$ corresponding to 1996. Use the model to estimate the sales in the years (a) 2008 and (b) 2014. *(Source: Starbucks Corp.)*

33. **Property Value**   Suppose that the value of a piece of property doubles every 15 years. If you buy the property for $64,000, its value $t$ years after the date of purchase should be $V(t) = 64,000(2)^{t/15}$. Use the model to approximate the value of the property (a) 5 years and (b) 20 years after it is purchased.

34. **Depreciation**   After $t$ years, the value of a car that originally cost $16,000 depreciates so that each year it is worth $\frac{3}{4}$ of its value for the previous year. Find a model for $V(t)$, the value of the car after $t$ years. Sketch a graph of the model and determine the value of the car 4 years after it was purchased.

35. **Inflation Rate**   Suppose that the annual rate of inflation averages 4% over the next 10 years. With this rate of inflation, the approximate cost $C$ of goods or services during any year in that decade will be given by

$$C(t) = P(1.04)^t, \quad 0 \le t \le 10$$

where $t$ is time in years and $P$ is the present cost. If the price of an oil change for your car is presently $24.95, estimate the price 10 years from now.

36. **Inflation Rate**   Repeat Exercise 35 assuming that the annual rate of inflation is 10% over the next 10 years and the approximate cost $C$ of goods or services will be given by

$$C(t) = P(1.10)^t, \quad 0 \le t \le 10.$$

(T) 37. *MAKE A DECISION: MEDIAN SALES PRICES*   For the years 1998 through 2005, the median sales prices $y$ (in dollars) of one-family homes in the United States are shown in the table. *(Source: U.S. Census Bureau and U.S. Department of Housing and Urban Development)*

| Year  | 1998    | 1999    | 2000    | 2001    |
|-------|---------|---------|---------|---------|
| Price | 152,500 | 161,000 | 169,000 | 175,200 |

| Year  | 2002    | 2003    | 2004    | 2005    |
|-------|---------|---------|---------|---------|
| Price | 187,600 | 195,000 | 221,000 | 240,900 |

A model for this data is given by $y = 90,120(1.0649)^t$, where $t$ represents the year, with $t = 8$ corresponding to 1998.

(a) Compare the actual prices with those given by the model. Does the model fit the data? Explain your reasoning.

(b) Use a graphing utility to graph the model.

(c) Use the *zoom* and *trace* features of a graphing utility to predict during which year the median sales price of one-family homes will reach $300,000.
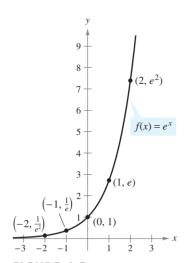
## Section 4.2

# Natural Exponential Functions

■ Evaluate and graph functions involving the natural exponential function.
■ Solve compound interest problems.
■ Solve present value problems.

## Natural Exponential Functions

In Section 4.1, exponential functions were introduced using an unspecified base $a$. In calculus, the most convenient (or natural) choice for a base is the irrational number $e$, whose decimal approximation is

$$e \approx 2.71828182846.$$

Although this choice of base may seem unusual, its convenience will become apparent as the rules for differentiating exponential functions are developed in Section 4.3. In that development, you will encounter the limit used in the definition of $e$.

**Limit Definition of e**

The irrational number $e$ is defined to be the limit of $(1 + x)^{1/x}$ as $x \to 0$. That is,

$$\lim_{x \to 0} (1 + x)^{1/x} = e.$$

### *Example 1*  Graphing the Natural Exponential Function

Sketch the graph of $f(x) = e^x$.

**SOLUTION**   Begin by evaluating the function for several values of $x$, as shown in the table.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $e^{-2} \approx 0.135$ | $e^{-1} \approx 0.368$ | $e^0 \approx 1$ | $e^1 \approx 2.718$ | $e^2 \approx 7.389$ |

The graph of $f(x) = e^x$ is shown in Figure 4.5. Note that $e^x$ is positive for all values of $x$. Moreover, the graph has the $x$-axis as a horizontal asymptote to the left. That is,

$$\lim_{x \to -\infty} e^x = 0.$$



**FIGURE 4.5**

### ✓ CHECKPOINT 1
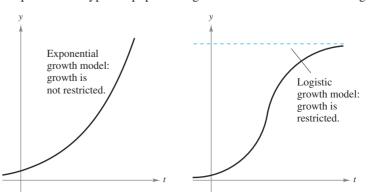
Complete the table of values for $f(x) = e^{-x}$. Sketch the graph of the function.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | | | | | |

Exponential functions are often used to model the growth of a quantity or a population. When the quantity's growth *is not* restricted, an exponential model is often used. When the quantity's growth *is* restricted, the best model is often a **logistic growth function** of the form

$$f(t) = \frac{a}{1 + be^{-kt}}.$$

Graphs of both types of population growth models are shown in Figure 4.6.



When a culture is grown in a dish, the size of the dish and the available food limit the culture's growth.

**FIGURE 4.6**

### Growth of Bacterial Culture



**FIGURE 4.7**

✓ **CHECKPOINT 2**

A bacterial culture is growing according to the model

$$y = \frac{1.50}{1 + 0.2e^{-0.5t}}, \quad t \ge 0$$

where $y$ is the culture weight (in grams) and $t$ is the time (in hours). Find the weight of the culture after 0 hours, 1 hour, and 10 hours. What is the limit of the model as $t$ increases without bound? ■

*Example 2*
**MAKE A DECISION** Modeling a Population

A bacterial culture is growing according to the *logistic growth model*

$$y = \frac{1.25}{1 + 0.25e^{-0.4t}}, \quad t \ge 0$$

where $y$ is the culture weight (in grams) and $t$ is the time (in hours). Find the weight of the culture after 0 hours, 1 hour, and 10 hours. What is the limit of the model as $t$ increases without bound? According to the model, will the weight of the culture reach 1.5 grams?

**SOLUTION**

$$y = \frac{1.25}{1 + 0.25e^{-0.4(0)}} = 1 \text{ gram} \qquad \text{Weight when } t = 0$$

$$y = \frac{1.25}{1 + 0.25e^{-0.4(1)}} \approx 1.071 \text{ grams} \qquad \text{Weight when } t = 1$$

$$y = \frac{1.25}{1 + 0.25e^{-0.4(10)}} \approx 1.244 \text{ grams} \qquad \text{Weight when } t = 10$$

As $t$ approaches infinity, the limit of $y$ is

$$\lim_{t \to \infty} \frac{1.25}{1 + 0.25e^{-0.4t}} = \lim_{t \to \infty} \frac{1.25}{1 + (0.25/e^{0.4t})} = \frac{1.25}{1 + 0} = 1.25.$$

So, as $t$ increases without bound, the weight of the culture approaches 1.25 grams. According to the model, the weight of the culture will not reach 1.5 grams. The graph of the model is shown in Figure 4.7.

## Extended Application: Compound Interest

If $P$ dollars is deposited in an account at an annual interest rate of $r$ (in decimal form), what is the balance after 1 year? The answer depends on the number of times the interest is compounded, according to the formula

$$A = P\left(1 + \frac{r}{n}\right)^n$$

where $n$ is the number of compoundings per year. The balances for a deposit of $1000 at 8%, at various compounding periods, are shown in the table.

| Number of times compounded per year, $n$ | Balance (in dollars), $A$ |
|---|---|
| Annually, $n = 1$ | $A = 1000\left(1 + \frac{0.08}{1}\right)^1 = \$1080.00$ |
| Semiannually, $n = 2$ | $A = 1000\left(1 + \frac{0.08}{2}\right)^2 = \$1081.60$ |
| Quarterly, $n = 4$ | $A = 1000\left(1 + \frac{0.08}{4}\right)^4 \approx \$1082.43$ |
| Monthly, $n = 12$ | $A = 1000\left(1 + \frac{0.08}{12}\right)^{12} \approx \$1083.00$ |
| Daily, $n = 365$ | $A = 1000\left(1 + \frac{0.08}{365}\right)^{365} \approx \$1083.28$ |

You may be surprised to discover that as $n$ increases, the balance $A$ approaches a limit, as indicated in the following development. In this development, let $x = r/n$. Then $x \to 0$ as $n \to \infty$, and you have

$$A = \lim_{n \to \infty} P\left(1 + \frac{r}{n}\right)^n$$

$$= P \lim_{n \to \infty} \left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^r$$

$$= P\left[\lim_{x \to 0} (1 + x)^{1/x}\right]^r \qquad \text{Substitute } x \text{ for } r/n.$$

$$= Pe^r.$$

This limit is the balance after 1 year of **continuous compounding.** So, for a deposit of $1000 at 8%, compounded continuously, the balance at the end of the year would be

$$A = 1000e^{0.08}$$

$$\approx \$1083.29.$$

**DISCOVERY**

Use a spreadsheet software program or the *table* feature of a graphing utility to evaluate the expression

$$\left(1 + \frac{1}{n}\right)^n$$

for each value of $n$.

| $n$ | $(1 + 1/n)^n$ |
|---|---|
| 10 | |
| 100 | |
| 1000 | |
| 10,000 | |
| 100,000 | |

What can you conclude? Try the same thing for negative values of $n$.

---

**Summary of Compound Interest Formulas**

Let $P$ be the amount deposited, $t$ the number of years, $A$ the balance, and $r$ the annual interest rate (in decimal form).

**1.** Compounded $n$ times per year: $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$

**2.** Compounded continuously: $A = Pe^{rt}$

The average interest rates paid by banks on savings accounts have varied greatly during the past 30 years. At times, savings accounts have earned as much as 12% annual interest and at times they have earned as little as 3%. The next example shows how the annual interest rate can affect the balance of an account.

### Example 3
**MAKE A DECISION** **Finding Account Balances** Ⓡ

You are creating a trust fund for your newborn nephew. You deposit $12,000 in an account, with instructions that the account be turned over to your nephew on his 25th birthday. Compare the balances in the account for each situation. Which account should you choose?

**a.** 7%, compounded continuously

**b.** 7%, compounded quarterly

**c.** 11%, compounded continuously

**d.** 11%, compounded quarterly

**SOLUTION**

**a.** $12{,}000e^{0.07(25)} \approx 69{,}055.23$                    7%, compounded continuously

**b.** $12{,}000\left(1 + \dfrac{0.07}{4}\right)^{4(25)} \approx 68{,}017.87$                    7%, compounded quarterly
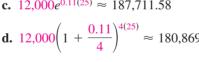
**c.** $12{,}000e^{0.11(25)} \approx 187{,}711.58$                    11%, compounded continuously

**d.** $12{,}000\left(1 + \dfrac{0.11}{4}\right)^{4(25)} \approx 180{,}869.07$                    11%, compounded quarterly

The growth of the account for parts (a) and (c) is shown in Figure 4.8. Notice the dramatic difference between the balances at 7% and 11%. You should choose the account described in part (c) because it earns more money than the other accounts.



**Account Balances**

$A = 12{,}000e^{0.11t}$

$A = 12{,}000e^{0.07t}$

(25, 187,711.58)

(25, 69,055.23)

Account balance (in dollars) / Time (in years)
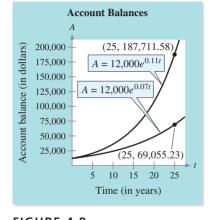
**FIGURE 4.8**

### ✓ CHECKPOINT 3

Find the balance in an account if $2000 is deposited for 10 years at an interest rate of 9%, compounded as follows. Compare the results and make a general statement about compounding.

**a.** quarterly          **b.** monthly

**c.** daily              **d.** continuously ■

In Example 3, note that the interest earned depends on the frequency with which the interest is compounded. The annual percentage rate is called the **stated rate** or **nominal rate.** However, the nominal rate does not reflect the actual rate at which interest is earned, which means that the compounding produced an **effective rate** that is larger than the nominal rate. In general, the effective rate corresponding to a nominal rate of $r$ that is compounded $n$ times per year is

$$\text{Effective rate} = r_{eff} = \left(1 + \frac{r}{n}\right)^{n} - 1.$$

*Example 4*   **Finding the Effective Rate of Interest**   Ⓡ

Find the effective rate of interest corresponding to a nominal rate of 6% per year compounded (a) annually, (b) quarterly, and (c) monthly.

**SOLUTION**

**a.** $r_{eff} = \left(1 + \dfrac{r}{n}\right)^n - 1$                Formula for effective rate of interest

$\quad = \left(1 + \dfrac{0.06}{1}\right)^1 - 1$                Substitute for $r$ and $n$.

$\quad = 1.06 - 1$                Simplify.

$\quad = 0.06$

So, the effective rate is 6% per year.

**b.** $r_{eff} = \left(1 + \dfrac{r}{n}\right)^n - 1$                Formula for effective rate of interest

$\quad = \left(1 + \dfrac{0.06}{4}\right)^4 - 1$                Substitute for $r$ and $n$.

$\quad = (1.015)^4 - 1$                Simplify.

$\quad \approx 0.0614$

So, the effective rate is about 6.14% per year.

**c.** $r_{eff} = \left(1 + \dfrac{r}{n}\right)^n - 1$                Formula for effective rate of interest

$\quad = \left(1 + \dfrac{0.06}{12}\right)^{12} - 1$                Substitute for $r$ and $n$.

$\quad = (1.005)^{12} - 1$                Simplify.

$\quad \approx 0.0617$

So, the effective rate is about 6.17% per year.

✓**CHECKPOINT 4**

Find the effective rate of interest corresponding to a nominal rate of 7% per year compounded (a) semiannually and (b) daily. ■

## Present Value

In planning for the future, this problem often arises: "How much money $P$ should be deposited now, at a fixed rate of interest $r$, in order to have a balance of $A$, $t$ years from now?" The answer to this question is given by the **present value** of $A$.

To find the present value of a future investment, use the formula for compound interest as shown.

$$A = P\left(1 + \dfrac{r}{n}\right)^{nt}$$                Formula for compound interest

Solving for $P$ gives a present value of

$$P = \frac{A}{\left(1 + \dfrac{r}{n}\right)^{nt}} \quad \text{or} \quad P = \frac{A}{(1 + i)^N}$$

where $i = r/n$ is the interest rate per compounding period and $N = nt$ is the total number of compounding periods. You will learn another way to find the present value of a future investment in Section 6.1.

**Example 5**    **Finding Present Value**    Ⓡ

An investor is purchasing a 12-year certificate of deposit that pays an annual percentage rate of 8%, compounded monthly. How much should the person invest in order to obtain a balance of $15,000 at maturity?

**SOLUTION**    Here, $A = 15{,}000$, $r = 0.08$, $n = 12$, and $t = 12$. Using the formula for present value, you obtain

$$P = \frac{15{,}000}{\left(1 + \dfrac{0.08}{12}\right)^{12(12)}} \qquad \text{Substitute for } A, r, n, \text{ and } t.$$

$$\approx 5761.72. \qquad\qquad\qquad \text{Simplify.}$$

So, the person should invest $5761.72 in the certificate of deposit.

✓ **CHECKPOINT 5**

How much money should be deposited in an account paying 6% interest compounded monthly in order to have a balance of $20,000 after 3 years? ■

┌─ **CONCEPT CHECK** ─┐

1. Can the number $e$ be written as the ratio of two integers? Explain.

2. When a quantity's growth is not restricted, which model is more often used: an exponential model or a logistic growth model?

3. When a quantity's growth is restricted, which model is more often used: an exponential model or a logistic growth model?

4. Write the formula for the balance $A$ in an account after $t$ years with principal $P$ and an annual interest rate $r$ compounded continuously.

**Skills Review 4.2**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.6 and 3.6.

In Exercises 1–4, discuss the continuity of the function.

**1.** $f(x) = \dfrac{3x^2 + 2x + 1}{x^2 + 1}$

**2.** $f(x) = \dfrac{x + 1}{x^2 - 4}$

**3.** $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 3}$

**4.** $g(x) = \dfrac{x^2 - 9x + 20}{x - 4}$

In Exercises 5–12, find the limit.

**5.** $\lim\limits_{x\to\infty} \dfrac{25}{1 + 4x}$

**6.** $\lim\limits_{x\to\infty} \dfrac{16x}{3 + x^2}$

**7.** $\lim\limits_{x\to\infty} \dfrac{8x^3 + 2}{2x^3 + x}$

**8.** $\lim\limits_{x\to\infty} \dfrac{x}{2x}$

**9.** $\lim\limits_{x\to\infty} \dfrac{3}{2 + (1/x)}$

**10.** $\lim\limits_{x\to\infty} \dfrac{6}{1 + x^{-2}}$

**11.** $\lim\limits_{x\to\infty} 2^{-x}$

**12.** $\lim\limits_{x\to\infty} \dfrac{7}{1 + 5x}$

## Exercises 4.2
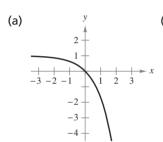
In Exercises 1–4, use the properties of exponents to simplify the expression.
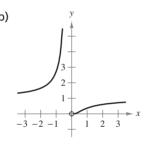
**1.** (a) $(e^3)(e^4)$      (b) $(e^3)^4$

  (c) $(e^3)^{-2}$      (d) $e^0$

**2.** (a) $\left(\dfrac{1}{e}\right)^{-2}$      (b) $\left(\dfrac{e^5}{e^2}\right)^{-1}$
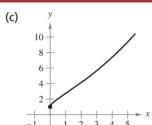
  (c) $\dfrac{e^5}{e^3}$      (d) $\dfrac{1}{e^{-3}}$
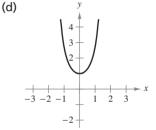
**3.** (a) $(e^2)^{5/2}$      (b) $(e^2)(e^{1/2})$

  (c) $(e^{-2})^{-3}$      (d) $\dfrac{e^5}{e^{-2}}$

**4.** (a) $(e^{-3})^{2/3}$      (b) $\dfrac{e^4}{e^{-1/2}}$

  (c) $(e^{-2})^{-4}$      (d) $(e^{-4})(e^{-3/2})$
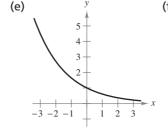
In Exercises 5–10, match the function with its graph. [The graphs are labeled (a)–(f).]

(a)



(b)



(c)



(d)



(e)



(f)
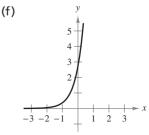


**5.** $f(x) = e^{2x+1}$      **6.** $f(x) = e^{-x/2}$

**7.** $f(x) = e^{x^2}$      **8.** $f(x) = e^{-1/x}$

**9.** $f(x) = e^{\sqrt{x}}$      **10.** $f(x) = -e^x + 1$

In Exercises 11–14, sketch the graph of the function.

**11.** $h(x) = e^{x-3}$      **12.** $f(x) = e^{2x}$

**13.** $g(x) = e^{1-x}$      **14.** $j(x) = e^{-x+2}$

(T) In Exercises 15–18, use a graphing utility to graph the function. Be sure to choose an appropriate viewing window.

**15.** $N(t) = 500e^{-0.2t}$

**16.** $A(t) = 500e^{0.15t}$

**17.** $g(x) = \dfrac{2}{1 + e^{x^2}}$

**18.** $g(x) = \dfrac{10}{1 + e^{-x}}$

(T) In Exercises 19–22, use a graphing utility to graph the function. Determine whether the function has any horizontal asymptotes and discuss the continuity of the function.

**19.** $f(x) = \dfrac{e^x + e^{-x}}{2}$

**20.** $f(x) = \dfrac{e^x - e^{-x}}{2}$

**21.** $f(x) = \dfrac{2}{1 + e^{1/x}}$

**22.** $f(x) = \dfrac{2}{1 + 2e^{-0.2x}}$

(T) **23.** Use a graphing utility to graph $f(x) = e^x$ and the given function in the same viewing window. How are the two graphs related?

(a) $g(x) = e^{x-2}$

(b) $h(x) = -\dfrac{1}{2}e^x$

(c) $q(x) = e^x + 3$

(T) **24.** Use a graphing utility to graph the function. Describe the shape of the graph for very large and very small values of $x$.

(a) $f(x) = \dfrac{8}{1 + e^{-0.5x}}$

(b) $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

(S) **Compound Interest**  In Exercises 25–28, use a spreadsheet to complete the table to determine the balance $A$ for $P$ dollars invested at rate $r$ for $t$ years, compounded $n$ times per year.

| $n$ | 1 | 2 | 4 | 12 | 365 | Continuous compounding |
|---|---|---|---|---|---|---|
| $A$ | | | | | | |

**25.** $P = \$1000$, $r = 3\%$, $t = 10$ years

**26.** $P = \$2500$, $r = 2.5\%$, $t = 20$ years

**27.** $P = \$1000$, $r = 4\%$, $t = 20$ years

**28.** $P = \$2500$, $r = 5\%$, $t = 40$ years

(S) **Compound Interest**  In Exercises 29–32, use a spreadsheet to complete the table to determine the amount of money $P$ that should be invested at rate $r$ to produce a final balance of $100,000 in $t$ years.

| $t$ | 1 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| $P$ | | | | | | |

**29.** $r = 4\%$, compounded continuously

**30.** $r = 3\%$, compounded continuously

**31.** $r = 5\%$, compounded monthly

**32.** $r = 6\%$, compounded daily

**33. Trust Fund**  On the day of a child's birth, a deposit of $20,000 is made in a trust fund that pays 8% interest, compounded continuously. Determine the balance in this account on the child's 21st birthday.

**34. Trust Fund**  A deposit of $10,000 is made in a trust fund that pays 7% interest, compounded continuously. It is specified that the balance will be given to the college from which the donor graduated after the money has earned interest for 50 years. How much will the college receive?

**35. Effective Rate**  Find the effective rate of interest corresponding to a nominal rate of 9% per year compounded (a) annually, (b) semiannually, (c) quarterly, and (d) monthly.

**36. Effective Rate**  Find the effective rate of interest corresponding to a nominal rate of 7.5% per year compounded (a) annually, (b) semiannually, (c) quarterly, and (d) monthly.

**37. Present Value**  How much should be deposited in an account paying 7.2% interest compounded monthly in order to have a balance of $15,503.77 three years from now?

**38. Present Value**  How much should be deposited in an account paying 7.8% interest compounded monthly in order to have a balance of $21,154.03 four years from now?

**39. Future Value**  Find the future value of an $8000 investment if the interest rate is 4.5% compounded monthly for 2 years.

**40. Future Value**  Find the future value of a $6500 investment if the interest rate is 6.25% compounded monthly for 3 years.

**41. Demand**  The demand function for a product is modeled by

$$p = 5000\left(1 - \dfrac{4}{4 + e^{-0.002x}}\right).$$

Find the price of the product if the quantity demanded is (a) $x = 100$ units and (b) $x = 500$ units. What is the limit of the price as $x$ increases without bound?

**42. Demand**  The demand function for a product is modeled by

$$p = 10,000\left(1 - \dfrac{3}{3 + e^{-0.001x}}\right).$$

Find the price of the product if the quantity demanded is (a) $x = 1000$ units and (b) $x = 1500$ units. What is the limit of the price as $x$ increases without bound?

**43. Probability** The average time between incoming calls at a switchboard is 3 minutes. If a call has just come in, the probability that the next call will come within the next $t$ minutes is $P(t) = 1 - e^{-t/3}$. Find the probability of each situation.

(a) A call comes in within $\frac{1}{2}$ minute.

(b) A call comes in within 2 minutes.

(c) A call comes in within 5 minutes.

**(S) 44. Consumer Awareness** An automobile gets 28 miles per gallon at speeds of up to and including 50 miles per hour. At speeds greater than 50 miles per hour, the number of miles per gallon drops at the rate of 12% for each 10 miles per hour. If $s$ is the speed (in miles per hour) and $y$ is the number of miles per gallon, then $y = 28e^{0.6 - 0.012s}$, $s > 50$. Use this information and a spreadsheet to complete the table. What can you conclude?

| Speed ($s$) | 50 | 55 | 60 | 65 | 70 |
|---|---|---|---|---|---|
| Miles per gallon ($y$) | | | | | |

**45. MAKE A DECISION: SALES** The sales $S$ (in millions of dollars) for Avon Products from 1998 through 2005 are shown in the table. *(Source: Avon Products Inc.)*

| $t$ | 8 | 9 | 10 | 11 |
|---|---|---|---|---|
| $S$ | 5212.7 | 5289.1 | 5673.7 | 5952.0 |

| $t$ | 12 | 13 | 14 | 15 |
|---|---|---|---|---|
| $S$ | 6170.6 | 6804.6 | 7656.2 | 8065.2 |

A model for this data is given by $S = 2962.6e^{0.0653t}$, where $t$ represents the year, with $t = 8$ corresponding to 1998.

(a) How well does the model fit the data?

(b) Find a linear model for the data. How well does the linear model fit the data? Which model, exponential or linear, is a better fit?

(c) Use the exponential growth model and the linear model from part (b) to predict when the sales will exceed 10 billion dollars.

**46. Population** The population $P$ (in thousands) of Las Vegas, Nevada from 1960 through 2005 can be modeled by $P = 68.4e^{0.0467t}$, where $t$ is the time in years, with $t = 0$ corresponding to 1960. *(Source: U.S. Census Bureau)*

(a) Find the populations in 1960, 1970, 1980, 1990, 2000, and 2005.

(b) Explain why the data do not fit a linear model.

(c) Use the model to estimate when the population will exceed 900,000.

**47. Biology** The population $y$ of a bacterial culture is modeled by the logistic growth function $y = 925/(1 + e^{-0.3t})$, where $t$ is the time in days.

**(T)** (a) Use a graphing utility to graph the model.

(b) Does the population have a limit as $t$ increases without bound? Explain your answer.

(c) How would the limit change if the model were $y = 1000/(1 + e^{-0.3t})$? Explain your answer. Draw some conclusions about this type of model.

**(T) 48. Biology: Cell Division** Suppose that you have a single **(B)** imaginary bacterium able to divide to form two new cells every 30 seconds. Make a table of values for the number of individuals in the population over 30-second intervals up to 5 minutes. Graph the points and use a graphing utility to fit an exponential model to the data. *(Source: Adapted from Levine/Miller, Biology: Discovering Life, Second Edition)*

**(T) 49. Learning Theory** In a learning theory project, the proportion $P$ of correct responses after $n$ trials can be modeled by

$$P = \frac{0.83}{1 + e^{-0.2n}}.$$

(a) Use a graphing utility to estimate the proportion of correct responses after 10 trials. Verify your result analytically.

(b) Use a graphing utility to estimate the number of trials required to have a proportion of correct responses of 0.75.

(c) Does the proportion of correct responses have a limit as $n$ increases without bound? Explain your answer.

**(T) 50. Learning Theory** In a typing class, the average number $N$ of words per minute typed after $t$ weeks of lessons can be modeled by

$$N = \frac{95}{1 + 8.5e^{-0.12t}}.$$

(a) Use a graphing utility to estimate the average number of words per minute typed after 10 weeks. Verify your result analytically.

(b) Use a graphing utility to estimate the number of weeks required to achieve an average of 70 words per minute.

(c) Does the number of words per minute have a limit as $t$ increases without bound? Explain your answer.

**51. MAKE A DECISION: CERTIFICATE OF DEPOSIT** You want to invest $5000 in a certificate of deposit for 12 months. You are given the options below. Which would you choose? Explain.

(a) $r = 5.25\%$, quarterly compounding

(b) $r = 5\%$, monthly compounding

(c) $r = 4.75\%$, continuous compounding

## Section 4.3

# Derivatives of Exponential Functions

■ Find the derivatives of natural exponential functions.
■ Use calculus to analyze the graphs of functions that involve the natural exponential function.
■ Explore the normal probability density function.

## Derivatives of Exponential Functions

In Section 4.2, it was stated that the most convenient base for exponential functions is the irrational number $e$. The convenience of this base stems primarily from the fact that the function $f(x) = e^x$ *is its own derivative.* You will see that this is not true of other exponential functions of the form $y = a^x$ where $a \neq e$. To verify that $f(x) = e^x$ is its own derivative, notice that the limit

$$\lim_{\Delta x \to 0} (1 + \Delta x)^{1/\Delta x} = e$$

implies that for small values of $\Delta x$, $e \approx (1 + \Delta x)^{1/\Delta x}$, or $e^{\Delta x} \approx 1 + \Delta x$. This approximation is used in the following derivation.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta x \to 0} \frac{e^{x + \Delta x} - e^x}{\Delta x} \qquad \text{Use } f(x) = e^x.$$

$$= \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x} \qquad \text{Factor numerator.}$$

$$= \lim_{\Delta x \to 0} \frac{e^x[(1 + \Delta x) - 1]}{\Delta x} \qquad \text{Substitute } 1 + \Delta x \text{ for } e^{\Delta x}.$$

$$= \lim_{\Delta x \to 0} \frac{e^x(\Delta x)}{\Delta x} \qquad \text{Divide out like factor.}$$

$$= \lim_{\Delta x \to 0} e^x \qquad \text{Simplify.}$$

$$= e^x \qquad \text{Evaluate limit.}$$

If $u$ is a function of $x$, you can apply the Chain Rule to obtain the derivative of $e^u$ with respect to $x$. Both formulas are summarized below.

**Derivative of the Natural Exponential Function**

Let $u$ be a differentiable function of $x$.

**1.** $\dfrac{d}{dx}[e^x] = e^x$ **2.** $\dfrac{d}{dx}[e^u] = e^u \dfrac{du}{dx}$

**TECHNOLOGY**

Ⓣ Let $f(x) = e^x$. Use a graphing utility to evaluate $f(x)$ and the numerical derivative of $f(x)$ at each $x$-value. Explain the results.

**a.** $x = -2$ **b.** $x = 0$ **c.** $x = 2$

*Example 1*   **Interpreting a Derivative**

Find the slopes of the tangent lines to

$$f(x) = e^x \qquad \text{Original function}$$

at the points $(0, 1)$ and $(1, e)$. What conclusion can you make?

**SOLUTION**   Because the derivative of $f$ is

$$f'(x) = e^x \qquad \text{Derivative}$$

it follows that the slope of the tangent line to the graph of $f$ is

$$f'(0) = e^0 = 1 \qquad \text{Slope at point } (0, 1)$$
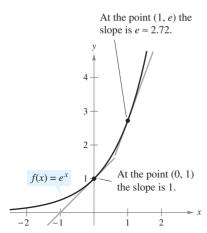
at the point $(0, 1)$ and

$$f'(1) = e^1 = e \qquad \text{Slope at point } (1, e)$$

at the point $(1, e)$, as shown in Figure 4.9. From this pattern, you can see that the slope of the tangent line to the graph of $f(x) = e^x$ at any point $(x, e^x)$ is equal to the $y$-coordinate of the point.

At the point $(1, e)$ the slope is $e \approx 2.72$.

At the point $(0, 1)$ the slope is 1.

$f(x) = e^x$

**FIGURE 4.9**

✓ **CHECKPOINT 1**

Find the equations of the tangent lines to $f(x) = e^x$ at the points $(0, 1)$ and $(1, e)$. ■

**STUDY TIP**

In Example 2, notice that when you differentiate an exponential function, the exponent does not change. For instance, the derivative of $y = e^{3x}$ is $y' = 3e^{3x}$. In both the function and its derivative, the exponent is $3x$.

*Example 2*   **Differentiating Exponential Functions**

Differentiate each function.

**a.** $f(x) = e^{2x}$        **b.** $f(x) = e^{-3x^2}$

**c.** $f(x) = 6e^{x^3}$       **d.** $f(x) = e^{-x}$

**SOLUTION**

**a.** Let $u = 2x$. Then $du/dx = 2$, and you can apply the Chain Rule.

$$f'(x) = e^u \frac{du}{dx} = e^{2x}(2) = 2e^{2x}$$

**b.** Let $u = -3x^2$. Then $du/dx = -6x$, and you can apply the Chain Rule.

$$f'(x) = e^u \frac{du}{dx} = e^{-3x^2}(-6x) = -6xe^{-3x^2}$$

**c.** Let $u = x^3$. Then $du/dx = 3x^2$, and you can apply the Chain Rule.

$$f'(x) = 6e^u \frac{du}{dx} = 6e^{x^3}(3x^2) = 18x^2 e^{x^3}$$

**d.** Let $u = -x$. Then $du/dx = -1$, and you can apply the Chain Rule.

$$f'(x) = e^u \frac{du}{dx} = e^{-x}(-1) = -e^{-x}$$

✓ **CHECKPOINT 2**

Differentiate each function.

**a.** $f(x) = e^{3x}$

**b.** $f(x) = e^{-2x^3}$

**c.** $f(x) = 4e^{x^2}$

**d.** $f(x) = e^{-2x}$ ■

The differentiation rules that you studied in Chapter 2 can be used with exponential functions, as shown in Example 3.

### Example 3 Differentiating Exponential Functions

Differentiate each function.

**a.** $f(x) = xe^x$       **b.** $f(x) = \dfrac{e^x - e^{-x}}{2}$

**c.** $f(x) = \dfrac{e^x}{x}$       **d.** $f(x) = xe^x - e^x$

**SOLUTION**

**a.** $f(x) = xe^x$       Write original function.

    $f'(x) = xe^x + e^x(1)$       Product Rule

        $= xe^x + e^x$       Simplify.

**b.** $f(x) = \dfrac{e^x - e^{-x}}{2}$       Write original function.

        $= \frac{1}{2}(e^x - e^{-x})$       Rewrite.

    $f'(x) = \frac{1}{2}(e^x + e^{-x})$       Constant Multiple Rule

**c.** $f(x) = \dfrac{e^x}{x}$       Write original function.

    $f'(x) = \dfrac{xe^x - e^x(1)}{x^2}$       Quotient Rule

        $= \dfrac{e^x(x-1)}{x^2}$       Simplify.

**d.** $f(x) = xe^x - e^x$       Write original function.

    $f'(x) = [xe^x + e^x(1)] - e^x$       Product and Difference Rules

        $= xe^x + e^x - e^x$

        $= xe^x$       Simplify.

### ✓ CHECKPOINT 3

Differentiate each function.

**a.** $f(x) = x^2 e^x$       **b.** $f(x) = \dfrac{e^x + e^{-x}}{2}$

**c.** $f(x) = \dfrac{e^x}{x^2}$       **d.** $f(x) = x^2 e^x - e^x$ ∎

### TECHNOLOGY

(T) If you have access to a symbolic differentiation utility, try using it to find the derivatives of the functions in Example 3.

## Applications

In Chapter 3, you learned how to use derivatives to analyze the graphs of functions. The next example applies those techniques to a function composed of exponential functions. In the example, notice that $e^a = e^b$ implies that $a = b$.

### Example 4 Analyzing a Catenary ®

When a telephone wire is hung between two poles, the wire forms a U-shaped curve called a **catenary.** For instance, the function

$$y = 30(e^{x/60} + e^{-x/60}), \quad -30 \le x \le 30$$

models the shape of a telephone wire strung between two poles that are 60 feet apart ($x$ and $y$ are measured in feet). Show that the lowest point on the wire is midway between the two poles. How much does the wire sag between the two poles?

Utility wires strung between poles have the shape of a catenary.

© Don Hammond/Design Pics/Corbis

**SOLUTION**    The derivative of the function is

$$y' = 30\left[e^{x/60}\left(\tfrac{1}{60}\right) + e^{-x/60}\left(-\tfrac{1}{60}\right)\right]$$
$$= \tfrac{1}{2}(e^{x/60} - e^{-x/60}).$$

To find the critical numbers, set the derivative equal to zero.

| | |
|---|---|
| $\tfrac{1}{2}(e^{x/60} - e^{-x/60}) = 0$ | Set derivative equal to 0. |
| $e^{x/60} - e^{-x/60} = 0$ | Multiply each side by 2. |
| $e^{x/60} = e^{-x/60}$ | Add $e^{-x/60}$ to each side. |
| $\dfrac{x}{60} = -\dfrac{x}{60}$ | If $e^a = e^b$, then $a = b$. |
| $x = -x$ | Multiply each side by 60. |
| $2x = 0$ | Add $x$ to each side. |
| $x = 0$ | Divide each side by 2. |
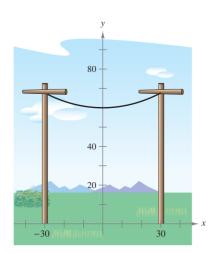
Using the First-Derivative Test, you can determine that the critical number $x = 0$ yields a relative minimum of the function. From the graph in Figure 4.10, you can see that this relative minimum is actually a minimum on the interval $[-30, 30]$. To find how much the wire sags between the two poles, you can compare its height at each pole with its height at the midpoint.

| | |
|---|---|
| $y = 30(e^{-30/60} + e^{-(-30)/60}) \approx 67.7$ feet | Height at left pole |
| $y = 30(e^{0/60} + e^{-(0)/60}) = 60$ feet | Height at midpoint |
| $y = 30(e^{30/60} + e^{-(30)/60}) \approx 67.7$ feet | Height at right pole |

From this, you can see that the wire sags about 7.7 feet.

**FIGURE 4.10**

### ✓ CHECKPOINT 4

Use a graphing utility to graph the function in Example 4. Verify the minimum value. Use the information in the example to choose an appropriate viewing window. ■

*Example 5*    **Finding a Maximum Revenue**    Ⓡ Ⓣ

The demand function for a product is modeled by

$$p = 56e^{-0.000012x}$$    Demand function

where $p$ is the price per unit (in dollars) and $x$ is the number of units. What price will yield a maximum revenue?

**SOLUTION**    The revenue function is
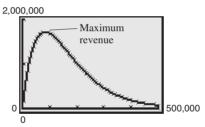
$$R = xp = 56xe^{-0.000012x}.$$    Revenue function

To find the maximum revenue *analytically*, you would set the marginal revenue, $dR/dx$, equal to zero and solve for $x$. In this problem, it is easier to use a *graphical* approach. After experimenting to find a reasonable viewing window, you can obtain a graph of $R$ that is similar to that shown in Figure 4.11. Using the *zoom* and *trace* features, you can conclude that the maximum revenue occurs when $x$ is about 83,300 units. To find the price that corresponds to this production level, substitute $x \approx 83,300$ into the demand function.

$$p \approx 56e^{-0.000012(83,300)} \approx \$20.61.$$

So, a price of about \$20.61 will yield a maximum revenue.



**FIGURE 4.11**    Use the *zoom* and *trace* features to approximate the *x*-value that corresponds to the maximum revenue.

✓**CHECKPOINT 5**

The demand function for a product is modeled by

$$p = 50e^{-0.0000125x}$$

where $p$ is the price per unit in dollars and $x$ is the number of units. What price will yield a maximum revenue? ∎

**STUDY TIP**

Try solving the problem in Example 5 analytically. When you do this, you obtain

$$\frac{dR}{dx} = 56xe^{-0.000012x}(-0.000012) + e^{-0.000012x}(56) = 0.$$
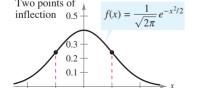
Explain how you would solve this equation. What is the solution?

## The Normal Probability Density Function

If you take a course in statistics or quantitative business analysis, you will spend quite a bit of time studying the characteristics and use of the **normal probability density function** given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$$

where $\sigma$ is the lowercase Greek letter sigma, and $\mu$ is the lowercase Greek letter mu. In this formula, $\sigma$ represents the *standard deviation* of the probability distribution, and $\mu$ represents the *mean* of the probability distribution.

Two points of inflection   $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$

**FIGURE 4.12**   The graph of the normal probability density function is bell-shaped.

### Example 6   Exploring a Probability Density Function

Show that the graph of the normal probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad \text{Original function}$$

has points of inflection at $x = \pm 1$.

**SOLUTION**   Begin by finding the second derivative of the function.

$$f'(x) = \frac{1}{\sqrt{2\pi}}(-x)e^{-x^2/2} \qquad \text{First derivative}$$

$$f''(x) = \frac{1}{\sqrt{2\pi}}[(-x)(-x)e^{-x^2/2} + (-1)e^{-x^2/2}] \qquad \text{Second derivative}$$

$$= \frac{1}{\sqrt{2\pi}}(e^{-x^2/2})(x^2 - 1) \qquad \text{Simplify.}$$

By setting the second derivative equal to 0, you can determine that $x = \pm 1$. By testing the concavity of the graph, you can then conclude that these $x$-values yield points of inflection, as shown in Figure 4.12.

✓ **CHECKPOINT 6**

Graph the normal probability density function

$$f(x) = \frac{1}{4\sqrt{2\pi}}e^{-x^2/32}$$

and approximate the points of inflection. ■

**CONCEPT CHECK**

1. What is the derivative of $f(x) = e^x$?
2. What is the derivative of $f(x) = e^u$? (Assume that $u$ is a differentiable function of $x$.)
3. If $e^a = e^b$, then $a$ is equal to what?
4. In the normal probability density function given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$$

identify what is represented by (a) $\sigma$ and (b) $\mu$.

**Skills Review 4.3**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4, 2.4, and 3.2.

In Exercises 1–4, factor the expression.

**1.** $x^2 e^x - \frac{1}{2} e^x$

**2.** $(xe^{-x})^{-1} + e^x$

**3.** $xe^x - e^{2x}$

**4.** $e^x - xe^{-x}$

In Exercises 5–8, find the derivative of the function.

**5.** $f(x) = \dfrac{3}{7x^2}$

**6.** $g(x) = 3x^2 - \dfrac{x}{6}$

**7.** $f(x) = (4x - 3)(x^2 + 9)$

**8.** $f(t) = \dfrac{t - 2}{\sqrt{t}}$

In Exercises 9 and 10, find the relative extrema of the function.

**9.** $f(x) = \frac{1}{8}x^3 - 2x$

**10.** $f(x) = x^4 - 2x^2 + 5$
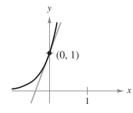
# Exercises 4.3
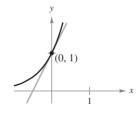
In Exercises 1–4, find the slope of the tangent line to the exponential function at the point (0, 1).

**1.** $y = e^{3x}$



(0, 1)

**2.** $y = e^{2x}$



(0, 1)

**3.** $y = e^{-x}$



(0, 1)

**4.** $y = e^{-2x}$



(0, 1)
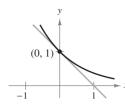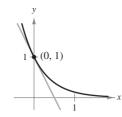
In Exercises 5–16, find the derivative of the function.

**5.** $y = e^{5x}$

**6.** $y = e^{1-x}$

**7.** $y = e^{-x^2}$

**8.** $f(x) = e^{1/x}$

**9.** $f(x) = e^{-1/x^2}$

**10.** $g(x) = e^{\sqrt{x}}$

**11.** $f(x) = (x^2 + 1)e^{4x}$

**12.** $y = 4x^3 e^{-x}$

**13.** $f(x) = \dfrac{2}{(e^x + e^{-x})^3}$

**14.** $f(x) = \dfrac{(e^x + e^{-x})^4}{2}$

**15.** $y = xe^x - 4e^{-x}$

**16.** $y = x^2 e^x - 2xe^x + 2e^x$

In Exercises 17–22, determine an equation of the tangent line to the function at the given point.

**17.** $y = e^{-2x+x^2}$, $(2, 1)$

**18.** $g(x) = e^{x^3}$, $\left(-1, \dfrac{1}{e}\right)$

**19.** $y = x^2 e^{-x}$, $\left(2, \dfrac{4}{e^2}\right)$

**20.** $y = \dfrac{x}{e^{2x}}$, $\left(1, \dfrac{1}{e^2}\right)$

**21.** $y = (e^{2x} + 1)^3$, $(0, 8)$

**22.** $y = (e^{4x} - 2)^2$, $(0, 1)$

In Exercises 23–26, find $dy/dx$ implicitly.

**23.** $xe^y - 10x + 3y = 0$

**24.** $x^2 y - e^y - 4 = 0$

**25.** $x^2 e^{-x} + 2y^2 - xy = 0$

**26.** $e^{xy} + x^2 - y^2 = 10$

In Exercises 27–30, find the second derivative.

**27.** $f(x) = 2e^{3x} + 3e^{-2x}$

**28.** $f(x) = (1 + 2x)e^{4x}$

**29.** $f(x) = 5e^{-x} - 2e^{-5x}$

**30.** $f(x) = (3 + 2x)e^{-3x}$

In Exercises 31–34, graph and analyze the function. Include extrema, points of inflection, and asymptotes in your analysis.

**31.** $f(x) = \dfrac{1}{2 - e^{-x}}$

**32.** $f(x) = \dfrac{e^x - e^{-x}}{2}$

**33.** $f(x) = x^2 e^{-x}$

**34.** $f(x) = xe^{-x}$

Ⓣ In Exercises 35 and 36, use a graphing utility to graph the function. Determine any asymptotes of the graph.

**35.** $f(x) = \dfrac{8}{1 + e^{-0.5x}}$

**36.** $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

In Exercises 37–40, solve the equation for *x*.

**37.** $e^{-3x} = e$

**38.** $e^x = 1$

**39.** $e^{\sqrt{x}} = e^3$

**40.** $e^{-1/x} = e^{1/2}$

**Depreciation** In Exercises 41 and 42, the value *V* (in dollars) of an item is a function of the time *t* (in years).

(a) Sketch the function over the interval [0, 10]. Use a graphing utility to verify your graph.

(b) Find the rate of change of *V* when *t* = 1.

(c) Find the rate of change of *V* when *t* = 5.

(d) Use the values (0, *V*(0)) and (10, *V*(10)) to find the linear depreciation model for the item.

(e) Compare the exponential function and the model from part (d). What are the advantages of each?

**41.** $V = 15,000e^{-0.6286t}$

**42.** $V = 500,000e^{-0.2231t}$

**43. Learning Theory** The average typing speed *N* (in words per minute) after *t* weeks of lessons is modeled by

$$N = \frac{95}{1 + 8.5e^{-0.12t}}.$$

Find the rates at which the typing speed is changing when (a) *t* = 5 weeks, (b) *t* = 10 weeks, and (c) *t* = 30 weeks.

**44. Compound Interest** The balance *A* (in dollars) in a savings account is given by $A = 5000e^{0.08t}$, where *t* is measured in years. Find the rates at which the balance is changing when (a) *t* = 1 year, (b) *t* = 10 years, and (c) *t* = 50 years.

**45. Ebbinghaus Model** The *Ebbinghaus Model* for human memory is $p = (100 - a)e^{-bt} + a$, where *p* is the percent retained after *t* weeks. (The constants *a* and *b* vary from one person to another.) If *a* = 20 and *b* = 0.5, at what rate is information being retained after 1 week? After 3 weeks?

**46. Agriculture** The yield *V* (in pounds per acre) for an orchard at age *t* (in years) is modeled by

$$V = 7955.6e^{-0.0458/t}.$$

At what rate is the yield changing when (a) *t* = 5 years, (b) *t* = 10 years, and (c) *t* = 25 years?

(T) **47. Employment** From 1996 through 2005, the numbers *y* (in millions) of employed people in the United States can be modeled by

$$y = 98.020 + 6.2472t - 0.24964t^2 + 0.000002e^t$$

where *t* represents the year, with *t* = 6 corresponding to 1996. *(Source: U.S. Bureau of Labor Statistics)*

(a) Use a graphing utility to graph the model.

(b) Use the graph to estimate the rates of change in the number of employed people in 1996, 2000, and 2005.

(c) Confirm the results of part (b) analytically.

(T) **48. Cell Sites** A cell site is a site where electronic communications equipment is placed in a cellular network for the use of mobile phones. From 1985 through 2006, the numbers *y* of cell sites can be modeled by

$$y = \frac{222,827}{1 + 2677e^{-0.377t}}$$

where *t* represents the year, with *t* = 5 corresponding to 1985. *(Source: Cellular Telecommunications & Internet Association)*

(a) Use a graphing utility to graph the model.

(b) Use the graph to estimate when the rate of change in the number of cell cites began to decrease.

(c) Confirm the result of part (b) analytically.

**49. Probability** A survey of high school seniors from a certain school district who took the SAT has determined that the mean score on the mathematics portion was 650 with a standard deviation of 12.5.

(a) Assuming the data can be modeled by a normal probability density function, find a model for these data.

(T) (b) Use a graphing utility to graph the model. Be sure to choose an appropriate viewing window.

(c) Find the derivative of the model.

(d) Show that $f' > 0$ for $x < \mu$ and $f' < 0$ for $x > \mu$.

**50. Probability** A survey of a college freshman class has determined that the mean height of females in the class is 64 inches with a standard deviation of 3.2 inches.

(a) Assuming the data can be modeled by a normal probability density function, find a model for these data.

(T) (b) Use a graphing utility to graph the model. Be sure to choose an appropriate viewing window.

(c) Find the derivative of the model.

(d) Show that $f' > 0$ for $x < \mu$ and $f' < 0$ for $x > \mu$.

(T) **51.** Use a graphing utility to graph the normal probability density function with $\mu = 0$ and $\sigma = 2, 3,$ and 4 in the same viewing window. What effect does the standard deviation $\sigma$ have on the function? Explain your reasoning.

(T) **52.** Use a graphing utility to graph the normal probability density function with $\sigma = 1$ and $\mu = -2, 1,$ and 3 in the same viewing window. What effect does the mean $\mu$ have on the function? Explain your reasoning.

(T) **53.** Use Example 6 as a model to show that the graph of the normal probability density function with $\mu = 0$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$$

has points of inflection at $x = \pm\sigma$. What is the maximum value of the function? Use a graphing utility to verify your answer by graphing the function for several values of $\sigma$.

## Mid-Chapter Quiz

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, evaluate each expression.

**1.** $4(4^2)$

**2.** $\left(\dfrac{2}{3}\right)^3$

**3.** $81^{1/3}$

**4.** $\left(\dfrac{4}{9}\right)^2$

In Exercises 5–12, use properties of exponents to simplify the expression.

**5.** $4^3(4^2)$

**6.** $\left(\dfrac{1}{6}\right)^{-3}$

**7.** $\dfrac{3^8}{3^5}$

**8.** $(5^{1/2})(3^{1/2})$

**9.** $(e^2)(e^5)$

**10.** $(e^{2/3})(e^3)$

**11.** $\dfrac{e^2}{e^{-4}}$

**12.** $(e^{-1})^{-3}$

(T) In Exercises 13–18, use a graphing utility to graph the function.

**13.** $f(x) = 3^x - 2$

**14.** $f(x) = 5^{-x} + 2$

**15.** $f(x) = 6^{x-3}$

**16.** $f(x) = e^{x+2}$

**17.** $f(x) = 250e^{0.15x}$

**18.** $f(x) = \dfrac{5}{1 + e^x}$

**19.** Suppose that the annual rate of inflation averages 4.5% over the next 10 years. With this rate of inflation, the approximate cost $C$ of goods or services during any year in that decade will be given by

$$C(t) = P(1.045)^t, \quad 0 \le t \le 10$$

where $t$ is time in years and $P$ is the present cost. If the price of a baseball game ticket is presently \$14.95, estimate the price 10 years from now.

**20.** For $P = \$3000$, $r = 3.5\%$, and $t = 5$ years, find the balance in an account if interest is compounded (a) monthly and (b) continuously.

In Exercises 21–24, find the derivative of the function.

**21.** $y = e^{5x}$

**22.** $y = e^{x-4}$

**23.** $y = 5e^{x+2}$

**24.** $y = 3e^x - xe^x$

**25.** Determine an equation of the tangent line to $y = e^{-2x}$ at the point $(0, 1)$.

**26.** Graph and analyze the function $f(x) = 0.5x^2e^{-0.5x}$. Include extrema, points of inflection, and asymptotes in your analysis.

## Section 4.4

# Logarithmic Functions

- Sketch the graphs of natural logarithmic functions.
- Use properties of logarithms to simplify, expand, and condense logarithmic expressions.
- Use inverse properties of exponential and logarithmic functions to solve exponential and logarithmic equations.
- Use properties of natural logarithms to answer questions about real-life situations.

## The Natural Logarithmic Function

From your previous algebra courses, you should be somewhat familiar with logarithms. For instance, the **common logarithm** $\log_{10} x$ is defined as

$$\log_{10} x = b \quad \text{if and only if} \quad 10^b = x.$$

The base of common logarithms is 10. In calculus, the most useful base for logarithms is the number $e$.

---

**Definition of the Natural Logarithmic Function**

The **natural logarithmic function,** denoted by $\ln x$, is defined as

$$\ln x = b \quad \text{if and only if} \quad e^b = x.$$

$\ln x$ is read as "el en of $x$" or as "the natural log of $x$."

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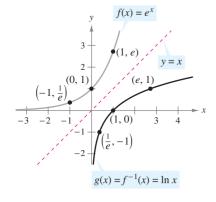
This definition implies that the natural logarithmic function and the natural exponential function are inverse functions. So, every logarithmic equation can be written in an equivalent exponential form and every exponential equation can be written in logarithmic form. Here are some examples.

| *Logarithmic form:* | *Exponential form:* |
|---|---|
| $\ln 1 = 0$ | $e^0 = 1$ |
| $\ln e = 1$ | $e^1 = e$ |
| $\ln \dfrac{1}{e} = -1$ | $e^{-1} = \dfrac{1}{e}$ |
| $\ln 2 \approx 0.693$ | $e^{0.693} \approx 2$ |

Because the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions, their graphs are reflections of each other in the line $y = x$. This reflective property is illustrated in Figure 4.13. The figure also contains a summary of several properties of the graph of the natural logarithmic function.

Notice that the domain of the natural logarithmic function is the set of *positive real numbers*—be sure you see that $\ln x$ is not defined for zero or for negative numbers. You can test this on your calculator. If you try evaluating $\ln(-1)$ or $\ln 0$, your calculator should indicate that the value is not a real number.



$g(x) = \ln x$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Intercept: $(1, 0)$
- Always increasing
- $\ln x \to \infty$ as $x \to \infty$
- $\ln x \to -\infty$ as $x \to 0^+$
- Continuous
- One-to-one

**FIGURE 4.13**

*Example 1* **Graphing Logarithmic Functions**

Sketch the graph of each function.

**a.** $f(x) = \ln(x + 1)$     **b.** $f(x) = 2 \ln(x - 2)$

**SOLUTION**

**a.** Because the natural logarithmic function is defined only for positive values, the domain of the function is $x + 1 > 0$, or

$$x > -1. \qquad \text{Domain}$$

To sketch the graph, begin by constructing a table of values, as shown below. Then plot the points in the table and connect them with a smooth curve, as shown in Figure 4.14(a).

| $x$ | $-0.5$ | $0$ | $0.5$ | $1$ | $1.5$ | $2$ |
|---|---|---|---|---|---|---|
| $\ln(x + 1)$ | $-0.693$ | $0$ | $0.405$ | $0.693$ | $0.916$ | $1.099$ |

**b.** The domain of this function is $x - 2 > 0$, or

$$x > 2. \qquad \text{Domain}$$

A table of values for the function is shown below, and its graph is shown in Figure 4.14(b).

| $x$ | $2.5$ | $3$ | $3.5$ | $4$ | $4.5$ | $5$ |
|---|---|---|---|---|---|---|
| $2 \ln(x - 2)$ | $-1.386$ | $0$ | $0.811$ | $1.386$ | $1.833$ | $2.197$ |

**✓CHECKPOINT 1**
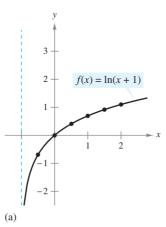
Use a graphing utility to complete the table and graph the function.

$$f(x) = \ln(x + 2)$$

| $x$ | $-1.5$ | $-1$ | $-0.5$ |
|---|---|---|---|
| $f(x)$ | | | |

| $x$ | $0$ | $0.5$ | $1$ |
|---|---|---|---|
| $f(x)$ | | | |



(a)



(b)
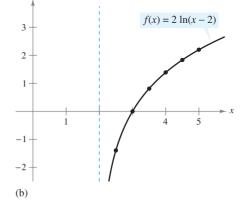
**FIGURE 4.14**

**STUDY TIP**

How does the graph of $f(x) = \ln(x + 1)$ relate to the graph of $y = \ln x$? The graph of $f$ is a translation of the graph of $y = \ln x$ one unit to the left.

## Properties of Logarithmic Functions

Recall from Section 1.4 that inverse functions have the property that

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

The properties listed below follow from the fact that the natural logarithmic function and the natural exponential function are inverse functions.

---

**Inverse Properties of Logarithms and Exponents**

**1.** $\ln e^x = x$          **2.** $e^{\ln x} = x$

---

*Example 2*  **Applying Inverse Properties**

Simplify each expression.

**a.** $\ln e^{\sqrt{2}}$      **b.** $e^{\ln 3x}$

**SOLUTION**

**a.** Because $\ln e^x = x$, it follows that

$$\ln e^{\sqrt{2}} = \sqrt{2}.$$

**b.** Because $e^{\ln x} = x$, it follows that

$$e^{\ln 3x} = 3x.$$

✓ **CHECKPOINT 2**

Simplify each expression.

**a.** $\ln e^3$      **b.** $e^{\ln(x+1)}$ ■

Most of the properties of exponential functions can be rewritten in terms of logarithmic functions. For instance, the property

$$e^x e^y = e^{x+y}$$

states that you can multiply two exponential expressions by adding their exponents. In terms of logarithms, this property becomes

$$\ln xy = \ln x + \ln y.$$

This property and two other properties of logarithms are summarized below.

**STUDY TIP**

There is no general property that can be used to rewrite $\ln(x + y)$. Specifically, $\ln(x + y)$ is not equal to $\ln x + \ln y$.

---

**Properties of Logarithms**

**1.** $\ln xy = \ln x + \ln y$          **2.** $\ln \dfrac{x}{y} = \ln x - \ln y$

**3.** $\ln x^n = n \ln x$

---

Rewriting a logarithm of a single quantity as the sum, difference, or multiple of logarithms is called *expanding* the logarithmic expression. The reverse procedure is called *condensing* a logarithmic expression.

**Example 3**  **Expanding Logarithmic Expressions**

Use the properties of logarithms to rewrite each expression as a sum, difference, or multiple of logarithms. (Assume $x > 0$ and $y > 0$.)

**a.** $\ln \dfrac{10}{9}$    **b.** $\ln \sqrt{x^2 + 1}$    **c.** $\ln \dfrac{xy}{5}$    **d.** $\ln[x^2(x + 1)]$

**SOLUTION**

**a.** $\ln \frac{10}{9} = \ln 10 - \ln 9$                         Property 2

**b.** $\ln \sqrt{x^2 + 1} = \ln(x^2 + 1)^{1/2}$                  Rewrite with rational exponent.

$\qquad\qquad = \frac{1}{2} \ln(x^2 + 1)$                          Property 3

**c.** $\ln \frac{xy}{5} = \ln(xy) - \ln 5$                       Property 2

$\qquad\quad = \ln x + \ln y - \ln 5$                            Property 1

**d.** $\ln[x^2(x + 1)] = \ln x^2 + \ln(x + 1)$                   Property 1

$\qquad\qquad\quad = 2 \ln x + \ln(x + 1)$                        Property 3

✓**CHECKPOINT 3**

Use the properties of logarithms to rewrite each expression as a sum, difference, or multiple of logarithms. (Assume $x > 0$ and $y > 0$.)

**a.** $\ln \dfrac{2}{5}$    **b.** $\ln \sqrt[3]{x + 2}$    **c.** $\ln \dfrac{x}{5y}$    **d.** $\ln x(x + 1)^2$ ■

**Example 4**  **Condensing Logarithmic Expressions**

Use the properties of logarithms to rewrite each expression as the logarithm of a single quantity. (Assume $x > 0$ and $y > 0$.)

**a.** $\ln x + 2 \ln y$

**b.** $2 \ln(x + 2) - 3 \ln x$

**SOLUTION**

**a.** $\ln x + 2 \ln y = \ln x + \ln y^2$                        Property 3

$\qquad\qquad\quad = \ln xy^2$                                    Property 1

**b.** $2 \ln(x + 2) - 3 \ln x = \ln(x + 2)^2 - \ln x^3$          Property 3

$\qquad\qquad\qquad\qquad = \ln \dfrac{(x + 2)^2}{x^3}$           Property 2

## Solving Exponential and Logarithmic Equations

The inverse properties of logarithms and exponents can be used to solve exponential and logarithmic equations, as shown in the next two examples.

**STUDY TIP**

In the examples on this page, note that the key step in solving an exponential equation is to take the log of each side, and the key step in solving a logarithmic equation is to exponentiate each side.

*Example 5*  **Solving Exponential Equations**

Solve each equation.

**a.** $e^x = 5$    **b.** $10 + e^{0.1t} = 14$

**SOLUTION**

**a.**   $e^x = 5$                   Write original equation.

   $\ln e^x = \ln 5$           Take natural log of each side.

   $x = \ln 5$             Inverse property: $\ln e^x = x$

**b.** $10 + e^{0.1t} = 14$         Write original equation.

   $e^{0.1t} = 4$           Subtract 10 from each side.

   $\ln e^{0.1t} = \ln 4$         Take natural log of each side.

   $0.1t = \ln 4$         Inverse property: $\ln e^{0.1t} = 0.1t$

   $t = 10 \ln 4$       Multiply each side by 10.

✓**CHECKPOINT 5**

Solve each equation.

**a.** $e^x = 6$       **b.** $5 + e^{0.2t} = 10$ ■

*Example 6*  **Solving Logarithmic Equations**

Solve each equation.

**a.** $\ln x = 5$    **b.** $3 + 2 \ln x = 7$

**SOLUTION**

**a.** $\ln x = 5$                   Write original equation.

   $e^{\ln x} = e^5$             Exponentiate each side.

   $x = e^5$             Inverse property: $e^{\ln x} = x$

**b.** $3 + 2 \ln x = 7$           Write original equation.

   $2 \ln x = 4$           Subtract 3 from each side.

   $\ln x = 2$           Divide each side by 2.

   $e^{\ln x} = e^2$           Exponentiate each side.

   $x = e^2$           Inverse property: $e^{\ln x} = x$

✓**CHECKPOINT 6**

Solve each equation.

**a.** $\ln x = 4$       **b.** $4 + 5 \ln x = 19$ ■

*Example 7* **Finding Doubling Time** Ⓡ

You deposit $P$ dollars in an account whose annual interest rate is $r$, compounded continuously. How long will it take for your balance to double?

**SOLUTION** The balance in the account after $t$ years is

$$A = Pe^{rt}.$$

So, the balance will have doubled when $Pe^{rt} = 2P$. To find the "doubling time," solve this equation for $t$.

| | |
|---|---|
| $Pe^{rt} = 2P$ | Balance in account has doubled. |
| $e^{rt} = 2$ | Divide each side by $P$. |
| $\ln e^{rt} = \ln 2$ | Take natural log of each side. |
| $rt = \ln 2$ | Inverse property: $\ln e^{rt} = rt$ |
| $t = \dfrac{1}{r}\ln 2$ | Divide each side by $r$. |

From this result, you can see that the time it takes for the balance to double is inversely proportional to the interest rate $r$. The table shows the doubling times for several interest rates. Notice that the doubling time decreases as the rate increases. The relationship between doubling time and the interest rate is shown graphically in Figure 4.15.

**Doubling Account Balances**



$t = \dfrac{1}{r}\ln 2$

Doubling time (in years)

Interest rate

**FIGURE 4.15**

| $r$ | 3% | 4% | 5% | 6% | 7% | 8% | 9% | 10% | 11% | 12% |
|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | 23.1 | 17.3 | 13.9 | 11.6 | 9.9 | 8.7 | 7.7 | 6.9 | 6.3 | 5.8 |
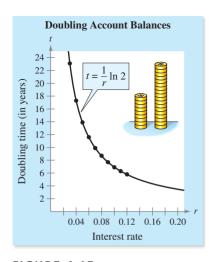
✓ **CHECKPOINT 7**

Use the equation found in Example 7 to determine the amount of time it would take for your balance to double at an interest rate of 8.75%. ■

**CONCEPT CHECK**

1. What are common logarithms and natural logarithms?
2. Write "logarithm of $x$ with base 3" symbolically.
3. What are the domain and range of $f(x) = \ln x$?
4. Explain the relationship between the functions $f(x) = \ln x$ and $g(x) = e^x$.

---

**Skills Review 4.4**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.1, 0.3, and 4.2.

In Exercises 1–8, use the properties of exponents to simplify the expression.

**1.** $(4^2)(4^{-3})$

**2.** $(2^3)^2$

**3.** $\dfrac{3^4}{3^{-2}}$

**4.** $\left(\dfrac{3}{2}\right)^{-3}$

**5.** $e^0$

**6.** $(3e)^4$

**7.** $\left(\dfrac{2}{e^3}\right)^{-1}$

**8.** $\left(\dfrac{4e^2}{25}\right)^{-3/2}$

In Exercises 9–12, solve for $x$.

**9.** $0 < x + 4$

**10.** $0 < x^2 + 1$

**11.** $0 < \sqrt{x^2 - 1}$

**12.** $0 < x - 5$

In Exercises 13 and 14, find the balance in the account after 10 years.

**13.** $P = \$1900,\ r = 6\%$, compounded continuously

**14.** $P = \$2500,\ r = 3\%$, compounded continuously

---

## Exercises 4.4

In Exercises 1–8, write the logarithmic equation as an exponential equation, or vice versa.

**1.** $\ln 2 = 0.6931 \ldots$

**2.** $\ln 9 = 2.1972 \ldots$

**3.** $\ln 0.2 = -1.6094 \ldots$

**4.** $\ln 0.05 = -2.9957 \ldots$

**5.** $e^0 = 1$

**6.** $e^2 = 7.3891 \ldots$

**7.** $e^{-3} = 0.0498 \ldots$

**8.** $e^{0.25} = 1.2840 \ldots$

In Exercises 9–12, match the function with its graph. [The graphs are labeled (a)–(d).]

(a)


(b)


(c)


(d)


**9.** $f(x) = 2 + \ln x$

**10.** $f(x) = -\ln x$

**11.** $f(x) = \ln(x + 2)$

**12.** $f(x) = -\ln(x - 1)$

In Exercises 13–18, sketch the graph of the function.

**13.** $y = \ln(x - 1)$
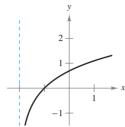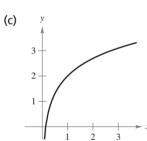
**14.** $y = \ln|x|$

**15.** $y = \ln 2x$

**16.** $y = 5 + \ln x$

**17.** $y = 3 \ln x$

**18.** $y = \frac{1}{4} \ln x$

Ⓣ In Exercises 19–22, analytically show that the functions are inverse functions. Then use a graphing utility to show this graphically.
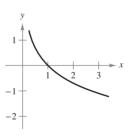
**19.** $f(x) = e^{2x}$
$g(x) = \ln \sqrt{x}$

**20.** $f(x) = e^x - 1$
$g(x) = \ln(x + 1)$

**21.** $f(x) = e^{2x-1}$
$g(x) = \frac{1}{2} + \ln \sqrt{x}$

**22.** $f(x) = e^{x/3}$
$g(x) = \ln x^3$

In Exercises 23–28, apply the inverse properties of logarithmic and exponential functions to simplify the expression.

**23.** $\ln e^{x^2}$

**24.** $\ln e^{2x-1}$

**25.** $e^{\ln(5x+2)}$

**26.** $e^{\ln \sqrt{x}}$

**27.** $-1 + \ln e^{2x}$

**28.** $-8 + e^{\ln x^3}$

In Exercises 29 and 30, use the properties of logarithms and the fact that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$ to approximate the logarithm. Then use a calculator to confirm your approximation.

**29.** (a) $\ln 6$    (b) $\ln \frac{3}{2}$    (c) $\ln 81$    (d) $\ln \sqrt{3}$

**30.** (a) $\ln 0.25$    (b) $\ln 24$    (c) $\ln \sqrt[3]{12}$    (d) $\ln \frac{1}{72}$

In Exercises 31–40, use the properties of logarithms to write the expression as a sum, difference, or multiple of logarithms.

**31.** $\ln \frac{2}{3}$

**32.** $\ln \frac{1}{5}$

**33.** $\ln xyz$

**34.** $\ln \frac{xy}{z}$

**35.** $\ln \sqrt{x^2 + 1}$

**36.** $\ln \sqrt{\dfrac{x^3}{x + 1}}$

**37.** $\ln [z(z - 1)^2]$

**38.** $\ln\left(x \sqrt[3]{x^2 + 1}\right)$

**39.** $\ln \dfrac{3x(x + 1)}{(2x + 1)^2}$

**40.** $\ln \dfrac{2x}{\sqrt{x^2 - 1}}$

In Exercises 41–50, write the expression as the logarithm of a single quantity.

**41.** $\ln(x - 2) - \ln(x + 2)$

**42.** $\ln(2x + 1) + \ln(2x - 1)$

**43.** $3 \ln x + 2 \ln y - 4 \ln z$

**44.** $2 \ln 3 - \frac{1}{2} \ln(x^2 + 1)$

**45.** $3[\ln x + \ln(x + 3) - \ln(x + 4)]$

**46.** $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$

**47.** $\frac{3}{2}[\ln x(x^2 + 1) - \ln(x + 1)]$

**48.** $2\left[\ln x + \frac{1}{4} \ln(x + 1)\right]$

**49.** $\frac{1}{3} \ln(x + 1) - \frac{2}{3} \ln(x - 1)$

**50.** $\frac{1}{2} \ln(x - 2) + \frac{3}{2} \ln(x + 2)$

In Exercises 51–74, solve for x or t.

**51.** $e^{\ln x} = 4$

**52.** $e^{\ln x^2} - 9 = 0$

**53.** $\ln x = 0$

**54.** $2 \ln x = 4$

**55.** $\ln 2x = 2.4$

**56.** $\ln 4x = 1$

**57.** $3 \ln 5x = 10$

**58.** $2 \ln 4x = 7$

**59.** $e^{x+1} = 4$

**60.** $e^{-0.5x} = 0.075$

**61.** $300e^{-0.2t} = 700$

**62.** $400e^{-0.0174t} = 1000$

**63.** $4e^{2x-1} - 1 = 5$

**64.** $2e^{-x+1} - 5 = 9$

**65.** $\dfrac{10}{1 + 4e^{-0.01x}} = 2.5$

**66.** $\dfrac{50}{1 + 12e^{-0.02x}} = 10.5$

**67.** $5^{2x} = 15$

**68.** $2^{1-x} = 6$

**69.** $500(1.07)^t = 1000$

**70.** $400(1.06)^t = 1300$

**71.** $\left(1 + \dfrac{0.07}{12}\right)^{12t} = 3$

**72.** $\left(1 + \dfrac{0.06}{12}\right)^{12t} = 5$

**73.** $\left(16 - \dfrac{0.878}{26}\right)^{3t} = 30$

**74.** $\left(4 - \dfrac{2.471}{40}\right)^{9t} = 21$

In Exercises 75 and 76, $3000$ is invested in an account at interest rate $r$, compounded continuously. Find the time required for the amount to (a) double and (b) triple.

**75.** $r = 0.085$

**76.** $r = 0.12$

**77. Compound Interest**  A deposit of $1000$ is made in an account that earns interest at an annual rate of 5%. How long will it take for the balance to double if the interest is compounded (a) annually, (b) monthly, (c) daily, and (d) continuously?

**78. Compound Interest**  Use a spreadsheet to complete the table, which shows the time $t$ necessary for $P$ dollars to triple if the interest is compounded continuously at the rate of $r$.

| $r$ | 2% | 4% | 6% | 8% | 10% | 12% | 14% |
|-----|----|----|----|----|-----|-----|-----|
| $t$ |    |    |    |    |     |     |     |

**79. Demand**  The demand function for a product is given by

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right)$$

where $p$ is the price per unit and $x$ is the number of units sold. Find the numbers of units sold for prices of (a) $p = \$200$ and (b) $p = \$800$.

**80. Demand**  The demand function for a product is given by

$$p = 10,000\left(1 - \frac{3}{3 + e^{-0.001x}}\right)$$

where $p$ is the price per unit and $x$ is the number of units sold. Find the numbers of units sold for prices of (a) $p = \$500$ and (b) $p = \$1500$.

**81. Population Growth**  The population $P$ (in thousands) of Orlando, Florida from 1980 through 2005 can be modeled by

$$P = 131e^{0.019t}$$

where $t = 0$ corresponds to 1980.  *(Source: U.S. Census Bureau)*

(a) According to this model, what was the population of Orlando in 2005?

(b) According to this model, in what year will Orlando have a population of 300,000?

**82. Population Growth**  The population $P$ (in thousands) of Houston, Texas from 1980 through 2005 can be modeled by $P = 1576e^{0.01t}$, where $t = 0$ corresponds to 1980. *(Source: U.S. Census Bureau)*

(a) According to this model, what was the population of Houston in 2005?

(b) According to this model, in what year will Houston have a population of 2,500,000?

**Carbon Dating** In Exercises 83–86, you are given the ratio of carbon atoms in a fossil. Use the information to estimate the age of the fossil. In living organic material, the ratio of radioactive carbon isotopes to the total number of carbon atoms is about 1 to $10^{12}$. (See Example 2 in Section 4.1.) When organic material dies, its radioactive carbon isotopes begin to decay, with a half-life of about 5715 years. So, the ratio $R$ of carbon isotopes to carbon-14 atoms is modeled by $R = 10^{-12}\left(\frac{1}{2}\right)^{t/5715}$, where $t$ is the time (in years) and $t = 0$ represents the time when the organic material died.
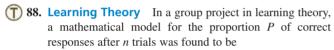
**83.** $R = 0.32 \times 10^{-12}$      **84.** $R = 0.27 \times 10^{-12}$

**85.** $R = 0.22 \times 10^{-12}$      **86.** $R = 0.13 \times 10^{-12}$

**87. Learning Theory** Students in a mathematics class were given an exam and then retested monthly with equivalent exams. The average scores $S$ (on a 100-point scale) for the class can be modeled by $S = 80 - 14 \ln(t + 1)$, $0 \le t \le 12$, where $t$ is the time in months.

(a) What was the average score on the original exam?

(b) What was the average score after 4 months?

(c) After how many months was the average score 46?

**(T) 88. Learning Theory** In a group project in learning theory, a mathematical model for the proportion $P$ of correct responses after $n$ trials was found to be

$$P = \frac{0.83}{1 + e^{-0.2n}}.$$

(a) Use a graphing utility to graph the function.

(b) Use the graph to determine any horizontal asymptotes of the graph of the function. Interpret the meaning of the upper asymptote in the context of the problem.

(c) After how many trials will 60% of the responses be correct?

**(T) 89. Agriculture** The yield $V$ (in pounds per acre) for an orchard at age $t$ (in years) is modeled by

$$V = 7955.6e^{-0.0458/t}.$$

(a) Use a graphing utility to graph the function.

(b) Determine the horizontal asymptote of the graph of the function. Interpret its meaning in the context of the problem.

(c) Find the time necessary to obtain a yield of 7900 pounds per acre.

**90. *MAKE A DECISION: FINANCE*** You are investing $P$ dollars at an annual interest rate of $r$, compounded continuously, for $t$ years, Which of the following options would you choose to get the highest value of the investment? Explain your reasoning.

(a) Double the amount you invest.

(b) Double your interest rate.

(c) Double the number of years.

**(S) 91.** Demonstrate that

$$\frac{\ln x}{\ln y} \ne \ln \frac{x}{y} = \ln x - \ln y$$

by using a spreadsheet to complete the table.

| $x$ | $y$ | $\dfrac{\ln x}{\ln y}$ | $\ln \dfrac{x}{y}$ | $\ln x - \ln y$ |
|-----|-----|-----|-----|-----|
| 1 | 2 | | | |
| 3 | 4 | | | |
| 10 | 5 | | | |
| 4 | 0.5 | | | |

**(S) 92.** Use a spreadsheet to complete the table using $f(x) = \dfrac{\ln x}{x}$.

| $x$ | 1 | 5 | 10 | $10^2$ | $10^4$ | $10^6$ |
|-----|---|---|----|--------|--------|--------|
| $f(x)$ | | | | | | |

(a) Use the table to estimate the limit: $\lim\limits_{x \to \infty} f(x)$.

**(T)** (b) Use a graphing utility to estimate the relative extrema of $f$.

**(T)** In Exercises 93 and 94, use a graphing utility to verify that the functions are equivalent for $x > 0$.

**93.** $f(x) = \ln \dfrac{x^2}{4}$

$g(x) = 2 \ln x - \ln 4$

**94.** $f(x) = \ln \sqrt{x(x^2 + 1)}$

$g(x) = \frac{1}{2}[\ln x + \ln(x^2 + 1)]$

**True or False?** In Exercises 95–100, determine whether the statement is true or false given that $f(x) = \ln x$. If it is false, explain why or give an example that shows it is false.

**95.** $f(0) = 0$

**96.** $f(ax) = f(a) + f(x), \quad a > 0, x > 0$

**97.** $f(x - 2) = f(x) - f(2), \quad x > 2$

**98.** $\sqrt{f(x)} = \frac{1}{2}f(x)$

**99.** If $f(u) = 2f(v)$, then $v = u^2$.

**100.** If $f(x) < 0$, then $0 < x < 1$.

**(T) 101. Research Project** Use a graphing utility to graph

$$y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{10}\right) - \sqrt{100 - x^2}$$

over the interval $(0, 10]$. This graph is called a *tractrix* or *pursuit curve*. Use your school's library, the Internet, or some other reference source to find information about a tractrix. Explain how such a curve can arise in a real-life setting.

# Derivatives of Logarithmic Functions

■ Find derivatives of natural logarithmic functions.
■ Use calculus to analyze the graphs of functions that involve the natural logarithmic function.
■ Use the definition of logarithms and the change-of-base formula to evaluate logarithmic expressions involving other bases.
■ Find derivatives of exponential and logarithmic functions involving other bases.

## Derivatives of Logarithmic Functions

*DISCOVERY*

Sketch the graph of $y = \ln x$ on a piece of paper. Draw tangent lines to the graph at various points. How do the slopes of these tangent lines change as you move to the right? Is the slope ever equal to zero? Use the formula for the derivative of the logarithmic function to confirm your conclusions.

Implicit differentiation can be used to develop the derivative of the natural logarithmic function.

$$y = \ln x \qquad \text{Natural logarithmic function}$$

$$e^y = x \qquad \text{Write in exponential form.}$$

$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x] \qquad \text{Differentiate with respect to } x.$$

$$e^y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{e^y} \qquad \text{Divide each side by } e^y.$$

$$\frac{dy}{dx} = \frac{1}{x} \qquad \text{Substitute } x \text{ for } e^y.$$

This result and its Chain Rule version are summarized below.

---

**Derivative of the Natural Logarithmic Function**

Let $u$ be a differentiable function of $x$.

**1.** $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$   **2.** $\dfrac{d}{dx}[\ln u] = \dfrac{1}{u}\dfrac{du}{dx}$

---

*Example 1*  **Differentiating a Logarithmic Function**

Find the derivative of

$$f(x) = \ln 2x.$$

**SOLUTION**  Let $u = 2x$. Then $du/dx = 2$, and you can apply the Chain Rule as shown.

$$f'(x) = \frac{1}{u}\frac{du}{dx} = \frac{1}{2x}(2) = \frac{1}{x}$$

✓ **CHECKPOINT 1**

Find the derivative of $f(x) = \ln 5x$. ■

Example 2   **Differentiating Logarithmic Functions**

Find the derivative of each function.

**a.** $f(x) = \ln(2x^2 + 4)$     **b.** $f(x) = x \ln x$     **c.** $f(x) = \dfrac{\ln x}{x}$

**SOLUTION**

**a.** Let $u = 2x^2 + 4$. Then $du/dx = 4x$, and you can apply the Chain Rule.

$$f'(x) = \frac{1}{u}\frac{du}{dx} \qquad \text{Chain Rule}$$

$$= \frac{1}{2x^2 + 4}(4x)$$

$$= \frac{2x}{x^2 + 2} \qquad \text{Simplify.}$$

**b.** Using the Product Rule, you can find the derivative.

$$f'(x) = x\frac{d}{dx}[\ln x] + (\ln x)\frac{d}{dx}[x] \qquad \text{Product Rule}$$

$$= x\left(\frac{1}{x}\right) + (\ln x)(1)$$

$$= 1 + \ln x \qquad \text{Simplify.}$$

**c.** Using the Quotient Rule, you can find the derivative.

$$f'(x) = \frac{x\dfrac{d}{dx}[\ln x] - (\ln x)\dfrac{d}{dx}[x]}{x^2} \qquad \text{Quotient Rule}$$

$$= \frac{x\left(\dfrac{1}{x}\right) - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2} \qquad \text{Simplify.}$$

**STUDY TIP**

When you are differentiating logarithmic functions, it is often helpful to use the properties of logarithms to rewrite the function *before* differentiating. To see the advantage of rewriting before differentiating, try using the Chain Rule to differentiate $f(x) = \ln\sqrt{x + 1}$ and compare your work with that shown in Example 3.

✓ **CHECKPOINT 2**

Find the derivative of each function.

**a.** $f(x) = \ln(x^2 - 4)$

**b.** $f(x) = x^2 \ln x$

**c.** $f(x) = -\dfrac{\ln x}{x^2}$ ■

Example 3   **Rewriting Before Differentiating**

Find the derivative of $f(x) = \ln\sqrt{x + 1}$.

**SOLUTION**

$$f(x) = \ln\sqrt{x + 1} \qquad \text{Write original function.}$$

$$= \ln(x + 1)^{1/2} \qquad \text{Rewrite with rational exponent.}$$

$$= \frac{1}{2}\ln(x + 1) \qquad \text{Property of logarithms}$$

$$f'(x) = \frac{1}{2}\left(\frac{1}{x + 1}\right) \qquad \text{Differentiate.}$$

$$= \frac{1}{2(x + 1)} \qquad \text{Simplify.}$$

✓ **CHECKPOINT 3**

Find the derivative of $f(x) = \ln\sqrt[3]{x + 1}$. ■

The next example is an even more dramatic illustration of the benefit of rewriting a function before differentiating.

### Example 4   Rewriting Before Differentiating

Find the derivative of $f(x) = \ln[x(x^2 + 1)^2]$.

**SOLUTION**

$$f(x) = \ln[x(x^2 + 1)^2] \qquad \text{Write original function.}$$
$$= \ln x + \ln(x^2 + 1)^2 \qquad \text{Logarithmic properties}$$
$$= \ln x + 2\ln(x^2 + 1) \qquad \text{Logarithmic properties}$$
$$f'(x) = \frac{1}{x} + 2\left(\frac{2x}{x^2 + 1}\right) \qquad \text{Differentiate.}$$
$$= \frac{1}{x} + \frac{4x}{x^2 + 1} \qquad \text{Simplify.}$$

✓**CHECKPOINT 4**

Find the derivative of $f(x) = \ln[x^2\sqrt{x^2 + 1}]$. ■

**STUDY TIP**

Finding the derivative of the function in Example 4 without first rewriting would be a formidable task.

$$f'(x) = \frac{1}{x(x^2 + 1)^2}\frac{d}{dx}[x(x^2 + 1)^2]$$

You might try showing that this yields the same result obtained in Example 4, but be careful—the algebra is messy.

**TECHNOLOGY**

(T) A symbolic differentiation utility will not generally list the derivative of the logarithmic function in the form obtained in Example 4. Use a symbolic differentiation utility to find the derivative of the function in Example 4. Show that the two forms are equivalent by rewriting the answer obtained in Example 4.

## Applications

**Example 5** Analyzing a Graph (T)

Analyze the graph of the function $f(x) = \dfrac{x^2}{2} - \ln x$.

**SOLUTION** From Figure 4.16, it appears that the function has a minimum at $x = 1$. To find the minimum analytically, find the critical numbers by setting the derivative of $f$ equal to zero and solving for $x$.

$$f(x) = \frac{x^2}{2} - \ln x \qquad \text{Write original function.}$$

$$f'(x) = x - \frac{1}{x} \qquad \text{Differentiate.}$$

$$x - \frac{1}{x} = 0 \qquad \text{Set derivative equal to 0.}$$

$$x = \frac{1}{x} \qquad \text{Add } 1/x \text{ to each side.}$$
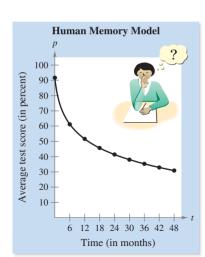
$$x^2 = 1 \qquad \text{Multiply each side by } x.$$

$$x = \pm 1 \qquad \text{Take square root of each side.}$$

Of these two possible critical numbers, only the positive one lies in the domain of $f$. By applying the First-Derivative Test, you can confirm that the function has a relative minimum when $x = 1$.



**FIGURE 4.16**

✓ **CHECKPOINT 5**

Determine the relative extrema of the function

$$f(x) = x - 2 \ln x. \ \blacksquare$$

**Example 6** Finding a Rate of Change ®

A group of 200 college students was tested every 6 months over a four-year period. The group was composed of students who took Spanish during the fall semester of their freshman year and did not take subsequent Spanish courses. The average test score $p$ (in percent) is modeled by

$$p = 91.6 - 15.6 \ln(t + 1), \quad 0 \le t \le 48$$

where $t$ is the time in months, as shown in Figure 4.17. At what rate was the average score changing after 1 year?

**SOLUTION** The rate of change is

$$\frac{dp}{dt} = -\frac{15.6}{t + 1}.$$

When $t = 12$, $dp/dt = -1.2$, which means that the average score was decreasing at the rate of 1.2% per month.



**FIGURE 4.17**

✓ **CHECKPOINT 6**

Suppose the average test score $p$ in Example 6 was modeled by $p = 92.3 - 16.9 \ln(t + 1)$, where $t$ is the time in months. How would the rate at which the average test score changed after 1 year compare with that of the model in Example 6? ■

## Other Bases

This chapter began with a definition of a general exponential function

$$f(x) = a^x$$

where $a$ is a positive number such that $a \neq 1$. The corresponding **logarithm to the base $a$** is defined by

$$\log_a x = b \quad \text{if and only if} \quad a^b = x.$$

As with the natural logarithmic function, the domain of the logarithmic function to the base $a$ is the set of positive numbers.

### Example 7   Evaluating Logarithms

Evaluate each logarithm without using a calculator.

**a.** $\log_2 8$    **b.** $\log_{10} 100$    **c.** $\log_{10} \frac{1}{10}$    **d.** $\log_3 81$

**SOLUTION**

**a.** $\log_2 8 = 3$      $2^3 = 8$

**b.** $\log_{10} 100 = 2$      $10^2 = 100$

**c.** $\log_{10} \frac{1}{10} = -1$      $10^{-1} = \frac{1}{10}$

**d.** $\log_3 81 = 4$      $3^4 = 81$

✓**CHECKPOINT 7**

Evaluate each logarithm without using a calculator.

**a.** $\log_2 16$

**b.** $\log_{10} \frac{1}{100}$

**c.** $\log_2 \frac{1}{32}$

**d.** $\log_5 125$ ◼

Logarithms to the base 10 are called **common logarithms.** Most calculators have only two logarithm keys—a natural logarithm key denoted by ⃞LN and a common logarithm key denoted by ⃞LOG. Logarithms to other bases can be evaluated with the following change-of-base formula.

$$\log_a x = \frac{\ln x}{\ln a} \qquad \text{Change-of-base formula}$$

### Example 8   Evaluating Logarithms

Use the change-of-base formula and a calculator to evaluate each logarithm.

**a.** $\log_2 3$    **b.** $\log_3 6$    **c.** $\log_2(-1)$

**SOLUTION**   In each case, use the change-of-base formula and a calculator.

**a.** $\log_2 3 = \frac{\ln 3}{\ln 2} \approx 1.585$      $\log_a x = \frac{\ln x}{\ln a}$

**b.** $\log_3 6 = \frac{\ln 6}{\ln 3} \approx 1.631$      $\log_a x = \frac{\ln x}{\ln a}$

**c.** $\log_2(-1)$ is not defined.

✓**CHECKPOINT 8**

Use the change-of-base formula and a calculator to evaluate each logarithm.

**a.** $\log_2 5$

**b.** $\log_3 18$

**c.** $\log_4 80$

**d.** $\log_{16} 0.25$ ◼

To find derivatives of exponential or logarithmic functions to bases other than $e$, you can either convert to base $e$ or use the differentiation rules shown on the next page.

**Other Bases and Differentiation**

Let $u$ be a differentiable function of $x$.

1. $\dfrac{d}{dx}[a^x] = (\ln a)a^x$        2. $\dfrac{d}{dx}[a^u] = (\ln a)a^u\dfrac{du}{dx}$

3. $\dfrac{d}{dx}[\log_a x] = \left(\dfrac{1}{\ln a}\right)\dfrac{1}{x}$        4. $\dfrac{d}{dx}[\log_a u] = \left(\dfrac{1}{\ln a}\right)\left(\dfrac{1}{u}\right)\dfrac{du}{dx}$

**PROOF**   By definition, $a^x = e^{(\ln a)x}$. So, you can prove the first rule by letting $u = (\ln a)x$ and differentiating with base $e$ to obtain

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{(\ln a)x}] = e^u\frac{du}{dx} = e^{(\ln a)x}(\ln a) = (\ln a)a^x.$$

*Example 9*   **Finding a Rate of Change**   ®

Radioactive carbon isotopes have a half-life of 5715 years. If 1 gram of the isotopes is present in an object now, the amount $A$ (in grams) that will be present after $t$ years is

$$A = \left(\frac{1}{2}\right)^{t/5715}.$$

At what rate is the amount changing when $t = 10{,}000$ years?

**SOLUTION**   The derivative of $A$ with respect to $t$ is

$$\frac{dA}{dt} = \left(\ln\frac{1}{2}\right)\left(\frac{1}{2}\right)^{t/5715}\left(\frac{1}{5715}\right).$$

When $t = 10{,}000$, the rate at which the amount is changing is

$$\left(\ln\frac{1}{2}\right)\left(\frac{1}{2}\right)^{10{,}000/5715}\left(\frac{1}{5715}\right) \approx -0.000036$$

which implies that the amount of isotopes in the object is decreasing at the rate of 0.000036 gram per year.

**✓ CHECKPOINT 9**

Use a graphing utility to graph the model in Example 9. Describe the rate at which the amount is changing as time $t$ increases. ■

**CONCEPT CHECK**

1. What is the derivative of $f(x) = \ln x$?

2. What is the derivative of $f(x) = \ln u$? (Assume $u$ is a differentiable function of $x$.)

3. Complete the following: The change-of-base formula for base $e$ is given by $\log_a x = $ _____.

4. Logarithms to the base $e$ are called natural logarithms. What are logarithms to the base 10 called?

**Skills Review 4.5**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.6, 2.7, and 4.4.

In Exercises 1–6, expand the logarithmic expression.

**1.** $\ln(x + 1)^2$

**2.** $\ln x(x + 1)$

**3.** $\ln \dfrac{x}{x + 1}$

**4.** $\ln\left(\dfrac{x}{x - 3}\right)^3$

**5.** $\ln \dfrac{4x(x - 7)}{x^2}$

**6.** $\ln x^3(x + 1)$

In Exercises 7 and 8, find $dy/dx$ implicitly.

**7.** $y^2 + xy = 7$

**8.** $x^2y - xy^2 = 3x$

In Exercises 9 and 10, find the second derivative of $f$.

**9.** $f(x) = x^2(x + 1) - 3x^3$
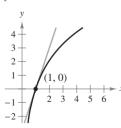
**10.** $f(x) = -\dfrac{1}{x^2}$

## Exercises 4.5
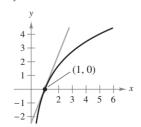
In Exercises 1–4, find the slope of the tangent line to the graph of the function at the point (1, 0).

**1.** $y = \ln x^3$
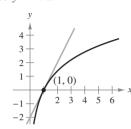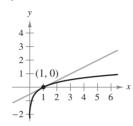
**2.** $y = \ln x^{5/2}$





**3.** $y = \ln x^2$

**4.** $y = \ln x^{1/2}$





In Exercises 5–26, find the derivative of the function.

**5.** $y = \ln x^2$

**6.** $f(x) = \ln 2x$

**7.** $y = \ln(x^2 + 3)$

**8.** $f(x) = \ln(1 - x^2)$

**9.** $y = \ln\sqrt{x - 4}$

**10.** $y = \ln(1 - x)^{3/2}$

**11.** $y = (\ln x)^4$

**12.** $y = (\ln x^2)^2$

**13.** $f(x) = 2x \ln x$

**14.** $y = \dfrac{\ln x}{x^2}$

**15.** $y = \ln\left(x\sqrt{x^2 - 1}\right)$

**16.** $y = \ln \dfrac{x}{x^2 + 1}$

**17.** $y = \ln \dfrac{x}{x + 1}$

**18.** $y = \ln \dfrac{x^2}{x^2 + 1}$

**19.** $y = \ln \sqrt[3]{\dfrac{x - 1}{x + 1}}$

**20.** $y = \ln \sqrt{\dfrac{x + 1}{x - 1}}$

**21.** $y = \ln \dfrac{\sqrt{4 + x^2}}{x}$

**22.** $y = \ln\left(x\sqrt{4 + x^2}\right)$

**23.** $g(x) = e^{-x} \ln x$

**24.** $f(x) = x \ln e^{x^2}$

**25.** $g(x) = \ln \dfrac{e^x + e^{-x}}{2}$

**26.** $f(x) = \ln \dfrac{1 + e^x}{1 - e^x}$

In Exercises 27–30, write the expression with base e.

**27.** $2^x$

**28.** $3^x$

**29.** $\log_4 x$

**30.** $\log_3 x$

In Exercises 31–38, use a calculator to evaluate the logarithm. Round to three decimal places.

**31.** $\log_4 7$

**32.** $\log_6 10$

**33.** $\log_2 48$

**34.** $\log_5 12$

**35.** $\log_3 \frac{1}{2}$

**36.** $\log_7 \frac{2}{9}$

**37.** $\log_{1/5} 31$

**38.** $\log_{2/3} 32$

In Exercises 39–48, find the derivative of the function.

**39.** $y = 3^x$

**40.** $y = \left(\frac{1}{4}\right)^x$

**41.** $f(x) = \log_2 x$

**42.** $g(x) = \log_5 x$

**43.** $h(x) = 4^{2x-3}$

**44.** $y = 6^{5x}$

**45.** $y = \log_{10}(x^2 + 6x)$

**46.** $f(x) = 10^{x^2}$

**47.** $y = x2^x$

**48.** $y = x3^{x+1}$

In Exercises 49–52, determine an equation of the tangent line to the function at the given point.

|   Function | Point |
|---|---|
| **49.** $y = x \ln x$ | $(1, 0)$ |
| **50.** $y = \dfrac{\ln x}{x}$ | $\left(e, \dfrac{1}{e}\right)$ |
| **51.** $y = \log_3 x$ | $(27, 3)$ |
| **52.** $g(x) = \log_{10} 2x$ | $(5, 1)$ |

In Exercises 53–56, find $dy/dx$ implicitly.

**53.** $x^2 - 3 \ln y + y^2 = 10$

**54.** $\ln xy + 5x = 30$

**55.** $4x^3 + \ln y^2 + 2y = 2x$

**56.** $4xy + \ln(x^2 y) = 7$

In Exercises 57 and 58, use implicit differentiation to find an equation of the tangent line to the graph at the given point.

**57.** $x + y - 1 = \ln(x^2 + y^2)$,   $(1, 0)$

**58.** $y^2 + \ln(xy) = 2$,   $(e, 1)$

In Exercises 59–64, find the second derivative of the function.

**59.** $f(x) = x \ln \sqrt{x} + 2x$

**60.** $f(x) = 3 + 2 \ln x$

**61.** $f(x) = 2 + x \ln x$

**62.** $f(x) = \dfrac{\ln x}{x} + x$

**63.** $f(x) = 5^x$

**64.** $f(x) = \log_{10} x$

**65. Sound Intensity**   The relationship between the number of decibels $\beta$ and the intensity of a sound $I$ in watts per square centimeter is given by

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-16}}\right).$$

Find the rate of change in the number of decibels when the intensity is $10^{-4}$ watt per square centimeter.

**66. Chemistry**   The temperatures $T$ (°F) at which water boils at selected pressures $p$ (pounds per square inch) can be modeled by

$$T = 87.97 + 34.96 \ln p + 7.91 \sqrt{p}.$$

Find the rate of change of the temperature when the pressure is 60 pounds per square inch.

In Exercises 67–72, find the slope of the graph at the indicated point. Then write an equation of the tangent line to the graph of the function at the given point.

**67.** $f(x) = 1 + 2x \ln x$,   $(1, 1)$

**68.** $f(x) = 2 \ln x^3$,   $(e, 6)$

**69.** $f(x) = \ln \dfrac{5(x + 2)}{x}$,   $(-2.5, 0)$

**70.** $f(x) = \ln(x\sqrt{x + 3})$,   $(1.2, 0.9)$

**71.** $f(x) = x \log_2 x$,   $(1, 0)$    **72.** $f(x) = x^2 \log_3 x$,   $(1, 0)$

In Exercises 73–78, graph and analyze the function. Include any relative extrema and points of inflection in your analysis. Use a graphing utility to verify your results.

**73.** $y = x - \ln x$

**74.** $y = \dfrac{x}{\ln x}$

**75.** $y = \dfrac{\ln x}{x}$

**76.** $y = x \ln x$
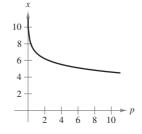
**77.** $y = x^2 \ln \dfrac{x}{4}$
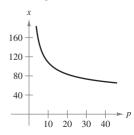
**78.** $y = (\ln x)^2$

**Demand**   In Exercises 79 and 80, find $dx/dp$ for the demand function. Interpret this rate of change when the price is \$10.

**79.** $x = \ln \dfrac{1000}{p}$

**80.** $x = \dfrac{500}{\ln(p^2 + 1)}$



**81. Demand**   Solve the demand function in Exercise 79 for $p$. Use the result to find $dp/dx$. Then find the rate of change when $p = \$10$. What is the relationship between this derivative and $dx/dp$?

**82. Demand**   Solve the demand function in Exercise 80 for $p$. Use the result to find $dp/dx$. Then find the rate of change when $p = \$10$. What is the relationship between this derivative and $dx/dp$?

**83. Minimum Average Cost**   The cost of producing $x$ units of a product is modeled by

$$C = 500 + 300x - 300 \ln x, \quad x \ge 1.$$

(a) Find the average cost function $\overline{C}$.

(b) Analytically find the minimum average cost. Use a graphing utility to confirm your result.

**84. Minimum Average Cost**    The cost of producing $x$ units of a product is modeled by

$$C = 100 + 25x - 120 \ln x, \quad x \geq 1.$$

(a) Find the average cost function $\overline{C}$.

(b) Analytically find the minimum average cost. Use a graphing utility to confirm your result.

**85. Consumer Trends**    The retail sales $S$ (in billions of dollars per year) of e-commerce companies in the United States from 1999 through 2004 are shown in the table.

| $t$ | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|
| $S$ | 14.5 | 27.8 | 34.5 | 45.0 | 56.6 | 70.9 |

The data can be modeled by $S = -254.9 + 121.95 \ln t$, where $t = 9$ corresponds to 1999. *(Source: U.S. Census Bureau)*

(T) (a) Use a graphing utility to plot the data and graph $S$ over the interval $[9, 14]$.

(b) At what rate were the sales changing in 2002?

**86. Home Mortgage**    The term $t$ (in years) of a $200,000 home mortgage at 7.5% interest can be approximated by

$$t = -13.375 \ln \frac{x - 1250}{x}, \quad x > 1250$$

where $x$ is the monthly payment in dollars.

(T) (a) Use a graphing utility to graph the model.

(b) Use the model to approximate the term of a home mortgage for which the monthly payment is $1398.43. What is the total amount paid?

(c) Use the model to approximate the term of a home mortgage for which the monthly payment is $1611.19. What is the total amount paid?

(d) Find the instantaneous rate of change of $t$ with respect to $x$ when $x = \$1398.43$ and $x = \$1611.19$.

(e) Write a short paragraph describing the benefit of the higher monthly payment.

**87. Earthquake Intensity**    On the Richter scale, the magnitude $R$ of an earthquake of intensity $I$ is given by

$$R = \frac{\ln I - \ln I_0}{\ln 10}$$

where $I_0$ is the minimum intensity used for comparison. Assume $I_0 = 1$.

(a) Find the intensity of the 1906 San Francisco earthquake for which $R = 8.3$.

(b) Find the intensity of the May 26, 2006 earthquake in Java, Indonesia for which $R = 6.3$.

(c) Find the factor by which the intensity is increased when the value of $R$ is doubled.

(d) Find $dR/dI$.

**88. Learning Theory**    Students in a learning theory study were given an exam and then retested monthly for 6 months with an equivalent exam. The data obtained in the study are shown in the table, where $t$ is the time in months after the initial exam and $s$ is the average score for the class.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $s$ | 84.2 | 78.4 | 72.1 | 68.5 | 67.1 | 65.3 |

(a) Use these data to find a logarithmic equation that relates $t$ and $s$.

(T) (b) Use a graphing utility to plot the data and graph the model. How well does the model fit the data?

(c) Find the rate of change of $s$ with respect to $t$ when $t = 2$. Interpret the meaning in the context of the problem.

---

**Business Capsule**



AP/Wide World Photos

Lillian Vernon Corporation is a leading national catalog and online retailer that markets gift, household, children's, and fashion accessory products. Lilly Menasche founded the company in Mount Vernon, New York in 1951 using $2000 of wedding gift money. Today, headquartered in Virginia Beach, Virginia, Lillian Vernon's annual sales exceed $287 million. More than 3.3 million packages were shipped in 2006.

**89. Research Project**    Use your school's library, the Internet, or some other reference source to research information about a mail-order or e-commerce company, such as that mentioned above. Collect data about the company (sales or membership over a 20-year period, for example) and find a mathematical model to represent the data.

# Exponential Growth and Decay

■ Use exponential growth and decay to model real-life situations.

## Exponential Growth and Decay

In this section, you will learn to create models of *exponential growth and decay*. Real-life situations that involve exponential growth and decay deal with a substance or population whose *rate of change at any time t is proportional to the amount of the substance present at that time*. For example, the rate of decomposition of a radioactive substance is proportional to the amount of radioactive substance at a given instant. In its simplest form, this relationship is described by the equation below.

Rate of change of *y* is proportional to *y*.

$$\frac{dy}{dt} = ky$$

In this equation, *k* is a constant and *y* is a function of *t*. The solution of this equation is shown below.

---

**Law of Exponential Growth and Decay**

If *y* is a positive quantity whose rate of change with respect to time is proportional to the quantity present at any time *t*, then *y* is of the form

$$y = Ce^{kt}$$

where *C* is the **initial value** and *k* is the **constant of proportionality.** **Exponential growth** is indicated by $k > 0$ and **exponential decay** by $k < 0$.

---

**PROOF**   Because the rate of change of *y* is proportional to *y*, you can write

$$\frac{dy}{dt} = ky.$$

You can see that $y = Ce^{kt}$ is a solution of this equation by differentiating to obtain $dy/dt = kCe^{kt}$ and substituting

$$\frac{dy}{dt} = kCe^{kt} = k(Ce^{kt}) = ky.$$

---

**STUDY TIP**

In the model $y = Ce^{kt}$, *C* is called the "initial value" because when $t = 0$

$$y = Ce^{k(0)} = C(1) = C.$$

Much of the cost of nuclear energy is the cost of disposing of radioactive waste. Because of the long half-life of the waste, it must be stored in containers that will remain undisturbed for thousands of years.

## Applications

Radioactive decay is measured in terms of **half-life,** the number of years required for half of the atoms in a sample of radioactive material to decay. The half-lives of some common radioactive isotopes are as shown.

| | |
|---|---:|
| Uranium ($^{238}$U) | 4,470,000,000  years |
| Plutonium ($^{239}$Pu) | 24,100  years |
| Carbon ($^{14}$C) | 5,715  years |
| Radium ($^{226}$Ra) | 1,599  years |
| Einsteinium ($^{254}$Es) | 276  days |
| Nobelium ($^{257}$No) | 25  seconds |

### Example 1
### MAKE A DECISION    Modeling Radioactive Decay (R)

A sample contains 1 gram of radium. Will more than 0.5 gram of radium remain after 1000 years?

**SOLUTION**   Let $y$ represent the mass (in grams) of the radium in the sample. Because the rate of decay is proportional to $y$, you can apply the Law of Exponential Decay to conclude that $y$ is of the form $y = Ce^{kt}$, where $t$ is the time in years. From the given information, you know that $y = 1$ when $t = 0$. Substituting these values into the model produces

$$1 = Ce^{k(0)} \qquad \text{Substitute 1 for } y \text{ and 0 for } t.$$

which implies that $C = 1$. Because radium has a half-life of 1599 years, you know that $y = \frac{1}{2}$ when $t = 1599$. Substituting these values into the model allows you to solve for $k$.

$$y = e^{kt} \qquad \text{Exponential decay model}$$
$$\frac{1}{2} = e^{k(1599)} \qquad \text{Substitute } \tfrac{1}{2} \text{ for } y \text{ and 1599 for } t.$$
$$\ln \frac{1}{2} = 1599k \qquad \text{Take natural log of each side.}$$
$$\frac{1}{1599}\ln \frac{1}{2} = k \qquad \text{Divide each side by 1599.}$$

So, $k \approx -0.0004335$, and the exponential decay model is $y = e^{-0.0004335t}$. To find the amount of radium remaining in the sample after 1000 years, substitute $t = 1000$ into the model. This produces
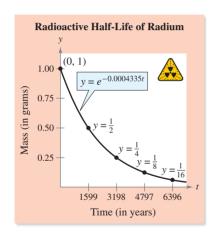
$$y = e^{-0.0004335(1000)} \approx 0.648 \text{ gram.}$$

Yes, more than 0.5 gram of radium will remain after 1000 years. The graph of the model is shown in Figure 4.18.



**Radioactive Half-Life of Radium**

**FIGURE 4.18**

Note: Instead of approximating the value of $k$ in Example 1, you could leave the value exact and obtain

$$y = e^{\ln[(1/2)^{(t/1599)}]} = \frac{1}{2}^{(t/1599)}.$$

This version of the model clearly shows the "half-life." When $t = 1599$, the value of $y$ is $\frac{1}{2}$. When $t = 2(1599)$, the value of $y$ is $\frac{1}{4}$, and so on.

✓ **CHECKPOINT 1**

Use the model in Example 1 to determine the number of years required for a one-gram sample of radium to decay to 0.4 gram. ■

**Guidelines for Modeling Exponential Growth and Decay**

1. Use the given information to write *two* sets of conditions involving $y$ and $t$.

2. Substitute the given conditions into the model $y = Ce^{kt}$ and use the results to solve for the constants $C$ and $k$. (If one of the conditions involves $t = 0$, substitute that value first to solve for $C$.)

3. Use the model $y = Ce^{kt}$ to answer the question.

*Example 2* **Modeling Population Growth** ®

### Algebra Review

For help with the algebra in Example 2, see Example 1(c) in the *Chapter 4 Algebra Review* on page 344.
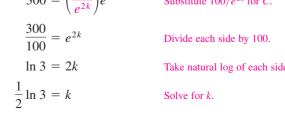
In a research experiment, a population of fruit flies is increasing in accordance with the exponential growth model. After 2 days, there are 100 flies, and after 4 days, there are 300 flies. How many flies will there be after 5 days?

**SOLUTION** Let $y$ be the number of flies at time $t$. From the given information, you know that $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$. Substituting this information into the model $y = Ce^{kt}$ produces

$$100 = Ce^{2k} \quad \text{and} \quad 300 = Ce^{4k}.$$

To solve for $k$, solve for $C$ in the first equation and substitute the result into the second equation.

$$300 = Ce^{4k} \qquad \text{Second equation}$$

$$300 = \left(\frac{100}{e^{2k}}\right)e^{4k} \qquad \text{Substitute } 100/e^{2k} \text{ for } C.$$

$$\frac{300}{100} = e^{2k} \qquad \text{Divide each side by 100.}$$

$$\ln 3 = 2k \qquad \text{Take natural log of each side.}$$

$$\frac{1}{2}\ln 3 = k \qquad \text{Solve for } k.$$
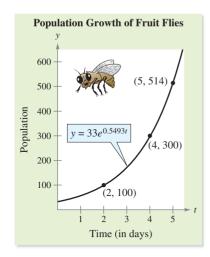
Using $k = \frac{1}{2}\ln 3 \approx 0.5493$, you can determine that $C \approx 100/e^{2(0.5493)} \approx 33$. So, the exponential growth model is

$$y = 33e^{0.5493t}$$

as shown in Figure 4.19. This implies that, after 5 days, the population is

$$y = 33e^{0.5493(5)} \approx 514 \text{ flies.}$$

**Population Growth of Fruit Flies**



$y = 33e^{0.5493t}$

**FIGURE 4.19**

✓**CHECKPOINT 2**

Find the exponential growth model if a population of fruit flies is 100 after 2 days and 400 after 4 days. ■

*Example 3*  **Modeling Compound Interest**    ®

Money is deposited in an account for which the interest is compounded continuously. The balance in the account doubles in 6 years. What is the annual interest rate?

**SOLUTION**    The balance $A$ in an account with continuously compounded interest is given by the exponential growth model

$$A = Pe^{rt} \qquad\qquad \text{Exponential growth model}$$
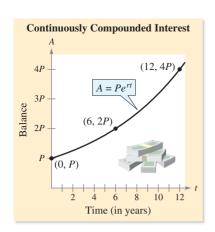
where $P$ is the original deposit, $r$ is the annual interest rate (in decimal form), and $t$ is the time (in years). From the given information, you know that $A = 2P$ when $t = 6$, as shown in Figure 4.20. Use this information to solve for $r$.

$$
\begin{aligned}
A &= Pe^{rt} && \text{Exponential growth model} \\
2P &= Pe^{r(6)} && \text{Substitute } 2P \text{ for } A \text{ and } 6 \text{ for } t. \\
2 &= e^{6r} && \text{Divide each side by } P. \\
\ln 2 &= 6r && \text{Take natural log of each side.} \\
\tfrac{1}{6}\ln 2 &= r && \text{Divide each side by } 6.
\end{aligned}
$$

So, the annual interest rate is

$$r = \tfrac{1}{6}\ln 2$$
$$\approx 0.1155$$

or about 11.55%.

**Continuously Compounded Interest**

**FIGURE 4.20**

✓**CHECKPOINT 3**

Find the annual interest rate if the balance in an account doubles in 8 years where the interest is compounded continuously. ■

Each of the examples in this section uses the exponential growth model in which the base is $e$. Exponential growth, however, can be modeled with *any* base. That is, the model

$$y = Ca^{bt}$$

also represents exponential growth. (To see this, note that the model can be written in the form $y = Ce^{(\ln a)bt}$.) In some real-life settings, bases other than $e$ are more convenient. For instance, in Example 1, knowing that the half-life of radium is 1599 years, you can immediately write the exponential decay model as

$$y = \left(\frac{1}{2}\right)^{t/1599}.$$

Using this model, the amount of radium left in the sample after 1000 years is

$$y = \left(\frac{1}{2}\right)^{1000/1599} \approx 0.648 \text{ gram}$$

which is the same answer obtained in Example 1.

**STUDY TIP**

Can you see why you can immediately write the model $y = \left(\frac{1}{2}\right)^{t/1599}$ for the radioactive decay described in Example 1? Notice that when $t = 1599$, the value of $y$ is $\frac{1}{2}$, when $t = 3198$, the value of $y$ is $\frac{1}{4}$, and so on.

**TECHNOLOGY**

### Fitting an Exponential Model to Data

(T) Most graphing utilities have programs that allow you to find the *least squares regression exponential model* for data. Depending on the type of graphing utility, you can fit the data to a model of the form

$$y = ab^x$$   Exponential model with base $b$

or

$$y = ae^{bx}.$$   Exponential model with base $e$

To see how to use such a program, consider the example below.

The cash flow per share $y$ for Harley-Davidson, Inc. from 1998 through 2005 is shown in the table.   *(Source: Harley-Davidson, Inc.)*

| $x$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|
| $y$ | $0.98 | $1.26 | $1.59 | $1.95 | $2.50 | $3.18 | $3.75 | $4.25 |

In the table, $x = 8$ corresponds to 1998. To fit an exponential model to these data, enter the coordinates listed below into the statistical data bank of a graphing utility.

(8, 0.98), (9, 1.26), (10, 1.59), (11, 1.95),

(12, 2.50), (13, 3.18), (14, 3.75), (15, 4.25)

After running the exponential regression program with a graphing utility that uses the model $y = ab^x$, the display should read $a \approx 0.183$ and $b \approx 1.2397$. (The coefficient of determination of $r^2 \approx 0.993$ tells you that the fit is very good.) So, a model for the data is

$$y = 0.183(1.2397)^x.$$   Exponential model with base $b$

If you use a graphing utility that uses the model $y = ae^{bx}$, the display should read $a \approx 0.183$ and $b \approx 0.2149$. The corresponding model is
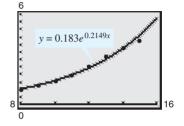
$$y = 0.183e^{0.2149x}.$$   Exponential model with base $e$

The graph of the second model is shown at the right. Notice that one way to interpret the model is that the cash flow per share increased by about 21.5% each year from 1998 through 2005.

You can use either model to predict the cash flow per share in future years. For instance, in 2006 ($x = 16$), the cash flow per share is predicted to be
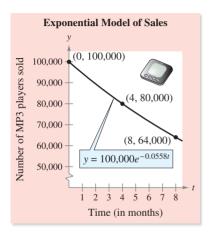
$$y = 0.183e^{(0.2149)(16)}$$
$$\approx \$5.70$$



$y = 0.183e^{0.2149x}$

Graph the model $y = 0.183(1.2397)^x$ and use the model to predict the cash flow for 2006. Compare your results with those obtained using the model $y = 0.183e^{0.2149x}$. What do you notice?

*Example 4*    **Modeling Sales**    Ⓡ

Four months after discontinuing advertising on national television, a manufacturer
notices that sales have dropped from 100,000 MP3 players per month to 80,000
MP3 players. If the sales follow an exponential pattern of decline, what will they
be after another 4 months?

**SOLUTION**    Let $y$ represent the number of MP3 players, let $t$ represent the time
(in months), and consider the exponential decay model

$$y = Ce^{kt}.\qquad\qquad\text{Exponential decay model}$$

From the given information, you know that $y = 100{,}000$ when $t = 0$. Using this
information, you have

$$100{,}000 = Ce^0$$

which implies that $C = 100{,}000$. To solve for $k$, use the fact that $y = 80{,}000$
when $t = 4$.

$$y = 100{,}000e^{kt}\qquad\qquad\text{Exponential decay model}$$

$$80{,}000 = 100{,}000e^{k(4)}\qquad\qquad\text{Substitute 80,000 for } y \text{ and 4 for } t.$$

$$0.8 = e^{4k}\qquad\qquad\text{Divide each side by 100,000.}$$

$$\ln 0.8 = 4k\qquad\qquad\text{Take natural log of each side.}$$

$$\tfrac{1}{4}\ln 0.8 = k\qquad\qquad\text{Divide each side by 4.}$$

So, $k = \tfrac{1}{4}\ln 0.8 \approx -0.0558$, which means that the model is

$$y = 100{,}000e^{-0.0558t}.$$

After four more months $(t = 8)$, you can expect sales to drop to

$$y = 100{,}000e^{-0.0558(8)}$$

$$\approx 64{,}000 \text{ MP3 players}$$

as shown in Figure 4.21.



**Exponential Model of Sales**

**FIGURE 4.21**

✓**CHECKPOINT 4**

Use the model in Example 4 to determine when sales drop to 50,000
MP3 players. ▪

**CONCEPT CHECK**

1. Describe what the values of $C$ and $k$ represent in the exponential growth
   and decay model, $y = Ce^{kt}$.

2. For what values of $k$ is $y = Ce^{kt}$ an exponential growth model? an
   exponential decay model?

3. Can the base used in an exponential growth model be a number other
   than $e$?

4. In exponential growth, is the rate of growth constant? Explain why or
   why not.

| Skills Review 4.6 | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 4.3 and 4.4. |

In Exercises 1–4, solve the equation for $k$.

**1.** $12 = 24e^{4k}$      **2.** $10 = 3e^{5k}$      **3.** $25 = 16e^{-0.01k}$      **4.** $22 = 32e^{-0.02k}$

In Exercises 5–8, find the derivative of the function.

**5.** $y = 32e^{0.23t}$      **6.** $y = 18e^{0.072t}$      **7.** $y = 24e^{-1.4t}$      **8.** $y = 25e^{-0.001t}$

In Exercises 9–12, simplify the expression.
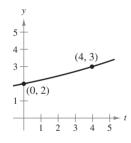
**9.** $e^{\ln 4}$      **10.** $4e^{\ln 3}$      **11.** $e^{\ln(2x+1)}$      **12.** $e^{\ln(x^2+1)}$

## Exercises 4.6
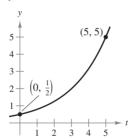
In Exercises 1–6, find the exponential function $y = Ce^{kt}$ that passes through the two given points.
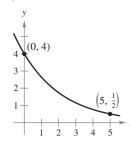
**1.** $y = Ce^{kt}$



**2.** $y = Ce^{kt}$



**3.** $y = Ce^{kt}$
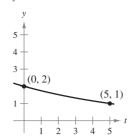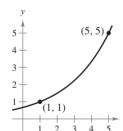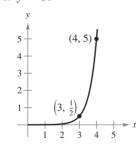


**4.** $y = Ce^{kt}$



**5.** $y = Ce^{kt}$



**6.** $y = Ce^{kt}$



In Exercises 7–10, use the given information to write an equation for $y$. Confirm your result analytically by showing that the function satisfies the equation $dy/dt = Cy$. Does the function represent exponential growth or exponential decay?

**7.** $\dfrac{dy}{dt} = 2y, \quad y = 10$ when $t = 0$

**8.** $\dfrac{dy}{dt} = -\dfrac{2}{3}y, \quad y = 20$ when $t = 0$

**9.** $\dfrac{dy}{dt} = -4y, \quad y = 30$ when $t = 0$

**10.** $\dfrac{dy}{dt} = 5.2y, \quad y = 18$ when $t = 0$

**Radioactive Decay** In Exercises 11–16, complete the table for each radioactive isotope.

| Isotope | Half-life (in years) | Initial quantity | Amount after 1000 years | Amount after 10,000 years |
|---|---|---|---|---|
| **11.** $^{226}$Ra | 1599 | 10 grams | | |
| **12.** $^{226}$Ra | 1599 | | 1.5 grams | |
| **13.** $^{14}$C | 5715 | | | 2 grams |
| **14.** $^{14}$C | 5715 | 3 grams | | |
| **15.** $^{239}$Pu | 24,100 | | 2.1 grams | |
| **16.** $^{239}$Pu | 24,100 | | | 0.4 gram |

**17. Radioactive Decay** What percent of a present amount of radioactive radium ($^{226}$Ra) will remain after 900 years?

**18. Radioactive Decay** Find the half-life of a radioactive material if after 1 year 99.57% of the initial amount remains.

**19. Carbon Dating**  $^{14}$C dating assumes that the carbon dioxide on the Earth today has the same radioactive content as it did centuries ago. If this is true, then the amount of $^{14}$C absorbed by a tree that grew several centuries ago should be the same as the amount of $^{14}$C absorbed by a similar tree today. A piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? (The half-life of $^{14}$C is 5715 years.)

**20. Carbon Dating**  Repeat Exercise 19 for a piece of charcoal that contains 30% as much radioactive carbon as a modern piece.

In Exercises 21 and 22, find exponential models

$$y_1 = Ce^{k_1 t} \quad \text{and} \quad y_2 = C(2)^{k_2 t}$$
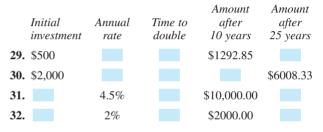
that pass through the points. Compare the values of $k_1$ and $k_2$. Briefly explain your results.

**21.** $(0, 5), (12, 20)$          **22.** $(0, 8), \left(20, \frac{1}{2}\right)$

**23. Population Growth**  The number of a certain type of bacteria increases continuously at a rate proportional to the number present. There are 150 present at a given time and 450 present 5 hours later.

  (a) How many will there be 10 hours after the initial time?

  (b) How long will it take for the population to double?

  (c) Does the answer to part (b) depend on the starting time? Explain your reasoning.

**24. School Enrollment**  In 1970, the total enrollment in public universities and colleges in the United States was 5.7 million students. By 2004, enrollment had risen to 13.7 million students. Assume enrollment can be modeled by exponential growth.  *(Source: U.S. Census Bureau)*

  (a) Estimate the total enrollments in 1980, 1990, and 2000.

  (b) How many years until the enrollment doubles from the 2004 figure?

  (c) By what percent is the enrollment increasing each year?

**Compound Interest**  In Exercises 25–32, complete the table for an account in which interest is compounded continuously.

| Initial investment | Annual rate | Time to double | Amount after 10 years | Amount after 25 years |
|---|---|---|---|---|
| **25.** $1,000 | 12% |  |  |  |
| **26.** $20,000 | $10\frac{1}{2}$% |  |  |  |
| **27.** $750 |  | 8 years |  |  |
| **28.** $10,000 |  | 10 years |  |  |

| Initial investment | Annual rate | Time to double | Amount after 10 years | Amount after 25 years |
|---|---|---|---|---|
| **29.** $500 |  |  | $1292.85 |  |
| **30.** $2,000 |  |  |  | $6008.33 |
| **31.** | 4.5% |  | $10,000.00 |  |
| **32.** | 2% |  | $2000.00 |  |

In Exercises 33 and 34, determine the principal $P$ that must be invested at interest rate $r$, compounded continuously, so that $1,000,000 will be available for retirement in $t$ years.

**33.** $r = 7.5\%, t = 40$          **34.** $r = 10\%, t = 25$

**35. Effective Yield**  The effective yield is the annual rate $i$ that will produce the same interest per year as the nominal rate $r$ compounded $n$ times per year.

  (a) For a rate $r$ that is compounded $n$ times per year, show that the effective yield is

$$i = \left(1 + \frac{r}{n}\right)^n - 1.$$

  (b) Find the effective yield for a nominal rate of 6%, compounded monthly.

**36. Effective Yield**  The effective yield is the annual rate $i$ that will produce the same interest per year as the nominal rate $r$.

  (a) For a rate $r$ that is compounded continuously, show that the effective yield is $i = e^r - 1$.

  (b) Find the effective yield for a nominal rate of 6%, compounded continuously.

**Effective Yield**  In Exercises 37 and 38, use the results of Exercises 35 and 36 to complete the table showing the effective yield for a nominal rate of $r$.

| Number of compoundings per year | 4 | 12 | 365 | Continuous |
|---|---|---|---|---|
| Effective yield |  |  |  |  |

**37.** $r = 5\%$          **38.** $r = 7\frac{1}{2}\%$

**39. Investment: Rule of 70**  Verify that the time necessary for an investment to double its value is approximately $70/r$, where $r$ is the annual interest rate entered as a percent.

**40. Investment: Rule of 70**  Use the Rule of 70 from Exercise 39 to approximate the times necessary for an investment to double in value if (a) $r = 10\%$ and (b) $r = 7\%$.

**41. MAKE A DECISION: REVENUE** The revenues for Sonic Corporation were $151.1 million in 1996 and $693.3 million in 2006. *(Source: Sonic Corporation)*

(a) Use an exponential growth model to estimate the revenue in 2011.

(b) Use a linear model to estimate the 2011 revenue.

(T) (c) Use a graphing utility to graph the models from parts (a) and (b). Which model is more accurate?

(T) **42. MAKE A DECISION: SALES** The sales for exercise equipment in the United States were $1824 million in 1990 and $5112 million in 2005. *(Source: National Sporting Goods Association)*

(a) Use the *regression* feature of a graphing utility to find an exponential growth model and a linear model for the data.

(b) Use the exponential growth model to estimate the sales in 2011.

(c) Use the linear model to estimate the sales in 2011.

(d) Use a graphing utility to graph the models from part (a). Which model is more accurate?

**43. Sales** The cumulative sales $S$ (in thousands of units) of a new product after it has been on the market for $t$ years are modeled by

$$S = Ce^{k/t}.$$

During the first year, 5000 units were sold. The saturation point for the market is 30,000 units. That is, the limit of $S$ as $t \to \infty$ is 30,000.

(a) Solve for $C$ and $k$ in the model.

(b) How many units will be sold after 5 years?

(T) (c) Use a graphing utility to graph the sales function.

**44. Sales** The cumulative sales $S$ (in thousands of units) of a new product after it has been on the market for $t$ years are modeled by

$$S = 30(1 - 3^{kt}).$$

During the first year, 5000 units were sold.

(a) Solve for $k$ in the model.

(b) What is the saturation point for this product?

(c) How many units will be sold after 5 years?

(T) (d) Use a graphing utility to graph the sales function.

**45. Learning Curve** The management of a factory finds that the maximum number of units a worker can produce in a day is 30. The learning curve for the number of units $N$ produced per day after a new employee has worked $t$ days is modeled by $N = 30(1 - e^{kt})$. After 20 days on the job, a worker is producing 19 units in a day. How many days should pass before this worker is producing 25 units per day?

**46. Learning Curve** The management in Exercise 45 requires that a new employee be producing at least 20 units per day after 30 days on the job.

(a) Find a learning curve model that describes this minimum requirement.

(b) Find the number of days before a minimal achiever is producing 25 units per day.

**47. Profit** Because of a slump in the economy, a company finds that its annual profits have dropped from $742,000 in 1998 to $632,000 in 2000. If the profit follows an exponential pattern of decline, what is the expected profit for 2003? (Let $t = 0$ correspond to 1998.)

**48. Revenue** A small business assumes that the demand function for one of its new products can be modeled by $p = Ce^{kx}$. When $p = \$45$, $x = 1000$ units, and when $p = \$40$, $x = 1200$ units.

(a) Solve for $C$ and $k$.

(b) Find the values of $x$ and $p$ that will maximize the revenue for this product.

**49. Revenue** Repeat Exercise 48 given that when $p = \$5$, $x = 300$ units, and when $p = \$4$, $x = 400$ units.

**50. Forestry** The value $V$ (in dollars) of a tract of timber can be modeled by $V = 100,000e^{0.75\sqrt{t}}$, where $t = 0$ corresponds to 1990. If money earns interest at a rate of 4%, compounded continuously, then the present value $A$ of the timber at any time $t$ is $A = Ve^{-0.04t}$. Find the year in which the timber should be harvested to maximize the present value.

**51. Forestry** Repeat Exercise 50 using the model

$$V = 100,000e^{0.6\sqrt{t}}.$$

(T) **52. MAKE A DECISION: MODELING DATA** The table shows the population $P$ (in millions) of the United States from 1960 through 2005. *(Source: U.S. Census Bureau)*

| Year | 1960 | 1970 | 1980 | 1990 | 2000 | 2005 |
|------|------|------|------|------|------|------|
| Population, $P$ | 181 | 205 | 228 | 250 | 282 | 297 |

(a) Use the 1960 and 1970 data to find an exponential model $P_1$ for the data. Let $t = 0$ represent 1960.

(b) Use a graphing utility to find an exponential model $P_2$ for the data. Let $t = 0$ represent 1960.

(c) Use a graphing utility to plot the data and graph both models in the same viewing window. Compare the actual data with the predictions. Which model is more accurate?

**53. Extended Application** To work an extended application analyzing the revenue per share for Target Corporation from 1990 through 2005, visit this text's website at *college.hmco.com.* *(Data Source: Target Corporation)*

## Algebra Review

### Solving Exponential and Logarithmic Equations

To find the extrema or points of inflection of an exponential or logarithmic function, you must know how to solve exponential and logarithmic equations. A few examples are given on page 321. Some additional examples are presented in this Algebra Review.

As with all equations, remember that your basic goal is to isolate the variable on one side of the equation. To do this, you use inverse operations. For instance, to get rid of an exponential expression such as $e^{2x}$, take the natural log of each side and use the property $\ln e^{2x} = 2x$. Similarly, to get rid of a logarithmic expression such as $\log_2 3x$, exponentiate each side and use the property $2^{\log_2 3x} = 3x$.

---

**_Example 1_**    **Solving Exponential Equations**

Solve each exponential equation.

**a.** $25 = 5e^{7t}$    **b.** $80{,}000 = 100{,}000e^{k(4)}$    **c.** $300 = \left(\dfrac{100}{e^{2k}}\right)e^{4k}$

**SOLUTION**

**a.**    $25 = 5e^{7t}$            Write original equation.

       $5 = e^{7t}$            Divide each side by 5.

     $\ln 5 = \ln e^{7t}$       Take natural log of each side.

     $\ln 5 = 7t$          Apply the property $\ln e^a = a$.

   $\frac{1}{7}\ln 5 = t$          Divide each side by 7.

**b.** $80{,}000 = 100{,}000e^{k(4)}$       Example 4, page 340

        $0.8 = e^{4k}$           Divide each side by 100,000.

     $\ln 0.8 = \ln e^{4k}$       Take natural log of each side.

     $\ln 0.8 = 4k$         Apply the property $\ln e^a = a$.

   $\frac{1}{4}\ln 0.8 = k$        Divide each side by 4.

**c.**    $300 = \left(\dfrac{100}{e^{2k}}\right)e^{4k}$       Example 2, page 337

    $300 = (100)\dfrac{e^{4k}}{e^{2k}}$       Rewrite product.

    $300 = 100e^{4k-2k}$       To divide powers, subtract exponents.

    $300 = 100e^{2k}$         Simplify.

      $3 = e^{2k}$           Divide each side by 100.

    $\ln 3 = \ln e^{2k}$        Take natural log of each side.

    $\ln 3 = 2k$          Apply the property $\ln e^a = a$.

  $\frac{1}{2}\ln 3 = k$          Divide each side by 2.

### *Example 2* Solving Logarithmic Equations

Solve each logarithmic equation.

**a.** $\ln x = 2$     **b.** $5 + 2 \ln x = 4$

**c.** $2 \ln 3x = 4$     **d.** $\ln x - \ln(x - 1) = 1$

**SOLUTION**

**a.**

| | |
|---|---|
| $\ln x = 2$ | Write original equation. |
| $e^{\ln x} = e^2$ | Exponentiate each side. |
| $x = e^2$ | Apply the property $e^{\ln a} = a$. |

**b.**

| | |
|---|---|
| $5 + 2 \ln x = 4$ | Write original equation. |
| $2 \ln x = -1$ | Subtract 5 from each side. |
| $\ln x = -\dfrac{1}{2}$ | Divide each side by 2. |
| $e^{\ln x} = e^{-1/2}$ | Exponentiate each side. |
| $x = e^{-1/2}$ | Apply the property $e^{\ln a} = a$. |

**c.**

| | |
|---|---|
| $2 \ln 3x = 4$ | Write original equation. |
| $\ln 3x = 2$ | Divide each side by 2. |
| $e^{\ln 3x} = e^2$ | Exponentiate each side. |
| $3x = e^2$ | Apply the property $e^{\ln a} = a$. |
| $x = \tfrac{1}{3}e^2$ | Divide each side by 3. |

**d.**

| | |
|---|---|
| $\ln x - \ln(x - 1) = 1$ | Write original equation. |
| $\ln \dfrac{x}{x - 1} = 1$ | $\ln m - \ln n = \ln(m/n)$ |
| $e^{\ln(x/x - 1)} = e^1$ | Exponentiate each side. |
| $\dfrac{x}{x - 1} = e^1$ | Apply the property $e^{\ln a} = a$. |
| $x = ex - e$ | Multiply each side by $x - 1$. |
| $x - ex = -e$ | Subtract $ex$ from each side. |
| $x(1 - e) = -e$ | Factor. |
| $x = \dfrac{-e}{1 - e}$ | Divide each side by $1 - e$. |
| $x = \dfrac{e}{e - 1}$ | Simplify. |

**STUDY TIP**

Because the domain of a logarithmic function generally does not include all real numbers, be sure to check for extraneous solutions.

# Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 348.
Answers to odd-numbered Review Exercises are given in the back of the text.*

| Section 4.1 | Review Exercises |
|---|---|
| ■ Use the properties of exponents to evaluate and simplify exponential expressions and functions. | *1–16* |

$$a^0 = 1, \qquad a^x a^y = a^{x+y}, \qquad \frac{a^x}{a^y} = a^{x-y}, \qquad (a^x)^y = a^{xy}$$

$$(ab)^x = a^x b^x, \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}, \qquad a^{-x} = \frac{1}{a^x}$$

| | |
|---|---|
| ■ Use properties of exponents to answer questions about real life. | *17, 18* |

| Section 4.2 | |
|---|---|
| ■ Sketch the graphs of exponential functions. | *19–28* |
| ■ Evaluate limits of exponential functions in real life. | *29, 30* |
| ■ Evaluate and graph functions involving the natural exponential function. | *31–34* |
| ■ Graph logistic growth functions. | *35, 36* |
| ■ Solve compound interest problems. | *37–40* |

$$A = P(1 + r/n)^{nt}, \quad A = Pe^{rt}$$

| | |
|---|---|
| ■ Solve effective rate of interest problems. | *41, 42* |

$$r_{eff} = (1 + r/n)^n - 1$$

| | |
|---|---|
| ■ Solve present value problems. | *43, 44* |

$$P = \frac{A}{(1 + r/n)^{nt}}$$

| | |
|---|---|
| ■ Answer questions involving the natural exponential function as a real-life model. | *45, 46* |

| Section 4.3 | |
|---|---|
| ■ Find the derivatives of natural exponential functions. | *47–54* |

$$\frac{d}{dx}[e^x] = e^x, \quad \frac{d}{dx}[e^u] = e^u \frac{du}{dx}$$

| | |
|---|---|
| ■ Use calculus to analyze the graphs of functions that involve the natural exponential function. | *55–62* |

| Section 4.4 | |
|---|---|
| ■ Use the definition of the natural logarithmic function to write exponential equations in logarithmic form, and vice versa. | *63–66* |

$$\ln x = b \quad \text{if and only if} \quad e^b = x.$$

---

* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

## Section 4.4   (continued)                                    Review Exercises

- Sketch the graphs of natural logarithmic functions.                              *67–70*

- Use properties of logarithms to expand and condense logarithmic expressions.      *71–76*

$$\ln xy = \ln x + \ln y, \quad \ln \frac{x}{y} = \ln x - \ln y, \quad \ln x^n = n \ln x$$

- Use inverse properties of exponential and logarithmic functions to solve exponential     *77–92*
  and logarithmic equations.

$$\ln e^x = x, \quad e^{\ln x} = x$$

- Use properties of natural logarithms to answer questions about real life.         *93, 94*

## Section 4.5

- Find the derivatives of natural logarithmic functions.                            *95–108*

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad \frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$$

- Use calculus to analyze the graphs of functions that involve the natural logarithmic    *109–112*
  function.

- Use the definition of logarithms to evaluate logarithmic expressions involving other bases.    *113–116*

$$\log_a x = b \quad \text{if and only if} \quad a^b = x$$

- Use the change-of-base formula to evaluate logarithmic expressions involving other bases.   *117–120*

$$\log_a x = \frac{\ln x}{\ln a}$$

- Find the derivatives of exponential and logarithmic functions involving other bases.    *121–124*

$$\frac{d}{dx}[a^x] = (\ln a)a^x, \quad \frac{d}{dx}[a^u] = (\ln a)a^u\frac{du}{dx}$$

$$\frac{d}{dx}[\log_a x] = \left(\frac{1}{\ln a}\right)\frac{1}{x}, \quad \frac{d}{dx}[\log_a u] = \left(\frac{1}{\ln a}\right)\left(\frac{1}{u}\right)\frac{du}{dx}$$

- Use calculus to answer questions about real-life rates of change.                 *125, 126*

## Section 4.6

- Use exponential growth and decay to model real-life situations.                   *127–132*

## Study Strategies

- **Classifying Differentiation Rules**   Differentiation rules fall into two basic classes: (1) general rules that apply to all differentiable functions; and (2) specific rules that apply to special types of functions. At this point in the course, you have studied six general rules: the Constant Rule, the Constant Multiple Rule, the Sum Rule, the Difference Rule, the Product Rule, and the Quotient Rule. Although these rules were introduced in the context of algebraic functions, remember that they can also be used with exponential and logarithmic functions. You have also studied three specific rules: the Power Rule, the derivative of the natural exponential function, and the derivative of the natural logarithmic function. Each of these rules comes in two forms: the "simple" version, such as $D_x[e^x] = e^x$, and the Chain Rule version, such as $D_x[e^u] = e^u(du/dx)$.

- **To Memorize or Not to Memorize?**   When studying mathematics, you need to memorize some formulas and rules. Much of this will come from practice—the formulas that you use most often will be committed to memory. Some formulas, however, are used only infrequently. With these, it is helpful to be able to *derive* the formula from a *known* formula. For instance, knowing the Log Rule for differentiation and the change-of-base formula, $\log_a x = (\ln x)/(\ln a)$, allows you to derive the formula for the derivative of a logarithmic function to base $a$.

## Review Exercises

In Exercises 1–4, evaluate the expression.

**1.** $32^{3/5}$

**2.** $25^{3/2}$

**3.** $\left(\frac{1}{16}\right)^{-3/2}$

**4.** $\left(\frac{27}{8}\right)^{-1/3}$

In Exercises 5–12, use the properties of exponents to simplify the expression.

**5.** $\left(\frac{9}{16}\right)^{0}$

**6.** $(9^{1/3})(3^{1/3})$

**7.** $\frac{6^3}{36^2}$

**8.** $\frac{1}{4}\left(\frac{1}{2}\right)^{-3}$

**9.** $(e^2)^5$

**10.** $\frac{e^6}{e^4}$

**11.** $(e^{-1})(e^4)$

**12.** $(e^{1/2})(e^3)$

(T) In Exercises 13–16, evaluate the function for the indicated value of $x$. If necessary, use a graphing utility, rounding your answers to three decimal places.

**13.** $f(x) = 2^{x+3}, \quad x = 4$

**14.** $f(x) = 4^{x-1}, \quad x = -2$

**15.** $f(x) = 1.02^x, \quad x = 10$

**16.** $f(x) = 1.12^x, \quad x = 1.3$

**17. Revenue**    The revenues $R$ (in millions of dollars) for California Pizza Kitchen from 1999 through 2005 can be modeled by

$$R = 39.615(1.183)^t$$

where $t = 9$ corresponds to 1999. *(Source: California Pizza Kitchen, Inc.)*

(a) Use this model to estimate the net profits in 1999, 2003, and 2005.

(b) Do you think the model will be valid for years beyond 2005? Explain your reasoning.

**18. Property Value**    Suppose that the value of a piece of property doubles every 12 years. If you buy the property for $55,000, its value $t$ years after the date of purchase should be

$$V(t) = 55,000(2)^{t/12}.$$

Use the model to approximate the value of the property (a) 4 years and (b) 25 years after it is purchased.

In Exercises 19–28, sketch the graph of the function.

**19.** $f(x) = 9^{x/2}$

**20.** $g(x) = 16^{3x/2}$

**21.** $f(t) = \left(\frac{1}{6}\right)^t$

**22.** $g(t) = \left(\frac{1}{3}\right)^{-t}$

**23.** $f(x) = \left(\frac{1}{2}\right)^{2x} + 4$

**24.** $g(x) = \left(\frac{2}{3}\right)^{2x} + 1$

**25.** $f(x) = e^{-x} + 1$

**26.** $g(x) = e^{2x} - 1$
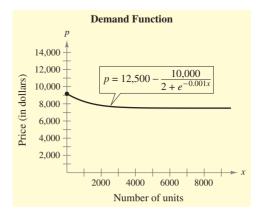
**27.** $f(x) = 1 - e^x$

**28.** $g(x) = 2 + e^{x-1}$

**29. Demand**    The demand function for a product is given by

$$p = 12{,}500 - \frac{10{,}000}{2 + e^{-0.001x}}$$
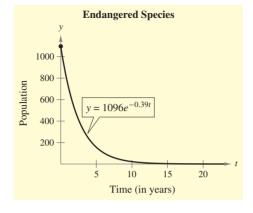
where $p$ is the price per unit and $x$ is the number of units produced (see figure). What is the limit of the price as $x$ increases without bound? Explain what this means in the context of the problem.

**Demand Function**



**30. Biology: Endangered Species**    Biologists consider a species of a plant or animal to be endangered if it is expected to become extinct in less than 20 years. The population $y$ of a certain species is modeled by

$$y = 1096e^{-0.39t}$$

(see figure). Is this species endangered? Explain your reasoning.

**Endangered Species**

In Exercises 31–34, evaluate the function at each indicated value.

**31.** $f(x) = 5e^{x-1}$

   (a) $x = 2$    (b) $x = \frac{1}{2}$    (c) $x = 10$

**32.** $f(t) = e^{4t} - 2$

   (a) $t = 0$    (b) $t = 2$    (c) $t = -\frac{3}{4}$

**33.** $g(t) = 6e^{-0.2t}$

   (a) $t = 17$    (b) $t = 50$    (c) $t = 100$

**34.** $g(x) = \dfrac{24}{1 + e^{-0.3x}}$

   (a) $x = 0$    (b) $x = 300$    (c) $x = 1000$

**35. Biology** A lake is stocked with 500 fish and the fish population $P$ begins to increase according to the logistic growth model

$$P = \frac{10,000}{1 + 19e^{-t/5}}, \quad t \geq 0$$

   where $t$ is measured in months.

(T) (a) Use a graphing utility to graph the function.

   (b) Estimate the number of fish in the lake after 4 months.

   (c) Does the population have a limit as $t$ increases without bound? Explain your reasoning.

   (d) After how many months is the population increasing most rapidly? Explain your reasoning.

**36. Medicine** On a college campus of 5000 students, the spread of a flu virus through the student body is modeled by

$$P = \frac{5000}{1 + 4999e^{-0.8t}}, \quad t \geq 0$$

   where $P$ is the total number of infected people and $t$ is the time, measured in days.

(T) (a) Use a graphing utility to graph the function.

   (b) How many students will be infected after 5 days?

   (c) According to this model, will all the students on campus become infected with the flu? Explain your reasoning.

In Exercises 37 and 38, complete the table to determine the balance $A$ when $P$ dollars is invested at an annual rate of $r$ for $t$ years, compounded $n$ times per year.

| $n$ | 1 | 2 | 4 | 12 | 365 | Continuous compounding |
|---|---|---|---|---|---|---|
| $A$ | | | | | | |

**37.** $P = \$1000$, $r = 4\%$, $t = 5$ years

**38.** $P = \$7000$, $r = 6\%$, $t = 20$ years

In Exercises 39 and 40, \$2000 is deposited in an account. Decide which account, (a) or (b), will have the greater balance after 10 years.

**39.** (a) 5%, compounded continuously

   (b) 6%, compounded quarterly

**40.** (a) $6\frac{1}{2}\%$, compounded monthly

   (b) $6\frac{1}{4}\%$, compounded continuously

**Effective Rate** In Exercises 41 and 42, find the effective rate of interest corresponding to a nominal rate $r$, compounded (a) quarterly and (b) monthly.

**41.** $r = 6\%$    **42.** $r = 8.25\%$

**43. Present Value** How much should be deposited in an account paying 5% interest compounded quarterly in order to have a balance of \$12,000 three years from now?

**44. Present Value** How much should be deposited in an account paying 8% interest compounded monthly in order to have a balance of \$20,000 five years from now?

**45. Vital Statistics** The population $P$ (in millions) of people 65 years old and over in the United States from 1990 through 2005 can be modeled by

$$P = 29.7e^{0.01t}, \quad 0 \leq t \leq 15$$

   where $t = 0$ corresponds to 1990. Use this model to estimate the populations of people 65 years old and over in 1990, 2000, and 2005. *(Source: U.S. Census Bureau)*

**46. Revenue** The revenues $R$ (in millions of dollars per year) for Papa John's International from 1998 through 2005 can be modeled by

$$R = -6310 + 1752.5t - 139.23t^2 + 3.634t^3$$
$$+ 0.000017e^t, \quad 8 \leq t \leq 15$$

   where $t = 8$ corresponds to 1998. Use this model to estimate the revenues for Papa John's in 1998, 2002, and 2005. *(Source: Papa John's International)*

In Exercises 47–54, find the derivative of the function.

**47.** $y = 4e^{x^2}$    **48.** $y = 4e^{\sqrt{x}}$

**49.** $y = \dfrac{x}{e^{2x}}$    **50.** $y = x^2e^x$

**51.** $y = \sqrt{4e^{4x}}$    **52.** $y = \sqrt[3]{2e^{3x}}$

**53.** $y = \dfrac{5}{1 + e^{2x}}$    **54.** $y = \dfrac{10}{1 - 2e^x}$

In Exercises 55–62, graph and analyze the function. Include any relative extrema, points of inflection, and asymptotes in your analysis.

**55.** $f(x) = 4e^{-x}$    **56.** $f(x) = 2e^{x^2}$

**57.** $f(x) = x^3e^x$    **58.** $f(x) = \dfrac{e^x}{x^2}$

**59.** $f(x) = \dfrac{1}{xe^x}$

**60.** $f(x) = \dfrac{x^2}{e^x}$

**61.** $f(x) = xe^{2x}$

**62.** $f(x) = xe^{-2x}$

In Exercises 63 and 64, write the logarithmic equation as an exponential equation.

**63.** $\ln 12 = 2.4849\ldots$

**64.** $\ln 0.6 = -0.5108\ldots$

In Exercises 65 and 66, write the exponential equation as a logarithmic equation.

**65.** $e^{1.5} = 4.4816\ldots$

**66.** $e^{-4} = 0.0183\ldots$

In Exercises 67–70, sketch the graph of the function.

**67.** $y = \ln(4 - x)$

**68.** $y = 5 + \ln x$

**69.** $y = \ln \dfrac{x}{3}$

**70.** $y = -2 \ln x$

In Exercises 71–76, use the properties of logarithms to write the expression as a sum, difference, or multiple of logarithms.

**71.** $\ln \sqrt{x^2(x - 1)}$

**72.** $\ln \sqrt[3]{x^2 - 1}$

**73.** $\ln \dfrac{x^2}{(x + 1)^3}$

**74.** $\ln \dfrac{x^2}{x^2 + 1}$

**75.** $\ln\left(\dfrac{1 - x}{3x}\right)^3$

**76.** $\ln\left(\dfrac{x - 1}{x + 1}\right)^2$

In Exercises 77–92, solve the equation for x.

**77.** $e^{\ln x} = 3$

**78.** $e^{\ln(x + 2)} = 5$

**79.** $\ln x = 3e^{-1}$

**80.** $\ln x = 2e^5$

**81.** $\ln 2x - \ln(3x - 1) = 0$

**82.** $\ln x - \ln(x + 1) = 2$

**83.** $e^{2x - 1} - 6 = 0$

**84.** $4e^{2x - 3} - 5 = 0$

**85.** $\ln x + \ln(x - 3) = 0$

**86.** $2 \ln x + \ln(x - 2) = 0$

**87.** $e^{-1.386x} = 0.25$

**88.** $e^{-0.01x} - 5.25 = 0$

**89.** $100(1.21)^x = 110$

**90.** $500(1.075)^{120x} = 100{,}000$

**91.** $\dfrac{40}{1 - 5e^{-0.01x}} = 200$

**92.** $\dfrac{50}{1 - 2e^{-0.001x}} = 1000$

**93.** *MAKE A DECISION: HOME MORTGAGE*   The monthly payment $M$ for a home mortgage of $P$ dollars for $t$ years at an annual interest rate $r$ is given by

$$M = P\left\{\dfrac{\dfrac{r}{12}}{1 - \left[\dfrac{1}{(r/12) + 1}\right]^{12t}}\right\}.$$

(T) (a) Use a graphing utility to graph the model when $P = \$150{,}000$ and $r = 0.075$.

(b) You are given a choice of a 20-year term or a 30-year term. Which would you choose? Explain your reasoning.

**94.** **Hourly Wages**   The average hourly wages $w$ in the United States from 1990 through 2005 can be modeled by

$$w = 8.25 + 0.681t - 0.0105t^2 + 1.94366e^{-t}$$

where $t = 0$ corresponds to 1990. *(Source: U.S. Bureau of Labor Statistics)*

(T) (a) Use a graphing utility to graph the model.

(b) Use the model to determine the year in which the average hourly wage was \$12.

(c) For how many years past 2005 do you think this equation might be a good model for the average hourly wage? Explain your reasoning.

In Exercises 95–108, find the derivative of the function.

**95.** $f(x) = \ln 3x^2$

**96.** $y = \ln \sqrt{x}$

**97.** $y = \ln \dfrac{x(x - 1)}{x - 2}$

**98.** $y = \ln \dfrac{x^2}{x + 1}$

**99.** $f(x) = \ln e^{2x + 1}$

**100.** $f(x) = \ln e^{x^2}$

**101.** $y = \dfrac{\ln x}{x^3}$

**102.** $y = \dfrac{x^2}{\ln x}$

**103.** $y = \ln(x^2 - 2)^{2/3}$

**104.** $y = \ln \sqrt[3]{x^3 + 1}$

**105.** $f(x) = \ln\left(x^2 \sqrt{x + 1}\right)$

**106.** $f(x) = \ln \dfrac{x}{\sqrt{x + 1}}$

**107.** $y = \ln \dfrac{e^x}{1 + e^x}$

**108.** $y = \ln\left(e^{2x} \sqrt{e^{2x} - 1}\right)$

In Exercises 109–112, graph and analyze the function. Include any relative extrema and points of inflection in your analysis.

**109.** $y = \ln(x + 3)$

**110.** $y = \dfrac{8 \ln x}{x^2}$

**111.** $y = \ln \dfrac{10}{x + 2}$

**112.** $y = \ln \dfrac{x^2}{9 - x^2}$

In Exercises 113–116, evaluate the logarithm.

**113.** $\log_7 49$

**114.** $\log_2 32$

**115.** $\log_{10} 1$

**116.** $\log_4 \frac{1}{64}$

In Exercises 117–120, use the change-of-base formula to evaluate the logarithm. Round the result to three decimal places.

**117.** $\log_5 13$

**118.** $\log_4 18$

**119.** $\log_{16} 64$

**120.** $\log_4 125$

In Exercises 121–124, find the derivative of the function.
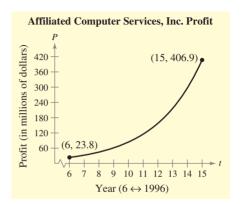
**121.** $y = \log_3(2x - 1)$

**122.** $y = \log_{10} \dfrac{3}{x}$

**123.** $y = \log_2 \dfrac{1}{x^2}$

**124.** $y = \log_{16}(x^2 - 3x)$

**125. Depreciation** After $t$ years, the value $V$ of a car purchased for \$25,000 is given by

$$V = 25{,}000(0.75)^t.$$

(a) Sketch a graph of the function and determine the value of the car 2 years after it was purchased.

(b) Find the rates of change of $V$ with respect to $t$ when $t = 1$ and when $t = 4$.

(c) After how many years will the car be worth \$5000?

**126. Inflation Rate** If the annual rate of inflation averages 4% over the next 10 years, then the approximate cost of goods or services $C$ during any year in that decade will be given by

$$C = P(1.04)^t$$
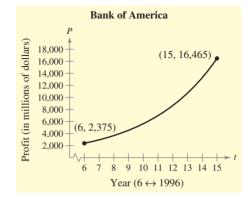
where $t$ is the time in years and $P$ is the present cost.

(a) The price of an oil change is presently \$24.95. Estimate the price of an oil change 10 years from now.

(b) Find the rate of change of $C$ with respect to $t$ when $t = 1$.

**127. Medical Science** A medical solution contains 500 milligrams of a drug per milliliter when the solution is prepared. After 40 days, it contains only 300 milligrams per milliliter. Assuming that the rate of decomposition is proportional to the concentration present, find an equation giving the concentration $A$ after $t$ days.

**128. Population Growth** A population is growing continuously at the rate of $2\frac{1}{2}\%$ per year. Find the time necessary for the population to (a) double in size and (b) triple in size.

**129. Radioactive Decay** A sample of radioactive waste is taken from a nuclear plant. The sample contains 50 grams of strontium-90 at time $t = 0$ years and 42.031 grams after 7 years. What is the half-life of strontium-90?

**130. Radioactive Decay** The half-life of cobalt-60 is 5.2 years. Find the time it would take for a sample of 0.5 gram of cobalt-60 to decay to 0.1 gram.

**131. Profit** The profit $P$ (in millions of dollars) for Affiliated Computer Services, Inc. was \$23.8 million in 1996 and \$406.9 million in 2005 (see figure). Use an exponential growth model to predict the profit in 2008. *(Source: Affiliated Computer Services, Inc.)*



**Affiliated Computer Services, Inc. Profit**

**132. Profit** The profit $P$ (in millions of dollars) for Bank of America was \$2375 million in 1996 and \$16,465 million in 2005 (see figure). Use an exponential growth model to predict the profit in 2008. *(Source: Bank of America)*



**Bank of America**

## Chapter Test

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, use properties of exponents to simplify the expression.

**1.** $3^2(3^{-2})$    **2.** $\left(\dfrac{2^3}{2^{-5}}\right)^{-1}$    **3.** $(e^{1/2})(e^4)$    **4.** $(e^3)(e^{-1})$

(T) In Exercises 5–10, use a graphing utility to graph the function.

**5.** $f(x) = 5^{x-2}$    **6.** $f(x) = 4^{-x}$    **7.** $f(x) = 3^{x-3}$

**8.** $f(x) = 8 + \ln x^2$    **9.** $f(x) = \ln(x - 5)$    **10.** $f(x) = 0.5 \ln x$

In Exercises 11–13, use the properties of logarithms to write the expression as a sum, difference , or multiple of logarithms.

**11.** $\ln \dfrac{3}{2}$    **12.** $\ln \sqrt{x + y}$    **13.** $\ln \dfrac{x + 1}{y}$

In Exercises 14–16, condense the logarithmic expression.

**14.** $\ln y + \ln(x + 1)$    **15.** $3 \ln 2 - 2 \ln(x - 1)$

**16.** $2 \ln x + \ln y - \ln(z + 4)$

In Exercises 17–19, solve the equation.

**17.** $e^{x-1} = 9$    **18.** $10e^{2x+1} = 900$    **19.** $50(1.06)^x = 1500$

**20.** A deposit of \$500 is made to an account that earns interest at an annual rate of 4%. How long will it take for the balance to double if the interest is compounded (a) annually, (b) monthly, (c) daily, and (d) continuously?

In Exercises 21–24, find the derivative of the function.

**21.** $y = e^{-3x} + 5$    **22.** $y = 7e^{x+2} + 2x$

**23.** $y = \ln(3 + x^2)$    **24.** $y = \ln \dfrac{5x}{x + 2}$

**25.** The gross revenues $R$ (in millions of dollars) of symphony orchestras in the United States from 1997 through 2004 can be modeled by

$$R = -93.4 + 349.36 \ln t$$

where $t = 7$ corresponds to 1997. *(Source: American Symphony Orchestra League, Inc.)*

(a) Use this model to estimate the gross revenues in 2004.

(b) At what rate were the gross revenues changing in 2004?

**26.** What percent of a present amount of radioactive radium ($^{226}$Ra) will remain after 1200 years? (The half-life of $^{266}$Ra is 1599 years.)

**27.** A population is growing continuously at the rate of 1.75% per year. Find the time necessary for the population to double in size.

# Integration and Its Applications

Natalie Fobes/Getty Images

Integration can be used to solve business problems, such as estimating the surface area of an oil spill. (See Chapter 5 Review Exercises, Exercise 101.)

## Applications

Integration has many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Make a Decision: Internet Users, Exercise 79, page 364
- Average Salary, Exercise 61, page 380
- Biology, Exercise 97, page 393
- Make a Decision: Budget Deficits, Exercise 46, page 401
- Consumer Trends, Exercise 51, page 402

# Antiderivatives and Indefinite Integrals

- Understand the definition of antiderivative.
- Use indefinite integral notation for antiderivatives.
- Use basic integration rules to find antiderivatives.
- Use initial conditions to find particular solutions of indefinite integrals.
- Use antiderivatives to solve real-life problems.

## Antiderivatives

Up to this point in the text, you have been concerned primarily with this problem: given a function, find its derivative. Many important applications of calculus involve the inverse problem: given the derivative of a function, find the function. For example, suppose you are given

$$f'(x) = 2, \quad g'(x) = 3x^2, \quad \text{and} \quad s'(t) = 4t.$$

Your goal is to determine the functions $f$, $g$, and $s$. By making educated guesses, you might come up with the following functions.

$$f(x) = 2x \quad \text{because} \quad \frac{d}{dx}[2x] = 2.$$

$$g(x) = x^3 \quad \text{because} \quad \frac{d}{dx}[x^3] = 3x^2.$$

$$s(t) = 2t^2 \quad \text{because} \quad \frac{d}{dt}[2t^2] = 4t.$$

This operation of determining the original function from its derivative is the inverse operation of differentiation. It is called **antidifferentiation.**

---

**Definition of Antiderivative**

A function $F$ is an **antiderivative** of a function $f$ if for every $x$ in the domain of $f$, it follows that $F'(x) = f(x)$.

---

If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$, where $C$ is any constant, is also an antiderivative of $f(x)$. For example,

$$F(x) = x^3, \quad G(x) = x^3 - 5, \quad \text{and} \quad H(x) = x^3 + 0.3$$

are all antiderivatives of $3x^2$ because the derivative of each is $3x^2$. As it turns out, *all* antiderivatives of $3x^2$ are of the form $x^3 + C$. So, the process of antidifferentiation does not determine a single function, but rather a *family* of functions, each differing from the others by a constant.

---

**STUDY TIP**

In this text, the phrase "$F(x)$ is an antiderivative of $f(x)$" is used synonymously with "$F$ is an antiderivative of $f$."
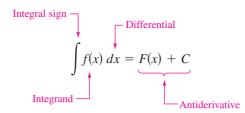
## Notation for Antiderivatives and Indefinite Integrals

The antidifferentiation process is also called **integration** and is denoted by the symbol

$$\int$$

Integral sign

which is called an **integral sign.** The symbol

$$\int f(x)\, dx$$

Indefinite integral

is the **indefinite integral** of $f(x)$, and it denotes the family of antiderivatives of $f(x)$. That is, if $F'(x) = f(x)$ for all $x$, then you can write

Integral sign ⎯

Differential

$$\int f(x)\, dx = \underbrace{F(x) + C}$$

Integrand ⎯

Antiderivative

where $f(x)$ is the **integrand** and $C$ is the **constant of integration.** The differential $dx$ in the indefinite integral identifies the variable of integration. That is, the symbol $\int f(x)\, dx$ denotes the "antiderivative of $f$ *with respect to x*" just as the symbol $dy/dx$ denotes the "derivative of *y with respect to x.*"

**Integral Notation of Antiderivatives**

The notation

$$\int f(x)\, dx = F(x) + C$$

where $C$ is an arbitrary constant, means that $F$ is an antiderivative of $f$. That is, $F'(x) = f(x)$ for all $x$ in the domain of $f$.

*Example 1*   **Notation for Antiderivatives**

Using integral notation, you can write the three antiderivatives from the beginning of this section as shown.

**a.** $\int 2\, dx = 2x + C$    **b.** $\int 3x^2\, dx = x^3 + C$    **c.** $\int 4t\, dt = 2t^2 + C$

✓**CHECKPOINT 1**

Rewrite each antiderivative using integral notation.

**a.** $\dfrac{d}{dx}[3x] = 3$    **b.** $\dfrac{d}{dx}[x^2] = 2x$    **c.** $\dfrac{d}{dt}[3t^3] = 9t^2$   ■

## Finding Antiderivatives

The inverse relationship between the operations of integration and differentiation can be shown symbolically, as follows.

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$$      Differentiation is the inverse of integration.

$$\int f'(x)\,dx = f(x) + C$$      Integration is the inverse of differentiation.

This inverse relationship between integration and differentiation allows you to obtain integration formulas directly from differentiation formulas. The following summary lists the integration formulas that correspond to some of the differentiation formulas you have studied.

---

### Basic Integration Rules

1. $\int k\,dx = kx + C, \quad k$ is a constant.      Constant Rule

2. $\int kf(x)\,dx = k\int f(x)\,dx$      Constant Multiple Rule

3. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$      Sum Rule

4. $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$      Difference Rule

5. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$      Simple Power Rule

---

**STUDY TIP**

You will study the General Power Rule for integration in Section 5.2 and the Exponential and Log Rules in Section 5.3.

**STUDY TIP**

In Example 2(b), the integral $\int 1\,dx$ is usually shortened to the form $\int dx$.

Be sure you see that the Simple Power Rule has the restriction that $n$ cannot be $-1$. So, you *cannot* use the Simple Power Rule to evaluate the integral

$$\int \frac{1}{x}\,dx.$$

To evaluate this integral, you need the Log Rule, which is described in Section 5.3.

### Example 2   Finding Indefinite Integrals

Find each indefinite integral.

**a.** $\int \frac{1}{2}\,dx$      **b.** $\int 1\,dx$      **c.** $\int -5\,dt$

**SOLUTION**

**a.** $\int \frac{1}{2}\,dx = \frac{1}{2}x + C$      **b.** $\int 1\,dx = x + C$      **c.** $\int -5\,dt = -5t + C$

✓ **CHECKPOINT 2**

Find each indefinite integral.

**a.** $\int 5\,dx$

**b.** $\int -1\,dr$

**c.** $\int 2\,dt$

*Example 3* **Finding an Indefinite Integral**

Find $\displaystyle\int 3x \, dx$.

**SOLUTION**

$$\int 3x \, dx = 3 \int x \, dx \qquad \text{Constant Multiple Rule}$$

$$= 3 \int x^1 \, dx \qquad \text{Rewrite } x \text{ as } x^1.$$

$$= 3\left(\frac{x^2}{2}\right) + C \qquad \text{Simple Power Rule with } n = 1$$

$$= \frac{3}{2}x^2 + C \qquad \text{Simplify.}$$

✓ **CHECKPOINT 3**

Find $\displaystyle\int 5x \, dx$.
■
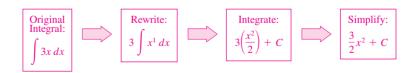
In finding indefinite integrals, a strict application of the basic integration rules tends to produce cumbersome constants of integration. For instance, in Example 3, you could have written

$$\int 3x \, dx = 3 \int x \, dx = 3\left(\frac{x^2}{2} + C\right) = \frac{3}{2}x^2 + 3C.$$

However, because $C$ represents *any* constant, it is unnecessary to write $3C$ as the constant of integration. You can simply write $\frac{3}{2}x^2 + C$.

In Example 3, note that the general pattern of integration is similar to that of differentiation.

| Original Integral: $\displaystyle\int 3x \, dx$ | $\Rightarrow$ | Rewrite: $3 \displaystyle\int x^1 \, dx$ | $\Rightarrow$ | Integrate: $3\left(\frac{x^2}{2}\right) + C$ | $\Rightarrow$ | Simplify: $\frac{3}{2}x^2 + C$ |
|---|---|---|---|---|---|---|

*Example 4* **Rewriting Before Integrating**

Find each indefinite integral.

**a.** $\displaystyle\int \frac{1}{x^3} \, dx$

**b.** $\displaystyle\int \sqrt{x} \, dx$

**SOLUTION**

| | Original Integral | Rewrite | Integrate | Simplify |
|---|---|---|---|---|
| **a.** | $\displaystyle\int \frac{1}{x^3} \, dx$ | $\displaystyle\int x^{-3} \, dx$ | $\dfrac{x^{-2}}{-2} + C$ | $-\dfrac{1}{2x^2} + C$ |
| **b.** | $\displaystyle\int \sqrt{x} \, dx$ | $\displaystyle\int x^{1/2} \, dx$ | $\dfrac{x^{3/2}}{3/2} + C$ | $\dfrac{2}{3}x^{3/2} + C$ |

✓ **CHECKPOINT 4**

Find each indefinite integral.

**a.** $\displaystyle\int \frac{1}{x^2} \, dx$ **b.** $\displaystyle\int \sqrt[3]{x} \, dx$
■

With the five basic integration rules, you can integrate *any* polynomial function, as demonstrated in the next example.

**Example 5**  **Integrating Polynomial Functions**

Find each indefinite integral.

**a.** $\int (x + 2)\, dx$    **b.** $\int (3x^4 - 5x^2 + x)\, dx$

**SOLUTION**

**a.** $\int (x + 2)\, dx = \int x\, dx + \int 2\, dx$    Apply Sum Rule.

$$= \frac{x^2}{2} + C_1 + 2x + C_2$$    Integrate.

$$= \frac{x^2}{2} + 2x + C$$    $C = C_1 + C_2$

The second line in the solution is usually omitted.

**b.** Try to identify each basic integration rule used to evaluate this integral.

$$\int (3x^4 - 5x^2 + x)\, dx = 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^3}{3}\right) + \frac{x^2}{2} + C$$

$$= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C$$

**STUDY TIP**

When integrating quotients, remember *not* to integrate the numerator and denominator separately. For instance, in Example 6, be sure you understand that

$$\int \frac{x + 1}{\sqrt{x}}\, dx = \frac{2}{3}\sqrt{x}(x + 3) + C$$

is not the same as

$$\frac{\int (x + 1)\, dx}{\int \sqrt{x}\, dx} = \frac{\frac{1}{2}x^2 + x + C_1}{\frac{2}{3}x\sqrt{x} + C_2}.$$

**Example 6**  **Rewriting Before Integrating**

Find $\int \dfrac{x + 1}{\sqrt{x}}\, dx$.

**SOLUTION**    Begin by rewriting the quotient in the integrand as a sum. Then rewrite each term using rational exponents.
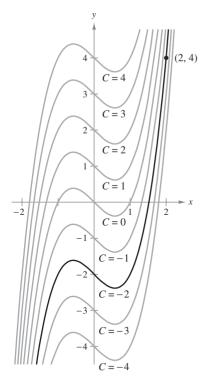
$$\int \frac{x + 1}{\sqrt{x}}\, dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx$$    Rewrite as a sum.

$$= \int (x^{1/2} + x^{-1/2})\, dx$$    Rewrite using rational exponents.

$$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$$    Apply Power Rule.

$$= \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$    Simplify.

$$= \frac{2}{3}\sqrt{x}(x + 3) + C$$    Factor.

**Algebra Review**

For help on the algebra in Example 6, see Example 1(a) in the *Chapter 5 Algebra Review*, on page 409.
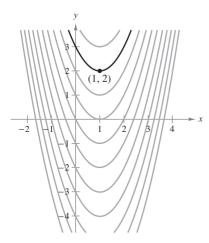
## Particular Solutions

You have already seen that the equation $y = \int f(x)\, dx$ has many solutions, each differing from the others by a constant. This means that the graphs of any two antiderivatives of $f$ are vertical translations of each other. For example, Figure 5.1 shows the graphs of several antiderivatives of the form

$$y = F(x) = \int (3x^2 - 1)\, dx = x^3 - x + C$$

for various integer values of $C$. Each of these antiderivatives is a solution of the *differential equation* $dy/dx = 3x^2 - 1$. A **differential equation** in $x$ and $y$ is an equation that involves $x, y,$ and derivatives of $y$. The **general solution** of $dy/dx = 3x^2 - 1$ is $F(x) = x^3 - x + C$.

In many applications of integration, you are given enough information to determine a **particular solution.** To do this, you only need to know the value of $F(x)$ for one value of $x$. (This information is called an **initial condition.**) For example, in Figure 5.1, there is only one curve that passes through the point $(2, 4)$. To find this curve, use the information below.

$$F(x) = x^3 - x + C \qquad \text{General solution}$$

$$F(2) = 4 \qquad \text{Initial condition}$$

By using the initial condition in the general solution, you can determine that $F(2) = 2^3 - 2 + C = 4$, which implies that $C = -2$. So, the particular solution is

$$F(x) = x^3 - x - 2. \qquad \text{Particular solution}$$



$$F(x) = x^3 - x + C$$

**FIGURE 5.1**

### Example 7    Finding a Particular Solution

Find the general solution of

$$F'(x) = 2x - 2$$

and find the particular solution that satisfies the initial condition $F(1) = 2$.

**SOLUTION**    Begin by integrating to find the general solution.

$$F(x) = \int (2x - 2)\, dx \qquad \text{Integrate } F'(x) \text{ to obtain } F(x).$$

$$= x^2 - 2x + C \qquad \text{General solution}$$

Using the initial condition $F(1) = 2$, you can write

$$F(1) = 1^2 - 2(1) + C = 2$$

which implies that $C = 3$. So, the particular solution is

$$F(x) = x^2 - 2x + 3. \qquad \text{Particular solution}$$

This solution is shown graphically in Figure 5.2. Note that each of the gray curves represents a solution of the equation $F'(x) = 2x - 2$. The black curve, however, is the only solution that passes through the point $(1, 2)$, which means that $F(x) = x^2 - 2x + 3$ is the only solution that satisfies the initial condition.



**FIGURE 5.2**

### ✓ CHECKPOINT 7

Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$. ■

## Applications

In Chapter 2, you used the general position function (neglecting air resistance) for a falling object

$$s(t) = -16t^2 + v_0 t + s_0$$

where $s(t)$ is the height (in feet) and $t$ is the time (in seconds). In the next example, integration is used to *derive* this function.

**Example 8**
**MAKE A DECISION**    **Deriving a Position Function**    Ⓡ

A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet, as shown in Figure 5.3. Derive the position function giving the height $s$ (in feet) as a function of the time $t$ (in seconds). Will the ball be in the air for more than 5 seconds?

**SOLUTION**    Let $t = 0$ represent the initial time. Then the two given conditions can be written as

$$s(0) = 80 \qquad \text{Initial height is 80 feet.}$$
$$s'(0) = 64. \qquad \text{Initial velocity is 64 feet per second.}$$

Because the acceleration due to gravity is $-32$ feet per second per second, you can integrate the acceleration function to find the velocity function as shown.

$$s''(t) = -32 \qquad \text{Acceleration due to gravity}$$

$$s'(t) = \int -32 \, dt \qquad \text{Integrate } s''(t) \text{ to obtain } s'(t).$$

$$= -32t + C_1 \qquad \text{Velocity function}$$

Using the initial velocity, you can conclude that $C_1 = 64$.

$$s'(t) = -32t + 64 \qquad \text{Velocity function}$$

$$s(t) = \int (-32t + 64) \, dt \qquad \text{Integrate } s'(t) \text{ to obtain } s(t).$$

$$= -16t^2 + 64t + C_2 \qquad \text{Position function}$$

Using the initial height, it follows that $C_2 = 80$. So, the position function is given by

$$s(t) = -16t^2 + 64t + 80. \qquad \text{Position function}$$

To find the time when the ball hits the ground, set the position function equal to 0 and solve for $t$.

$$-16t^2 + 64t + 80 = 0 \qquad \text{Set } s(t) \text{ equal to zero.}$$
$$-16(t + 1)(t - 5) = 0 \qquad \text{Factor.}$$
$$t = -1, \quad t = 5 \qquad \text{Solve for } t.$$

Because the time must be positive, you can conclude that the ball hits the ground 5 seconds after it is thrown. No, the ball was not in the air for more than 5 seconds.



$s(t) = -16t^2 + 64t + 80$
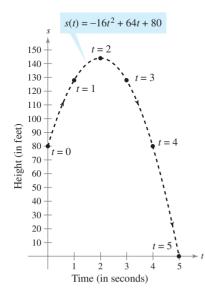
**FIGURE 5.3**

✓ **CHECKPOINT 8**

Derive the position function if a ball is thrown upward with an initial velocity of 32 feet per second from an initial height of 48 feet. When does the ball hit the ground? With what velocity does the ball hit the ground? ▪

*Example 9*   **Finding a Cost Function**   Ⓡ

The marginal cost for producing $x$ units of a product is modeled by

$$\frac{dC}{dx} = 32 - 0.04x. \qquad\qquad \text{\color{magenta}Marginal cost}$$

It costs $50 to produce one unit. Find the total cost of producing 200 units.

**SOLUTION**   To find the cost function, integrate the marginal cost function.

$$C = \int (32 - 0.04x)\, dx \qquad\qquad \text{\color{magenta}Integrate } \frac{dC}{dx} \text{ to obtain } C.$$

$$= 32x - 0.04\left(\frac{x^2}{2}\right) + K$$

$$= 32x - 0.02x^2 + K \qquad\qquad \text{\color{magenta}Cost function}$$

To solve for $K$, use the initial condition that $C = 50$ when $x = 1$.

$$50 = 32(1) - 0.02(1)^2 + K \qquad\qquad \text{\color{magenta}Substitute 50 for } C \text{ and 1 for } x.$$

$$18.02 = K \qquad\qquad \text{\color{magenta}Solve for } K.$$

So, the total cost function is given by

$$C = 32x - 0.02x^2 + 18.02 \qquad\qquad \text{\color{magenta}Cost function}$$

which implies that the cost of producing 200 units is

$$C = 32(200) - 0.02(200)^2 + 18.02$$

$$= \$5618.02.$$

✓**CHECKPOINT 9**

The marginal cost function for producing $x$ units of a product is modeled by

$$\frac{dC}{dx} = 28 - 0.02x.$$

It costs $40 to produce one unit. Find the cost of producing 200 units.  ◼

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**CONCEPT CHECK**

1. How can you check your answer to an antidifferentiation problem?

2. Write what is meant by the symbol $\int f(x)\, dx$ in words.

3. Given $\int (2x + 1)\, dx = x^2 + x + C$, identify (a) the integrand and (b) the antiderivative.

4. True or false: The antiderivative of a second-degree polynomial function is a third-degree polynomial function.

In Exercises 1–6, rewrite the expression using rational exponents.

**1.** $\dfrac{\sqrt{x}}{x}$

**2.** $\sqrt[3]{2x}(2x)$

**3.** $\sqrt{5x^3} + \sqrt{x^5}$

**4.** $\dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt[3]{x^2}}$

**5.** $\dfrac{(x+1)^3}{\sqrt{x+1}}$

**6.** $\dfrac{\sqrt{x}}{\sqrt[3]{x}}$

In Exercises 7–10, let $(x, y) = (2, 2)$, and solve the equation for $C$.

**7.** $y = x^2 + 5x + C$

**8.** $y = 3x^3 - 6x + C$

**9.** $y = -16x^2 + 26x + C$

**10.** $y = -\frac{1}{4}x^4 - 2x^2 + C$

---

## Exercises 5.1

In Exercises 1–8, verify the statement by showing that the derivative of the right side is equal to the integrand of the left side.

**1.** $\displaystyle\int \left(-\frac{9}{x^4}\right) dx = \frac{3}{x^3} + C$

**2.** $\displaystyle\int \frac{4}{\sqrt{x}}\, dx = 8\sqrt{x} + C$

**3.** $\displaystyle\int \left(4x^3 - \frac{1}{x^2}\right) dx = x^4 + \frac{1}{x} + C$

**4.** $\displaystyle\int \left(1 - \frac{1}{\sqrt[3]{x^2}}\right) dx = x - 3\sqrt[3]{x} + C$

**5.** $\displaystyle\int 2\sqrt{x}(x - 3)\, dx = \frac{4x^{3/2}(x-5)}{5} + C$

**6.** $\displaystyle\int 4\sqrt{x}(x^2 - 2)\, dx = \frac{8x^{3/2}(3x^2 - 14)}{21} + C$

**7.** $\displaystyle\int (x-2)(x+2)\, dx = \frac{1}{3}x^3 - 4x + C$

**8.** $\displaystyle\int \frac{x^2 - 1}{x^{3/2}}\, dx = \frac{2(x^2 + 3)}{3\sqrt{x}} + C$

In Exercises 9–20, find the indefinite integral and check your result by differentiation.

**9.** $\displaystyle\int 6\, dx$

**10.** $\displaystyle\int -4\, dx$

**11.** $\displaystyle\int 5t^2\, dt$

**12.** $\displaystyle\int 3t^4\, dt$

**13.** $\displaystyle\int 5x^{-3}\, dx$

**14.** $\displaystyle\int 4y^{-3}\, dy$

**15.** $\displaystyle\int du$

**16.** $\displaystyle\int dr$

**17.** $\displaystyle\int e\, dt$

**18.** $\displaystyle\int e^3\, dy$

**19.** $\displaystyle\int y^{3/2}\, dy$

**20.** $\displaystyle\int v^{-1/2}\, dv$

In Exercises 21–26, complete the table.

| Original Integral | Rewrite | Integrate | Simplify |
|---|---|---|---|
| **21.** $\displaystyle\int \sqrt[3]{x}\, dx$ | | | |
| **22.** $\displaystyle\int \frac{1}{x^2}\, dx$ | | | |
| **23.** $\displaystyle\int \frac{1}{x\sqrt{x}}\, dx$ | | | |
| **24.** $\displaystyle\int x(x^2 + 3)\, dx$ | | | |
| **25.** $\displaystyle\int \frac{1}{2x^3}\, dx$ | | | |
| **26.** $\displaystyle\int \frac{1}{(3x)^2}\, dx$ | | | |

In Exercises 27–38, find the indefinite integral and check your result by differentiation.

**27.** $\displaystyle\int (x + 3)\, dx$

**28.** $\displaystyle\int (5 - x)\, dx$

**29.** $\displaystyle\int (x^3 + 2)\, dx$

**30.** $\displaystyle\int (x^3 - 4x + 2)\, dx$

**31.** $\displaystyle\int \left(\sqrt[3]{x} - \frac{1}{2\sqrt[3]{x}}\right) dx$

**32.** $\displaystyle\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx$

**33.** $\displaystyle\int \sqrt[3]{x^2}\, dx$

**34.** $\displaystyle\int \left(\sqrt[4]{x^3} + 1\right) dx$

**35.** $\displaystyle\int \frac{1}{x^4}\, dx$

**36.** $\displaystyle\int \frac{1}{4x^2}\, dx$

**37.** $\displaystyle\int \frac{2x^3 + 1}{x^3}\, dx$

**38.** $\displaystyle\int \frac{t^2 + 2}{t^2}\, dt$

Ⓣ In Exercises 39– 44, use a symbolic integration utility to find the indefinite integral.

**39.** $\displaystyle\int u(3u^2 + 1)\, du$

**40.** $\displaystyle\int \sqrt{x}(x + 1)\, dx$

**41.** $\displaystyle\int (x + 1)(3x - 2)\, dx$

**42.** $\displaystyle\int (2t^2 - 1)^2\, dt$
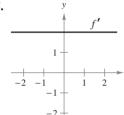
**43.** $\displaystyle\int y^2 \sqrt{y}\, dy$
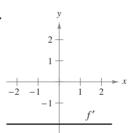
**44.** $\displaystyle\int (1 + 3t)t^2\, dt$

In Exercises 45–48, the graph of the derivative of a function is given. Sketch the graphs of *two* functions that have the given derivative. (There is more than one correct answer.)
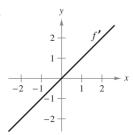
**45.**
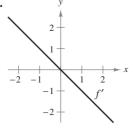


**46.**



**47.**



**48.**



In Exercises 49–54, find the particular solution $y = f(x)$ that satisfies the differential equation and initial condition.

**49.** $f'(x) = 4x; \quad f(0) = 6$

**50.** $f'(x) = \frac{1}{5}x - 2; \quad f(10) = -10$

**51.** $f'(x) = 2(x - 1); \quad f(3) = 2$

**52.** $f'(x) = (2x - 3)(2x + 3); \quad f(3) = 0$
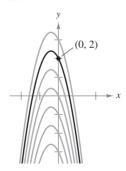
**53.** $f'(x) = \dfrac{2 - x}{x^3}, \ x > 0; \quad f(2) = \dfrac{3}{4}$
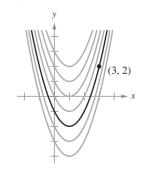
**54.** $f'(x) = \dfrac{x^2 - 5}{x^2}, \ x > 0; \quad f(1) = 2$

In Exercises 55 and 56, find the equation for y, given the derivative and the indicated point on the curve.

**55.** $\dfrac{dy}{dx} = -5x - 2$



**56.** $\dfrac{dy}{dx} = 2(x - 1)$



In Exercises 57 and 58, find the equation of the function f whose graph passes through the point.

| Derivative | Point |
|---|---|
| **57.** $f'(x) = 2x$ | $(-2, -2)$ |
| **58.** $f'(x) = 2\sqrt{x}$ | $(4, 12)$ |

In Exercises 59–62, find a function f that satisfies the conditions.

**59.** $f''(x) = 2, \quad f'(2) = 5, \quad f(2) = 10$

**60.** $f''(x) = x^2, \quad f'(0) = 6, \quad f(0) = 3$

**61.** $f''(x) = x^{-2/3}, \quad f'(8) = 6, \quad f(0) = 0$

**62.** $f''(x) = x^{-3/2}, \quad f'(1) = 2, \quad f(9) = -4$

**Cost**  In Exercises 63–66, find the cost function for the marginal cost and fixed cost.

| Marginal Cost | Fixed Cost $(x = 0)$ |
|---|---|
| **63.** $\dfrac{dC}{dx} = 85$ | \$5500 |
| **64.** $\dfrac{dC}{dx} = \dfrac{1}{50}x + 10$ | \$1000 |
| **65.** $\dfrac{dC}{dx} = \dfrac{1}{20\sqrt{x}} + 4$ | \$750 |
| **66.** $\dfrac{dC}{dx} = \dfrac{\sqrt[4]{x}}{10} + 10$ | \$2300 |

**Demand Function**  In Exercises 67 and 68, find the revenue and demand functions for the given marginal revenue. (Use the fact that $R = 0$ when $x = 0$.)

**67.** $\dfrac{dR}{dx} = 225 - 3x$

**68.** $\dfrac{dR}{dx} = 310 - 4x$

**Profit** In Exercises 69–72, find the profit function for the given marginal profit and initial condition.

| Marginal Profit | Initial Condition |
|---|---|

**69.** $\dfrac{dP}{dx} = -18x + 1650$  $P(15) = \$22{,}725$

**70.** $\dfrac{dP}{dx} = -40x + 250$  $P(5) = \$650$

**71.** $\dfrac{dP}{dx} = -24x + 805$  $P(12) = \$8000$

**72.** $\dfrac{dP}{dx} = -30x + 920$  $P(8) = \$6500$

**Vertical Motion** In Exercises 73 and 74, use $a(t) = -32$ feet per second per second as the acceleration due to gravity.

**73.** The Grand Canyon is 6000 feet deep at the deepest part. A rock is dropped from this height. Express the height $s$ of the rock as a function of the time $t$ (in seconds). How long will it take the rock to hit the canyon floor?

**74.** With what initial velocity must an object be thrown upward from the ground to reach the height of the Washington Monument (550 feet)?

**75. Cost** A company produces a product for which the marginal cost of producing $x$ units is modeled by $dC/dx = 2x - 12$, and the fixed costs are \$125.

(a) Find the total cost function and the average cost function.

(b) Find the total cost of producing 50 units.

(c) In part (b), how much of the total cost is fixed? How much is variable? Give examples of fixed costs associated with the manufacturing of a product. Give examples of variable costs.

**76. Tree Growth** An evergreen nursery usually sells a certain shrub after 6 years of growth and shaping. The growth rate during those 6 years is approximated by $dh/dt = 1.5t + 5$, where $t$ is the time in years and $h$ is the height in centimeters. The seedlings are 12 centimeters tall when planted ($t = 0$).

(a) Find the height after $t$ years.

(b) How tall are the shrubs when they are sold?

**77. MAKE A DECISION: POPULATION GROWTH** The growth rate of Horry County in South Carolina can be modeled by $dP/dt = 105.46t + 2642.7$, where $t$ is the time in years, with $t = 0$ corresponding to 1970. The county's population was 226,992 in 2005. *(Source: U.S. Census Bureau)*

(a) Find the model for Horry County's population.

(b) Use the model to predict the population in 2012. Does your answer seem reasonable? Explain your reasoning.

**78. MAKE A DECISION: VITAL STATISTICS** The rate of increase of the number of married couples $M$ (in thousands) in the United States from 1970 to 2005 can be modeled by

$$\frac{dM}{dt} = 1.218t^2 - 44.72t + 709.1$$

where $t$ is the time in years, with $t = 0$ corresponding to 1970. The number of married couples in 2005 was 59,513 thousand. *(Source: U.S. Census Bureau)*

(a) Find the model for the number of married couples in the United States.

(b) Use the model to predict the number of married couples in the United States in 2012. Does your answer seem reasonable? Explain your reasoning.

**79. MAKE A DECISION: INTERNET USERS** The rate of growth of the number of Internet users $I$ (in millions) in the world from 1991 to 2004 can be modeled by

$$\frac{dI}{dt} = -0.25t^3 + 5.319t^2 - 19.34t + 21.03$$

where $t$ is the time in years, with $t = 1$ corresponding to 1991. The number of Internet users in 2004 was 863 million. *(Source: International Telecommunication Union)*

(a) Find the model for the number of Internet users in the world.

(b) Use the model to predict the number of Internet users in the world in 2012. Does your answer seem reasonable? Explain your reasoning.

**80. Economics: Marginal Benefits and Costs** The table gives the marginal benefit and marginal cost of producing $x$ units of a product for a given company. Plot the points in each column and use the *regression* feature of a graphing utility to find a linear model for marginal benefit and a quadratic model for marginal cost. Then use integration to find the benefit $B$ and cost $C$ equations. Assume $B(0) = 0$ and $C(0) = 425$. Finally, find the intervals in which the benefit exceeds the cost of producing $x$ units, and make a recommendation for how many units the company should produce based on your findings. *(Source: Adapted from Taylor, Economics, Fifth Edition)*

| Number of units | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Marginal benefit | 330 | 320 | 290 | 270 | 250 |
| Marginal cost | 150 | 120 | 100 | 110 | 120 |

| Number of units | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| Marginal benefit | 230 | 210 | 190 | 170 | 160 |
| Marginal cost | 140 | 160 | 190 | 250 | 320 |

## Section 5.2

# Integration by Substitution and the General Power Rule

- Use the General Power Rule to find indefinite integrals.
- Use substitution to find indefinite integrals.
- Use the General Power Rule to solve real-life problems.

## The General Power Rule

In Section 5.1, you used the Simple Power Rule

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

to find antiderivatives of functions expressed as powers of $x$ alone. In this section, you will study a technique for finding antiderivatives of more complicated functions.

To begin, consider how you might find the antiderivative of $2x(x^2 + 1)^3$. Because you are hunting for a function whose derivative is $2x(x^2 + 1)^3$, you might discover the antiderivative as shown.

$$\frac{d}{dx}[(x^2 + 1)^4] = 4(x^2 + 1)^3(2x) \qquad \text{Use Chain Rule.}$$

$$\frac{d}{dx}\left[\frac{(x^2 + 1)^4}{4}\right] = (x^2 + 1)^3(2x) \qquad \text{Divide both sides by 4.}$$

$$\frac{(x^2 + 1)^4}{4} + C = \int 2x(x^2 + 1)^3 \, dx \qquad \text{Write in integral form.}$$

The key to this solution is the presence of the factor $2x$ in the integrand. In other words, this solution works because $2x$ is precisely the derivative of $(x^2 + 1)$. Letting $u = x^2 + 1$, you can write

$$\int \overbrace{(x^2 + 1)^3}^{u^3} \underbrace{2x \, dx}_{du} = \int u^3 \, du$$

$$= \frac{u^4}{4} + C.$$

This is an example of the **General Power Rule** for integration.

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**General Power Rule for Integration**

If $u$ is a differentiable function of $x$, then

$$\int u^n \frac{du}{dx} \, dx = \int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

---

When using the General Power Rule, you must first identify a factor $u$ of the integrand that is raised to a power. Then, you must show that its derivative $du/dx$ is also a factor of the integrand. This is demonstrated in Example 1.

**Example 1**    **Applying the General Power Rule**

Find each indefinite integral.

**a.** $\int 3(3x - 1)^4 \, dx$    **b.** $\int (2x + 1)(x^2 + x) \, dx$

**c.** $\int 3x^2 \sqrt{x^3 - 2} \, dx$    **d.** $\int \frac{-4x}{(1 - 2x^2)^2} \, dx$

**SOLUTION**

**STUDY TIP**

Example 1(b) illustrates a case of the General Power Rule that is sometimes overlooked—when the power is $n = 1$. In this case, the rule takes the form

$$\int u \frac{du}{dx} \, dx = \frac{u^2}{2} + C.$$

**a.** $\int 3(3x - 1)^4 \, dx = \int \overbrace{(3x - 1)^4}^{u^n} \overbrace{(3)}^{\frac{du}{dx}} \, dx$    Let $u = 3x - 1$.

$\qquad = \frac{(3x - 1)^5}{5} + C$    General Power Rule

**b.** $\int (2x + 1)(x^2 + x) \, dx = \int \overbrace{(x^2 + x)}^{u^n} \overbrace{(2x + 1)}^{\frac{du}{dx}} \, dx$    Let $u = x^2 + x$.

$\qquad = \frac{(x^2 + x)^2}{2} + C$    General Power Rule

**c.** $\int 3x^2 \sqrt{x^3 - 2} \, dx = \int \overbrace{(x^3 - 2)^{1/2}}^{u^n} \overbrace{(3x^2)}^{\frac{du}{dx}} \, dx$    Let $u = x^3 - 2$.

$\qquad = \frac{(x^3 - 2)^{3/2}}{3/2} + C$    General Power Rule

$\qquad = \frac{2}{3}(x^3 - 2)^{3/2} + C$    Simplify.

**STUDY TIP**

Remember that you can verify the result of an indefinite integral by differentiating the function. Check the answer to Example 1(d) by differentiating the function

$$F(x) = -\frac{1}{1 - 2x^2} + C.$$

$$\frac{d}{dx}\left[ -\frac{1}{1 - 2x^2} + C \right]$$

$$= \frac{-4x}{(1 - 2x^2)^2}$$

**d.** $\int \frac{-4x}{(1 - 2x^2)^2} \, dx = \int \overbrace{(1 - 2x^2)^{-2}}^{u^n} \overbrace{(-4x)}^{\frac{du}{dx}} \, dx$    Let $u = 1 - 2x^2$.

$\qquad = \frac{(1 - 2x^2)^{-1}}{-1} + C$    General Power Rule

$\qquad = -\frac{1}{1 - 2x^2} + C$    Simplify.

✓**CHECKPOINT 1**

Find each indefinite integral.

**a.** $\int (3x^2 + 6)(x^3 + 6x)^2 \, dx$    **b.** $\int 2x\sqrt{x^2 - 2} \, dx$

Many times, part of the derivative $du/dx$ is missing from the integrand, and in *some* cases you can make the necessary adjustments to apply the General Power Rule.

**Algebra Review**

For help on the algebra in Example 2, see Example 1(b) in the *Chapter 5 Algebra Review*, on page 409.

**STUDY TIP**

Try using the Chain Rule to check the result of Example 2. After differentiating $-\frac{1}{24}(3 - 4x^2)^3$ and simplifying, you should obtain the original integrand.

**STUDY TIP**

In Example 3, be sure you see that you cannot factor variable quantities outside the integral sign. After all, if this were permissible, then you could move the entire integrand outside the integral sign and eliminate the need for all integration rules except the rule $\int dx = x + C$.

✓ **CHECKPOINT 3**

Find $\int 2(3x^4 + 1)^2\, dx.$ ▪

*Example 2*  **Multiplying and Dividing by a Constant**

Find $\int x(3 - 4x^2)^2\, dx.$

**SOLUTION**   Let $u = 3 - 4x^2$. To apply the General Power Rule, you need to create $du/dx = -8x$ as a factor of the integrand. You can accomplish this by multiplying and dividing by the constant $-8$.

$$\int x(3 - 4x^2)^2\, dx = \int \left(-\frac{1}{8}\right) \overbrace{(3 - 4x^2)^2}^{u^n} \overbrace{(-8x)}^{\frac{du}{dx}}\, dx \qquad \text{Multiply and divide by } -8.$$

$$= -\frac{1}{8}\int (3 - 4x^2)^2(-8x)\, dx \qquad \text{Factor } -\tfrac{1}{8} \text{ out of integrand.}$$

$$= \left(-\frac{1}{8}\right)\frac{(3 - 4x^2)^3}{3} + C \qquad \text{General Power Rule}$$

$$= -\frac{(3 - 4x^2)^3}{24} + C \qquad \text{Simplify.}$$

✓ **CHECKPOINT 2**

Find $\int x^3(3x^4 + 1)^2\, dx.$ ▪

*Example 3*  **A Failure of the General Power Rule**

Find $\int -8(3 - 4x^2)^2\, dx.$

**SOLUTION**   Let $u = 3 - 4x^2$. As in Example 2, to apply the General Power Rule you must create $du/dx = -8x$ as a factor of the integrand. In Example 2, you could do this by multiplying and dividing by a constant, and then factoring that constant out of the integrand. This strategy doesn't work with variables. That is,

$$\int -8(3 - 4x^2)^2\, dx \neq \frac{1}{x}\int (3 - 4x^2)^2(-8x)\, dx.$$

To find this indefinite integral, you can expand the integrand and use the Simple Power Rule.

$$\int -8(3 - 4x^2)^2\, dx = \int (-72 + 192x^2 - 128x^4)\, dx$$

$$= -72x + 64x^3 - \frac{128}{5}x^5 + C$$

When an integrand contains an extra constant factor that is not needed as part of $du/dx$, you can simply move the factor outside the integral sign, as shown in the next example.

### *Example 4*  Applying the General Power Rule

Find $\displaystyle\int 7x^2\sqrt{x^3 + 1}\, dx$.

**SOLUTION**   Let $u = x^3 + 1$. Then you need to create $du/dx = 3x^2$ by multiplying and dividing by 3. The constant factor $\frac{7}{3}$ is not needed as part of $du/dx$, and can be moved outside the integral sign.

$$\int 7x^2\sqrt{x^3 + 1}\, dx = \int 7x^2(x^3 + 1)^{1/2}\, dx \qquad \text{Rewrite with rational exponent.}$$

$$= \int \frac{7}{3}(x^3 + 1)^{1/2}(3x^2)\, dx \qquad \text{Multiply and divide by 3.}$$

$$= \frac{7}{3}\int (x^3 + 1)^{1/2}(3x^2)\, dx \qquad \text{Factor } \tfrac{7}{3} \text{ outside integral.}$$

$$= \frac{7}{3}\frac{(x^3 + 1)^{3/2}}{3/2} + C \qquad \text{General Power Rule}$$

$$= \frac{14}{9}(x^3 + 1)^{3/2} + C \qquad \text{Simplify.}$$

✓ **CHECKPOINT 4**

Find $\displaystyle\int 5x\sqrt{x^2 - 1}\, dx$. ∎

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**Algebra Review**

For help on the algebra in Example 4, see Example 1(c) in the *Chapter 5 Algebra Review*, on page 409.

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**TECHNOLOGY**

(T) If you use a symbolic integration utility to find indefinite integrals, you should be in for some surprises. This is true because integration is not nearly as straightforward as differentiation. By trying different integrands, you should be able to find several that the program cannot solve: in such situations, it may list a new indefinite integral. You should also be able to find several that have horrendous antiderivatives, some with functions that you may not recognize.

## Substitution

The integration technique used in Examples 1, 2, and 4 depends on your ability to recognize or create an integrand of the form $u^n \, du/dx$. With more complicated integrands, it is difficult to recognize the steps needed to fit the integrand to a basic integration formula. When this occurs, an alternative procedure called **substitution** or **change of variables** can be helpful. With this procedure, you completely rewrite the integral in terms of $u$ and $du$. That is, if $u = f(x)$, then $du = f'(x) \, dx$, and the General Power Rule takes the form

$$\int u^n \frac{du}{dx} \, dx = \int u^n \, du. \qquad \text{General Power Rule}$$

### Example 5 Integrating by Substitution

Find $\displaystyle\int \sqrt{1 - 3x} \, dx$.

**SOLUTION** Begin by letting $u = 1 - 3x$. Then, $du/dx = -3$ and $du = -3 \, dx$. This implies that $dx = -\frac{1}{3} \, du$, and you can find the indefinite integral as shown.

$$\int \sqrt{1 - 3x} \, dx = \int (1 - 3x)^{1/2} \, dx \qquad \text{Rewrite with rational exponent.}$$

$$= \int u^{1/2} \left( -\frac{1}{3} \, du \right) \qquad \text{Substitute for } x \text{ and } dx.$$

$$= -\frac{1}{3} \int u^{1/2} \, du \qquad \text{Factor } -\frac{1}{3} \text{ out of integrand.}$$

$$= -\frac{1}{3} \frac{u^{3/2}}{3/2} + C \qquad \text{Apply Power Rule.}$$

$$= -\frac{2}{9} u^{3/2} + C \qquad \text{Simplify.}$$

$$= -\frac{2}{9} (1 - 3x)^{3/2} + C \qquad \text{Substitute } 1 - 3x \text{ for } u.$$

**✓CHECKPOINT 5**

Find $\int \sqrt{1 - 2x} \, dx$ by the method of substitution. ■

The basic steps for integration by substitution are outlined in the guidelines below.

**Guidelines for Integration by Substitution**

1. Let $u$ be a function of $x$ (usually part of the integrand).

2. Solve for $x$ and $dx$ in terms of $u$ and $du$.

3. Convert the entire integral to $u$-variable form.

4. After integrating, rewrite the antiderivative as a function of $x$.

5. Check your answer by differentiating.

| Example 6 | Integration by Substitution

Find $\int x\sqrt{x^2 - 1}\ dx$.

**SOLUTION**    Consider the substitution $u = x^2 - 1$, which produces $du = 2x\ dx$. To create $2x\ dx$ as part of the integral, multiply and divide by 2.

$$\int x\sqrt{x^2 - 1}\ dx = \frac{1}{2}\int \overbrace{(x^2 - 1)^{1/2}}^{u^{1/n}}\ \overbrace{2x\ dx}^{du} \qquad \text{Multiply and divide by 2.}$$

$$= \frac{1}{2}\int u^{1/2}\ du \qquad \text{Substitute for } x \text{ and } dx.$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \qquad \text{Apply Power Rule.}$$

$$= \frac{1}{3}u^{3/2} + C \qquad \text{Simplify.}$$

$$= \frac{1}{3}(x^2 - 1)^{3/2} + C \qquad \text{Substitute for } u.$$

You can check this result by differentiating.

$$\frac{d}{dx}\left[\frac{1}{3}(x^2 - 1)^{3/2} + C\right] = \frac{1}{3}\left(\frac{3}{2}\right)(x^2 - 1)^{1/2}(2x)$$

$$= \frac{1}{2}(2x)(x^2 - 1)^{1/2}$$

$$= x\sqrt{x^2 - 1}$$

✓**CHECKPOINT 6**

Find $\int x\sqrt{x^2 + 4}\ dx$ by the method of substitution. ■

To become efficient at integration, you should learn to use *both* techniques discussed in this section. For simpler integrals, you should use pattern recognition and create $du/dx$ by multiplying and dividing by an appropriate constant. For more complicated integrals, you should use a formal change of variables, as shown in Examples 5 and 6. For the integrals in this section's exercise set, try working several of the problems twice—once with pattern recognition and once using formal substitution.

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### *DISCOVERY*

Suppose you were asked to evaluate the integrals below. Which one would you choose? Explain your reasoning.

$$\int \sqrt{x^2 + 1}\ dx \quad \text{or} \quad \int x\sqrt{x^2 + 1}\ dx$$
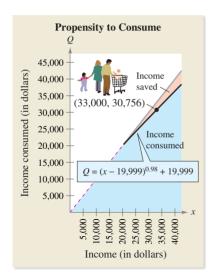
## Extended Application: Propensity to Consume

In 2005, the U.S. poverty level for a family of four was about $20,000. Families at or below the poverty level tend to consume 100% of their income—that is, they use all their income to purchase necessities such as food, clothing, and shelter. As income level increases, the average consumption tends to drop below 100%. For instance, a family earning $22,000 may be able to save $440 and so consume only $21,560 (98%) of their income. As the income increases, the ratio of consumption to savings tends to decrease. The rate of change of consumption with respect to income is called the **marginal propensity to consume.** *(Source: U.S. Census Bureau)*

### Example 7
**MAKE A DECISION**  Analyzing Consumption  Ⓡ

For a family of four in 2005, the marginal propensity to consume income $x$ can be modeled by

$$\frac{dQ}{dx} = \frac{0.98}{(x - 19,999)^{0.02}}, \quad x \geq 20,000$$

where $Q$ represents the income consumed. Use the model to estimate the amount consumed by a family of four whose 2005 income was $33,000. Would the family have consumed more than $30,000?

**SOLUTION**  Begin by integrating $dQ/dx$ to find a model for the consumption $Q$. Use the initial condition that $Q = 20,000$ and $x = 20,000$.

$$\frac{dQ}{dx} = \frac{0.98}{(x - 19,999)^{0.02}}$$ Marginal propensity to consume

$$Q = \int \frac{0.98}{(x - 19,999)^{0.02}} \, dx$$ Integrate to obtain $Q$.

$$= \int 0.98(x - 19,999)^{-0.02} \, dx$$ Rewrite.

$$= (x - 19,999)^{0.98} + C$$ General Power Rule

$$= (x - 19,999)^{0.98} + 19,999$$ Use initial condition to find $C$.

Using this model, you can estimate that a family of four with an income of $x = 33,000$ consumed about $30,756. So, a family of four would have consumed more than $30,000. The graph of $Q$ is shown in Figure 5.4.



**Propensity to Consume**

$Q = (x - 19,999)^{0.98} + 19,999$

(33,000, 30,756)

Income saved. Income consumed.

**FIGURE 5.4**

### STUDY TIP

When you use the initial condition to find the value of $C$ in Example 7, you substitute 20,000 for $Q$ and 20,000 for $x$.

$$Q = (x - 19,999)^{0.98} + C$$

$$20,000 = (20,000 - 19,999)^{0.98} + C$$

$$20,000 = 1 + C$$

$$19,999 = C$$

### ✓CHECKPOINT 7

According to the model in Example 7, at what income level would a family of four consume $30,000? ▪

### CONCEPT CHECK

1. When using the General Power Rule for an integrand that contains an extra constant factor that is not needed as part of $du/dx$, what can you do with the factor?
2. Write the General Power Rule for integration.
3. Write the guidelines for integration by substitution.
4. Explain why the General Power Rule works for finding $\int 2x\sqrt{x^2 + 1} \, dx$, but not for finding $\int 2\sqrt{x^2 + 1} \, dx$.

In Exercises 1–10, find the indefinite integral.

**1.** $\displaystyle\int (2x^3 + 1)\, dx$

**2.** $\displaystyle\int (x^{1/2} + 3x - 4)\, dx$

**3.** $\displaystyle\int \frac{1}{x^2}\, dx$

**4.** $\displaystyle\int \frac{1}{3t^3}\, dt$

**5.** $\displaystyle\int (1 + 2t)t^{3/2}\, dt$

**6.** $\displaystyle\int \sqrt{x}(2x - 1)\, dx$

**7.** $\displaystyle\int \frac{5x^3 + 2}{x^2}\, dx$

**8.** $\displaystyle\int \frac{2x^2 - 5}{x^4}\, dx$

**9.** $\displaystyle\int (x^2 + 1)^2\, dx$

**10.** $\displaystyle\int (x^3 - 2x + 1)^2\, dx$

In Exercises 11–14, simplify the expression.

**11.** $\left(-\dfrac{5}{4}\right)\dfrac{(x - 2)^4}{4}$

**12.** $\left(\dfrac{1}{6}\right)\dfrac{(x - 1)^{-2}}{-2}$

**13.** $(6)\dfrac{(x^2 + 3)^{2/3}}{2/3}$

**14.** $\left(\dfrac{5}{2}\right)\dfrac{(1 - x^3)^{-1/2}}{-1/2}$

## Exercises 5.2

In Exercises 1–8, identify $u$ and $du/dx$ for the integral $\int u^n(du/dx)\, dx$.

**1.** $\displaystyle\int (5x^2 + 1)^2(10x)\, dx$

**2.** $\displaystyle\int (3 - 4x^2)^3(-8x)\, dx$

**3.** $\displaystyle\int \sqrt{1 - x^2}\,(-2x)\, dx$

**4.** $\displaystyle\int 3x^2\sqrt{x^3 + 1}\, dx$

**5.** $\displaystyle\int \left(4 + \frac{1}{x^2}\right)^5\left(\frac{-2}{x^3}\right)\, dx$

**6.** $\displaystyle\int \frac{1}{(1 + 2x)^2}(2)\, dx$

**7.** $\displaystyle\int (1 + \sqrt{x})^3\left(\frac{1}{2\sqrt{x}}\right)\, dx$

**8.** $\displaystyle\int (4 - \sqrt{x})^2\left(\frac{-1}{2\sqrt{x}}\right)\, dx$

In Exercises 9–28, find the indefinite integral and check the result by differentiation.

**9.** $\displaystyle\int (1 + 2x)^4(2)\, dx$

**10.** $\displaystyle\int (x^2 - 1)^3(2x)\, dx$

**11.** $\displaystyle\int \sqrt{4x^2 - 5}\,(8x)\, dx$

**12.** $\displaystyle\int \sqrt[3]{1 - 2x^2}\,(-4x)\, dx$

**13.** $\displaystyle\int (x - 1)^4\, dx$

**14.** $\displaystyle\int (x - 3)^{5/2}\, dx$

**15.** $\displaystyle\int 2x(x^2 - 1)^7\, dx$

**16.** $\displaystyle\int x(1 - 2x^2)^3\, dx$

**17.** $\displaystyle\int \frac{x^2}{(1 + x^3)^2}\, dx$

**18.** $\displaystyle\int \frac{x^2}{(x^3 - 1)^2}\, dx$

**19.** $\displaystyle\int \frac{x + 1}{(x^2 + 2x - 3)^2}\, dx$

**20.** $\displaystyle\int \frac{6x}{(1 + x^2)^3}\, dx$

**21.** $\displaystyle\int \frac{x - 2}{\sqrt{x^2 - 4x + 3}}\, dx$

**22.** $\displaystyle\int \frac{4x + 6}{(x^2 + 3x + 7)^3}\, dx$

**23.** $\displaystyle\int 5u\sqrt[3]{1 - u^2}\, du$

**24.** $\displaystyle\int u^3\sqrt{u^4 + 2}\, du$

**25.** $\displaystyle\int \frac{4y}{\sqrt{1 + y^2}}\, dy$

**26.** $\displaystyle\int \frac{3x^2}{\sqrt{1 - x^3}}\, dx$

**27.** $\displaystyle\int \frac{-3}{\sqrt{2t + 3}}\, dt$

**28.** $\displaystyle\int \frac{t + 2t^2}{\sqrt{t}}\, dt$

(T) In Exercises 29–34, use a symbolic integration utility to find the indefinite integral.

**29.** $\displaystyle\int \frac{x^3}{\sqrt{1 - x^4}}\, dx$

**30.** $\displaystyle\int \frac{3x}{\sqrt{1 - 4x^2}}\, dx$

**31.** $\displaystyle\int \left(1 + \frac{4}{t^2}\right)^2\left(\frac{1}{t^3}\right)\, dt$

**32.** $\displaystyle\int \left(1 + \frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)\, dt$

**33.** $\displaystyle\int (x^3 + 3x + 9)(x^2 + 1)\, dx$

**34.** $\displaystyle\int (7 - 3x - 3x^2)(2x + 1)\, dx$

In Exercises 35–42, use formal substitution (as illustrated in Examples 5 and 6) to find the indefinite integral.

**35.** $\displaystyle\int 12x(6x^2 - 1)^3 \, dx$

**36.** $\displaystyle\int 3x^2(1 - x^3)^2 \, dx$

**37.** $\displaystyle\int x^2(2 - 3x^3)^{3/2} \, dx$

**38.** $\displaystyle\int t\sqrt{t^2 + 1} \, dt$

**39.** $\displaystyle\int \frac{x}{\sqrt{x^2 + 25}} \, dx$

**40.** $\displaystyle\int \frac{3}{\sqrt{2x + 1}} \, dx$

**41.** $\displaystyle\int \frac{x^2 + 1}{\sqrt{x^3 + 3x + 4}} \, dx$

**42.** $\displaystyle\int \sqrt{x}\,(4 - x^{3/2})^2 \, dx$

In Exercises 43–46, (a) perform the integration in two ways: once using the Simple Power Rule and once using the General Power Rule. (b) Explain the difference in the results. (c) Which method do you prefer? Explain your reasoning.

**43.** $\displaystyle\int (x - 1)^2 \, dx$

**44.** $\displaystyle\int (3 - x)^2 \, dx$

**45.** $\displaystyle\int x(x^2 - 1)^2 \, dx$

**46.** $\displaystyle\int x(2x^2 + 1)^2 \, dx$

**47.** Find the equation of the function $f$ whose graph passes through the point $\left(0, \frac{4}{3}\right)$ and whose derivative is
$$f'(x) = x\sqrt{1 - x^2}.$$

**48.** Find the equation of the function $f$ whose graph passes through the point $\left(0, \frac{7}{3}\right)$ and whose derivative is
$$f'(x) = x\sqrt{1 - x^2}.$$

**49. Cost**    The marginal cost of a product is modeled by
$$\frac{dC}{dx} = \frac{4}{\sqrt{x + 1}}. \text{ When } x = 15, C = 50.$$

(a) Find the cost function.

(T) (b) Use a graphing utility to graph $dC/dx$ and $C$ in the same viewing window.

**50. Cost**    The marginal cost of a product is modeled by
$$\frac{dC}{dx} = \frac{12}{\sqrt[3]{12x + 1}}.$$
When $x = 13$, $C = 100$.

(a) Find the cost function.

(T) (b) Use a graphing utility to graph $dC/dx$ and $C$ in the same viewing window.

**Supply Function**    In Exercises 51 and 52, find the supply function $x = f(p)$ that satisfies the initial conditions.

**51.** $\displaystyle\frac{dx}{dp} = p\sqrt{p^2 - 25}, \quad x = 600 \text{ when } p = \$13$

**52.** $\displaystyle\frac{dx}{dp} = \frac{10}{\sqrt{p - 3}}, \quad x = 100 \text{ when } p = \$3$

**Demand Function**    In Exercises 53 and 54, find the demand function $x = f(p)$ that satisfies the initial conditions.

**53.** $\displaystyle\frac{dx}{dp} = -\frac{6000p}{(p^2 - 16)^{3/2}}, \quad x = 5000 \text{ when } p = \$5$

**54.** $\displaystyle\frac{dx}{dp} = -\frac{400}{(0.02p - 1)^3}, \quad x = 10,000 \text{ when } p = \$100$

**55. Gardening**    An evergreen nursery usually sells a type of shrub after 5 years of growth and shaping. The growth rate during those 5 years is approximated by
$$\frac{dh}{dt} = \frac{17.6t}{\sqrt{17.6t^2 + 1}}$$
where $t$ is time in years and $h$ is height in inches. The seedlings are 6 inches tall when planted ($t = 0$).

(a) Find the height function.

(b) How tall are the shrubs when they are sold?

**56. Cash Flow**    The rate of disbursement $dQ/dt$ of a $4 million federal grant is proportional to the square of $100 - t$, where $t$ is the time (in days, $0 \leq t \leq 100$) and $Q$ is the amount that remains to be disbursed. Find the amount that remains to be disbursed after 50 days. Assume that the entire grant will be disbursed after 100 days.

(T)(S) **Marginal Propensity to Consume**    In Exercises 57 and 58, (a) use the marginal propensity to consume, $dQ/dx$, to write $Q$ as a function of $x$, where $x$ is the income (in dollars) and $Q$ is the income consumed (in dollars). Assume that 100% of the income is consumed for families that have annual incomes of $25,000 or less. (b) Use the result of part (a) and a spreadsheet to complete the table showing the income consumed and the income saved, $x - Q$, for various incomes. (c) Use a graphing utility to represent graphically the income consumed and saved.

| $x$ | 25,000 | 50,000 | 100,000 | 150,000 |
|-----|--------|--------|---------|---------|
| $Q$ |        |        |         |         |
| $x - Q$ |    |        |         |         |

**57.** $\displaystyle\frac{dQ}{dx} = \frac{0.95}{(x - 24{,}999)^{0.05}}, \quad x \geq 25{,}000$

**58.** $\displaystyle\frac{dQ}{dx} = \frac{0.93}{(x - 24{,}999)^{0.07}}, \quad x \geq 25{,}000$

(T) In Exercises 59 and 60, use a symbolic integration utility to find the indefinite integral. Verify the result by differentiating.

**59.** $\displaystyle\int \frac{1}{\sqrt{x} + \sqrt{x + 1}} \, dx$

**60.** $\displaystyle\int \frac{x}{\sqrt{3x + 2}} \, dx$

## Section 5.3

# Exponential and Logarithmic Integrals

■ Use the Exponential Rule to find indefinite integrals.
■ Use the Log Rule to find indefinite integrals.

## Using the Exponential Rule

Each of the differentiation rules for exponential functions has its corresponding integration rule.

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**Integrals of Exponential Functions**

Let $u$ be a differentiable function of $x$.

$$\int e^x \, dx = e^x + C \qquad\qquad \text{Simple Exponential Rule}$$

$$\int e^u \frac{du}{dx} \, dx = \int e^u \, du = e^u + C \qquad \text{General Exponential Rule}$$

---

**Example 1**  **Integrating Exponential Functions**

Find each indefinite integral.

**a.** $\displaystyle\int 2e^x \, dx$   **b.** $\displaystyle\int 2e^{2x} \, dx$   **c.** $\displaystyle\int (e^x + x) \, dx$

**SOLUTION**

**a.** $\displaystyle\int 2e^x \, dx = 2 \int e^x \, dx$         Constant Multiple Rule

$\qquad\qquad = 2e^x + C$         Simple Exponential Rule

**b.** $\displaystyle\int 2e^{2x} \, dx = \int e^{2x}(2) \, dx$         Let $u = 2x$, then $\dfrac{du}{dx} = 2$.

$\qquad\qquad = \int e^u \dfrac{du}{dx} \, dx$

$\qquad\qquad = e^{2x} + C$         General Exponential Rule

**c.** $\displaystyle\int (e^x + x) \, dx = \int e^x \, dx + \int x \, dx$         Sum Rule

$\qquad\qquad = e^x + \dfrac{x^2}{2} + C$         Simple Exponential and Power Rules

You can check each of these results by differentiating.

✓ **CHECKPOINT 1**

Find each indefinite integral.

**a.** $\displaystyle\int 3e^x \, dx$

**b.** $\displaystyle\int 5e^{5x} \, dx$

**c.** $\displaystyle\int (e^x - x) \, dx$

### *Example 2* Integrating an Exponential Function

Find $\displaystyle\int e^{3x+1}\, dx$.

**SOLUTION** Let $u = 3x + 1$, then $du/dx = 3$. You can introduce the missing factor of 3 in the integrand by multiplying and dividing by 3.

$$\int e^{3x+1}\, dx = \frac{1}{3}\int e^{3x+1}(3)\, dx \qquad \text{Multiply and divide by 3.}$$

$$= \frac{1}{3}\int e^{u}\,\frac{du}{dx}\, dx \qquad \text{Substitute } u \text{ and } du/dx.$$

$$= \frac{1}{3}e^{u} + C \qquad \text{General Exponential Rule}$$

$$= \frac{1}{3}e^{3x+1} + C \qquad \text{Substitute for } u.$$

✓**CHECKPOINT 2**

Find $\displaystyle\int e^{2x+3}\, dx$. ▪

### Algebra Review

For help on the algebra in Example 3, see Example 1(d) in the *Chapter 5 Algebra Review*, on page 409.

### *Example 3* Integrating an Exponential Function

Find $\displaystyle\int 5xe^{-x^2}\, dx$.

**SOLUTION** Let $u = -x^2$, then $du/dx = -2x$. You can create the factor $-2x$ in the integrand by multiplying and dividing by $-2$.

$$\int 5xe^{-x^2}\, dx = \int \left(-\frac{5}{2}\right)e^{-x^2}(-2x)\, dx \qquad \text{Multiply and divide by } -2.$$

$$= -\frac{5}{2}\int e^{-x^2}(-2x)\, dx \qquad \text{Factor } -\tfrac{5}{2} \text{ out of the integrand.}$$

$$= -\frac{5}{2}\int e^{u}\,\frac{du}{dx}\, dx \qquad \text{Substitute } u \text{ and } \frac{du}{dx}.$$

$$= -\frac{5}{2}e^{u} + C \qquad \text{General Exponential Rule}$$

$$= -\frac{5}{2}e^{-x^2} + C \qquad \text{Substitute for } u.$$

### STUDY TIP

Remember that you cannot introduce a missing *variable* in the integrand. For instance, you cannot find $\int e^{x^2}\, dx$ by multiplying and dividing by $2x$ and then factoring $1/(2x)$ out of the integral. That is,

$$\int e^{x^2}\, dx \neq \frac{1}{2x}\int e^{x^2}(2x)\, dx.$$

✓**CHECKPOINT 3**

Find $\displaystyle\int 4xe^{x^2}\, dx$. ▪

## Using the Log Rule

When the Power Rules for integration were introduced in Sections 5.1 and 5.2, you saw that they work for powers other than $n = -1$.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1 \qquad \text{Simple Power Rule}$$

$$\int u^n \frac{du}{dx} \, dx = \int u^n \, du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1 \qquad \text{General Power Rule}$$

The Log Rules for integration allow you to integrate functions of the form $\int x^{-1} \, dx$ and $\int u^{-1} \, du$.

**Integrals of Logarithmic Functions**

Let $u$ be a differentiable function of $x$.

$$\int \frac{1}{x} \, dx = \ln|x| + C \qquad \text{Simple Logarithmic Rule}$$

$$\int \frac{du/dx}{u} \, dx = \int \frac{1}{u} \, du = \ln|u| + C \qquad \text{General Logarithmic Rule}$$

You can verify each of these rules by differentiating. For instance, to verify that $d/dx[\ln|x|] = 1/x$, notice that

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \qquad \text{and} \qquad \frac{d}{dx}[\ln(-x)] = \frac{-1}{-x} = \frac{1}{x}.$$

**STUDY TIP**

Notice the absolute values in the Log Rules. For those special cases in which $u$ or $x$ cannot be negative, you can omit the absolute value. For instance, in Example 4(b), it is not necessary to write the antiderivative as $\ln|x^2| + C$ because $x^2$ cannot be negative.

### Example 4  Integrating Logarithmic Functions

Find each indefinite integral.

**a.** $\int \dfrac{4}{x} \, dx$    **b.** $\int \dfrac{2x}{x^2} \, dx$    **c.** $\int \dfrac{3}{3x+1} \, dx$

**SOLUTION**

**a.** $\displaystyle \int \frac{4}{x} \, dx = 4 \int \frac{1}{x} \, dx$    Constant Multiple Rule

$\displaystyle \phantom{\int \frac{4}{x} \, dx} = 4 \ln|x| + C$    Simple Logarithmic Rule

**b.** $\displaystyle \int \frac{2x}{x^2} \, dx = \int \frac{du/dx}{u} \, dx$    Let $u = x^2$, then $\dfrac{du}{dx} = 2x$.

$\displaystyle \phantom{\int \frac{2x}{x^2} \, dx} = \ln|u| + C$    General Logarithmic Rule

$\displaystyle \phantom{\int \frac{2x}{x^2} \, dx} = \ln x^2 + C$    Substitute for $u$.

**c.** $\displaystyle \int \frac{3}{3x+1} \, dx = \int \frac{du/dx}{u} \, dx$    Let $u = 3x + 1$, then $\dfrac{du}{dx} = 3$.

$\displaystyle \phantom{\int \frac{3}{3x+1} \, dx} = \ln|u| + C$    General Logarithmic Rule

$\displaystyle \phantom{\int \frac{3}{3x+1} \, dx} = \ln|3x+1| + C$    Substitute for $u$.

**✓ CHECKPOINT 4**

Find each indefinite integral.

**a.** $\int \dfrac{2}{x} \, dx$

**b.** $\int \dfrac{3x^2}{x^3} \, dx$

**c.** $\int \dfrac{2}{2x+1} \, dx$

**Example 5**   Using the Log Rule

Find $\int \dfrac{1}{2x - 1}\, dx$.

**SOLUTION**   Let $u = 2x - 1$, then $du/dx = 2$. You can create the necessary factor of 2 in the integrand by multiplying and dividing by 2.

$$\int \frac{1}{2x - 1}\, dx = \frac{1}{2} \int \frac{2}{2x - 1}\, dx \qquad\qquad \text{Multiply and divide by 2.}$$

$$= \frac{1}{2} \int \frac{du/dx}{u}\, dx \qquad\qquad \text{Substitute } u \text{ and } \frac{du}{dx}.$$

$$= \frac{1}{2} \ln|u| + C \qquad\qquad \text{General Log Rule}$$

$$= \frac{1}{2} \ln|2x - 1| + C \qquad\qquad \text{Substitute for } u.$$

✓ **CHECKPOINT 5**

Find $\int \dfrac{1}{4x + 1}\, dx$. ■

**Example 6**   Using the Log Rule

Find $\int \dfrac{6x}{x^2 + 1}\, dx$.

**SOLUTION**   Let $u = x^2 + 1$, then $du/dx = 2x$. You can create the necessary factor of $2x$ in the integrand by factoring a 3 out of the integrand.

$$\int \frac{6x}{x^2 + 1}\, dx = 3 \int \frac{2x}{x^2 + 1}\, dx \qquad\qquad \text{Factor 3 out of integrand.}$$

$$= 3 \int \frac{du/dx}{u}\, dx \qquad\qquad \text{Substitute } u \text{ and } \frac{du}{dx}.$$

$$= 3 \ln|u| + C \qquad\qquad \text{General Log Rule}$$

$$= 3 \ln(x^2 + 1) + C \qquad\qquad \text{Substitute for } u.$$

✓ **CHECKPOINT 6**

Find $\int \dfrac{3x}{x^2 + 4}\, dx$. ■

Integrals to which the Log Rule can be applied are often given in disguised form. For instance, if a rational function has a numerator of degree greater than or equal to that of the denominator, you should use long division to rewrite the integrand. Here is an example.

$$\int \frac{x^2 + 6x + 1}{x^2 + 1}\, dx = \int \left(1 + \frac{6x}{x^2 + 1}\right) dx$$

$$= x + 3 \ln(x^2 + 1) + C$$

**Algebra Review**

For help on the algebra in the integral at the right, see Example 2(d) in the *Chapter 5 Algebra Review*, on page 410.

The next example summarizes some additional situations in which it is helpful to rewrite the integrand in order to recognize the antiderivative.

### *Example 7*  Rewriting Before Integrating

Find each indefinite integral.

**a.** $\displaystyle\int \frac{3x^2 + 2x - 1}{x^2}\,dx$    **b.** $\displaystyle\int \frac{1}{1 + e^{-x}}\,dx$    **c.** $\displaystyle\int \frac{x^2 + x + 1}{x - 1}\,dx$

**SOLUTION**

**a.** Begin by rewriting the integrand as the sum of three fractions.

$$\int \frac{3x^2 + 2x - 1}{x^2}\,dx = \int \left(\frac{3x^2}{x^2} + \frac{2x}{x^2} - \frac{1}{x^2}\right) dx$$

$$= \int \left(3 + \frac{2}{x} - \frac{1}{x^2}\right) dx$$

$$= 3x + 2\ln|x| + \frac{1}{x} + C$$

**b.** Begin by rewriting the integrand by multiplying and dividing by $e^x$.

$$\int \frac{1}{1 + e^{-x}}\,dx = \int \left(\frac{e^x}{e^x}\right)\frac{1}{1 + e^{-x}}\,dx$$

$$= \int \frac{e^x}{e^x + 1}\,dx$$

$$= \ln(e^x + 1) + C$$

**c.** Begin by dividing the numerator by the denominator.

$$\int \frac{x^2 + x + 1}{x - 1}\,dx = \int \left(x + 2 + \frac{3}{x - 1}\right) dx$$

$$= \frac{x^2}{2} + 2x + 3\ln|x - 1| + C$$

✓**CHECKPOINT 7**

Find each indefinite integral.

**a.** $\displaystyle\int \frac{4x^2 - 3x + 2}{x^2}\,dx$

**b.** $\displaystyle\int \frac{2}{e^{-x} + 1}\,dx$

**c.** $\displaystyle\int \frac{x^2 + 2x + 4}{x + 1}\,dx$    ◼

**STUDY TIP**

The Exponential and Log Rules are necessary to solve certain real-life problems, such as population growth. You will see such problems in the exercise set for this section.

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**CONCEPT CHECK**

1.  Write the General Exponential Rule for integration.
2.  Write the General Logarithmic Rule for integration.
3.  Which integration rule allows you to integrate functions of the form

    $\displaystyle\int e^u \frac{du}{dx}\,dx$?

4.  Which integration rule allows you to integrate

    $\displaystyle\int x^{-1}\,dx$?

---

**Skills Review 5.3**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 4.4, 5.1, and 5.2.

In Exercises 1 and 2, find the domain of the function.

**1.** $y = \ln(2x - 5)$

**2.** $y = \ln(x^2 - 5x + 6)$

In Exercises 3–6, use long division to rewrite the quotient.

**3.** $\dfrac{x^2 + 4x + 2}{x + 2}$

**4.** $\dfrac{x^2 - 6x + 9}{x - 4}$

**5.** $\dfrac{x^3 + 4x^2 - 30x - 4}{x^2 - 4x}$

**6.** $\dfrac{x^4 - x^3 + x^2 + 15x + 2}{x^2 + 5}$

In Exercises 7–10, evaluate the integral.

**7.** $\displaystyle\int \left(x^3 + \dfrac{1}{x^2}\right) dx$

**8.** $\displaystyle\int \dfrac{x^2 + 2x}{x} dx$

**9.** $\displaystyle\int \dfrac{x^3 + 4}{x^2} dx$

**10.** $\displaystyle\int \dfrac{x + 3}{x^3} dx$

---

## Exercises 5.3

In Exercises 1–12, use the Exponential Rule to find the indefinite integral.

**1.** $\displaystyle\int 2e^{2x} dx$

**2.** $\displaystyle\int -3e^{-3x} dx$

**3.** $\displaystyle\int e^{4x} dx$

**4.** $\displaystyle\int e^{-0.25x} dx$

**5.** $\displaystyle\int 9xe^{-x^2} dx$

**6.** $\displaystyle\int 3xe^{0.5x^2} dx$

**7.** $\displaystyle\int 5x^2 e^{x^3} dx$

**8.** $\displaystyle\int (2x + 1)e^{x^2 + x} dx$

**9.** $\displaystyle\int (x^2 + 2x)e^{x^3 + 3x^2 - 1} dx$

**10.** $\displaystyle\int 3(x - 4)e^{x^2 - 8x} dx$

**11.** $\displaystyle\int 5e^{2-x} dx$

**12.** $\displaystyle\int 3e^{-(x+1)} dx$

In Exercises 13–28, use the Log Rule to find the indefinite integral.

**13.** $\displaystyle\int \dfrac{1}{x + 1} dx$

**14.** $\displaystyle\int \dfrac{1}{x - 5} dx$

**15.** $\displaystyle\int \dfrac{1}{3 - 2x} dx$

**16.** $\displaystyle\int \dfrac{1}{6x - 5} dx$

**17.** $\displaystyle\int \dfrac{2}{3x + 5} dx$

**18.** $\displaystyle\int \dfrac{5}{2x - 1} dx$

**19.** $\displaystyle\int \dfrac{x}{x^2 + 1} dx$

**20.** $\displaystyle\int \dfrac{x^2}{3 - x^3} dx$

**21.** $\displaystyle\int \dfrac{x^2}{x^3 + 1} dx$

**22.** $\displaystyle\int \dfrac{x}{x^2 + 4} dx$

**23.** $\displaystyle\int \dfrac{x + 3}{x^2 + 6x + 7} dx$

**24.** $\displaystyle\int \dfrac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x + 1} dx$

**25.** $\displaystyle\int \dfrac{1}{x \ln x} dx$

**26.** $\displaystyle\int \dfrac{1}{x(\ln x)^2} dx$

**27.** $\displaystyle\int \dfrac{e^{-x}}{1 - e^{-x}} dx$

**28.** $\displaystyle\int \dfrac{e^x}{1 + e^x} dx$

Ⓣ In Exercises 29–38, use a symbolic integration utility to find the indefinite integral.

**29.** $\displaystyle\int \dfrac{1}{x^2} e^{2/x} dx$

**30.** $\displaystyle\int \dfrac{1}{x^3} e^{1/4x^2} dx$

**31.** $\displaystyle\int \dfrac{1}{\sqrt{x}} e^{\sqrt{x}} dx$

**32.** $\displaystyle\int \dfrac{e^{1/\sqrt{x}}}{x^{3/2}} dx$

**33.** $\displaystyle\int (e^x - 2)^2 dx$

**34.** $\displaystyle\int (e^x - e^{-x})^2 dx$

**35.** $\displaystyle\int \dfrac{e^{-x}}{1 + e^{-x}} dx$

**36.** $\displaystyle\int \dfrac{3e^x}{2 + e^x} dx$

**37.** $\displaystyle\int \dfrac{4e^{2x}}{5 - e^{2x}} dx$

**38.** $\displaystyle\int \dfrac{-e^{3x}}{2 - e^{3x}} dx$

In Exercises 39–54, use any basic integration formula or formulas to find the indefinite integral. State which integration formula(s) you used to find the integral.

**39.** $\displaystyle \int \frac{e^{2x} + 2e^x + 1}{e^x}\, dx$

**40.** $\displaystyle \int (6x + e^x)\sqrt{3x^2 + e^x}\, dx$

**41.** $\displaystyle \int e^x\sqrt{1 - e^x}\, dx$

**42.** $\displaystyle \int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2}\, dx$

**43.** $\displaystyle \int \frac{1}{(x - 1)^2}\, dx$

**44.** $\displaystyle \int \frac{1}{\sqrt{x + 1}}\, dx$

**45.** $\displaystyle \int 4e^{2x-1}\, dx$

**46.** $\displaystyle \int (5e^{-2x} + 1)\, dx$

**47.** $\displaystyle \int \frac{x^3 - 8x}{2x^2}\, dx$

**48.** $\displaystyle \int \frac{x - 1}{4x}\, dx$

**49.** $\displaystyle \int \frac{2}{1 + e^{-x}}\, dx$

**50.** $\displaystyle \int \frac{3}{1 + e^{-3x}}\, dx$

**51.** $\displaystyle \int \frac{x^2 + 2x + 5}{x - 1}\, dx$

**52.** $\displaystyle \int \frac{x - 3}{x + 3}\, dx$

**53.** $\displaystyle \int \frac{1 + e^{-x}}{1 + xe^{-x}}\, dx$

**54.** $\displaystyle \int \frac{5}{e^{-5x} + 7}\, dx$

In Exercises 55 and 56, find the equation of the function $f$ whose graph passes through the point.

**55.** $f'(x) = \dfrac{x^2 + 4x + 3}{x - 1}; \quad (2, 4)$

**56.** $f'(x) = \dfrac{x^3 - 4x^2 + 3}{x - 3}; \quad (4, -1)$

**57. Biology** A population of bacteria is growing at the rate of

$$\frac{dP}{dt} = \frac{3000}{1 + 0.25t}$$

where $t$ is the time in days. When $t = 0$, the population is 1000.

(a) Write an equation that models the population $P$ in terms of the time $t$.

(b) What is the population after 3 days?

(c) After how many days will the population be 12,000?

**58. Biology** Because of an insufficient oxygen supply, the trout population in a lake is dying. The population's rate of change can be modeled by

$$\frac{dP}{dt} = -125e^{-t/20}$$

where $t$ is the time in days. When $t = 0$, the population is 2500.

(a) Write an equation that models the population $P$ in terms of the time $t$.

(b) What is the population after 15 days?

(c) According to this model, how long will it take for the entire trout population to die?

(T) **59. Demand**   The marginal price for the demand of a product can be modeled by $dp/dx = 0.1e^{-x/500}$, where $x$ is the quantity demanded. When the demand is 600 units, the price is $30.

(a) Find the demand function, $p = f(x)$.

(b) Use a graphing utility to graph the demand function. Does price increase or decrease as demand increases?

(c) Use the *zoom* and *trace* features of the graphing utility to find the quantity demanded when the price is $22.

(T) **60. Revenue**   The marginal revenue for the sale of a product can be modeled by

$$\frac{dR}{dx} = 50 - 0.02x + \frac{100}{x + 1}$$

where $x$ is the quantity demanded.

(a) Find the revenue function.

(b) Use a graphing utility to graph the revenue function.

(c) Find the revenue when 1500 units are sold.

(d) Use the *zoom* and *trace* features of the graphing utility to find the number of units sold when the revenue is $60,230.

**61. Average Salary**   From 2000 through 2005, the average salary for public school nurses $S$ (in dollars) in the United States changed at the rate of

$$\frac{dS}{dt} = 1724.1e^{-t/4.2}$$

where $t = 0$ corresponds to 2000. In 2005, the average salary for public school nurses was $40,520. *(Source: Educational Research Service)*

(a) Write a model that gives the average salary for public school nurses per year.

(b) Use the model to find the average salary for public school nurses in 2002.

**62. Sales**   The rate of change in sales for The Yankee Candle Company from 1998 through 2005 can be modeled by

$$\frac{dS}{dt} = 0.528t + \frac{597.2099}{t}$$

where $S$ is the sales (in millions) and $t = 8$ corresponds to 1998. In 1999, the sales for The Yankee Candle Company were $256.6 million. *(Source: The Yankee Candle Company)*

(a) Find a model for sales from 1998 through 2005.

(b) Find The Yankee Candle Company's sales in 2004.

**True or False?**   In Exercises 63 and 64, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**63.** $(\ln x)^{1/2} = \tfrac{1}{2}(\ln x)$

**64.** $\displaystyle \int \ln x = \left(\frac{1}{x}\right) + C$

# Mid-Chapter Quiz

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–9, find the indefinite integral and check your result by differentiation.

**1.** $\displaystyle\int 3\,dx$

**2.** $\displaystyle\int 10x\,dx$

**3.** $\displaystyle\int \frac{1}{x^5}\,dx$

**4.** $\displaystyle\int (x^2 - 2x + 15)\,dx$

**5.** $\displaystyle\int x(x + 4)\,dx$

**6.** $\displaystyle\int (6x + 1)^3(6)\,dx$

**7.** $\displaystyle\int (x^2 - 5x)(2x - 5)\,dx$

**8.** $\displaystyle\int \frac{3x^2}{(x^3 + 3)^3}\,dx$

**9.** $\displaystyle\int \sqrt{5x + 2}\,dx$

In Exercises 10 and 11, find the particular solution $y = f(x)$ that satisfies the differential equation and initial condition.

**10.** $f'(x) = 16x; f(0) = 1$

**11.** $f'(x) = 9x^2 + 4; f(1) = 5$

**12.** The marginal cost function for producing $x$ units of a product is modeled by

$$\frac{dC}{dx} = 16 - 0.06x.$$

It costs \$25 to produce one unit. Find (a) the fixed cost (when $x = 0$) and (b) the total cost of producing 500 units.

**13.** Find the equation of the function $f$ whose graph passes through the point $(0, 1)$ and whose derivative is

$$f'(x) = 2x^2 + 1.$$

In Exercises 14–16, use the Exponential Rule to find the indefinite integral. Check your result by differentiation.

**14.** $\displaystyle\int 5e^{5x+4}\,dx$

**15.** $\displaystyle\int (x + 2e^{2x})\,dx$

**16.** $\displaystyle\int 3x^2 e^{x^3}\,dx$

In Exercises 17–19, use the Log Rule to find the indefinite integral.

**17.** $\displaystyle\int \frac{2}{2x - 1}\,dx$

**18.** $\displaystyle\int \frac{-2x}{x^2 + 3}\,dx$

**19.** $\displaystyle\int \frac{3(3x^2 + 4x)}{x^3 + 2x^2}\,dx$
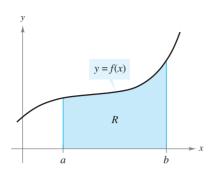
**20.** The number of bolts $B$ produced by a foundry changes according to the model

$$\frac{dB}{dt} = \frac{250t}{\sqrt{t^2 + 36}}, \quad 0 \le t \le 40$$

where $t$ is measured in hours. Find the number of bolts produced in (a) 8 hours and (b) 40 hours.

**Section 5.4**

# Area and the Fundamental Theorem of Calculus

■ Evaluate definite integrals.
■ Evaluate definite integrals using the Fundamental Theorem of Calculus.
■ Use definite integrals to solve marginal analysis problems.
■ Find the average values of functions over closed intervals.
■ Use properties of even and odd functions to help evaluate definite integrals.
■ Find the amounts of annuities.

## Area and Definite Integrals

From your study of geometry, you know that area is a number that defines the size of a bounded region. For simple regions, such as rectangles, triangles, and circles, area can be found using geometric formulas.

In this section, you will learn how to use calculus to find the areas of nonstandard regions, such as the region $R$ shown in Figure 5.5.



**FIGURE 5.5**    $\int_a^b f(x)\,dx = $ Area

---

**Definition of a Definite Integral**

Let $f$ be nonnegative and continuous on the closed interval $[a, b]$. The area of the region bounded by the graph of $f$, the $x$-axis, and the lines $x = a$ and $x = b$ is denoted by

$$\text{Area} = \int_a^b f(x)\,dx.$$

The expression $\int_a^b f(x)\,dx$ is called the **definite integral** from $a$ to $b$, where $a$ is the **lower limit of integration** and $b$ is the **upper limit of integration.**

---



**FIGURE 5.6**

*Example 1*    **Evaluating a Definite Integral**

Evaluate $\displaystyle\int_0^2 2x\,dx$.

**SOLUTION**    This definite integral represents the area of the region bounded by the graph of $f(x) = 2x$, the $x$-axis, and the line $x = 2$, as shown in Figure 5.6. The region is triangular, with a height of four units and a base of two units.
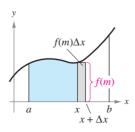
$$\int_0^2 2x\,dx = \frac{1}{2}(\text{base})(\text{height}) \qquad \text{Formula for area of triangle}$$

$$= \frac{1}{2}(2)(4) = 4 \qquad \text{Simplify.}$$

**✓CHECKPOINT 1**

Evaluate the definite integral using a geometric formula. Illustrate your answer with an appropriate sketch.

$$\int_0^3 4x\,dx \qquad ■$$

## The Fundamental Theorem of Calculus

Consider the function $A$, which denotes the area of the region shown in Figure 5.7. To discover the relationship between $A$ and $f$, let $x$ increase by an amount $\Delta x$. This increases the area by $\Delta A$. Let $f(m)$ and $f(M)$ denote the minimum and maximum values of $f$ on the interval $[x, x + \Delta x]$.



**FIGURE 5.7**    $A(x) = $ Area from $a$ to $x$
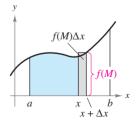


**FIGURE 5.8**

As indicated in Figure 5.8, you can write the inequality below.

| | | | |
|---|---|---|---|
| $f(m) \, \Delta x \leq$ | $\Delta A$ | $\leq f(M) \, \Delta x$ | See Figure 5.8. |
| $f(m) \leq$ | $\dfrac{\Delta A}{\Delta x}$ | $\leq f(M)$ | Divide each term by $\Delta x$. |
| $\lim\limits_{\Delta x \to 0} f(m) \leq$ | $\lim\limits_{\Delta x \to 0} \dfrac{\Delta A}{\Delta x}$ | $\leq \lim\limits_{\Delta x \to 0} f(M)$ | Take limit of each term. |
| $f(x) \leq$ | $A'(x)$ | $\leq f(x)$ | Definition of derivative of $A(x)$ |

So, $f(x) = A'(x)$, and $A(x) = F(x) + C$, where $F'(x) = f(x)$. Because $A(a) = 0$, it follows that $C = -F(a)$. So, $A(x) = F(x) - F(a)$, which implies that

$$A(b) = \int_a^b f(x) \, dx = F(b) - F(a).$$

This equation tells you that *if you can find an antiderivative for f*, then you can use the antiderivative to evaluate the definite integral $\int_a^b f(x) \, dx$. This result is called the **Fundamental Theorem of Calculus.**

---

**The Fundamental Theorem of Calculus**

If $f$ is nonnegative and continuous on the closed interval $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

where $F$ is any function such that $F'(x) = f(x)$ for all $x$ in $[a, b]$.

---

**STUDY TIP**

There are two basic ways to introduce the Fundamental Theorem of Calculus. One way uses an area function, as shown here. The other uses a summation process, as shown in Appendix A.

**Guidelines for Using the Fundamental Theorem of Calculus**

**1.** The Fundamental Theorem of Calculus describes a way of *evaluating* a definite integral, not a procedure for finding antiderivatives.

**2.** In applying the Fundamental Theorem, it is helpful to use the notation

$$\int_a^b f(x)\, dx = F(x)\Big]_a^b = F(b) - F(a).$$

**3.** The constant of integration $C$ can be dropped because

$$\int_a^b f(x)\, dx = \Big[F(x) + C\Big]_a^b$$
$$= [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a) + C - C$$
$$= F(b) - F(a).$$

In the development of the Fundamental Theorem of Calculus, $f$ was assumed to be nonnegative on the closed interval $[a, b]$. As such, the definite integral was defined as an area. Now, with the Fundamental Theorem, the definition can be extended to include functions that are negative on all or part of the closed interval $[a, b]$. Specifically, if $f$ is *any* function that is continuous on a closed interval $[a, b]$, then the **definite integral** of $f(x)$ from $a$ to $b$ is defined to be

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where $F$ is an antiderivative of $f$. Remember that definite integrals do not necessarily represent areas and can be negative, zero, or positive.

**Properties of Definite Integrals**

Let $f$ and $g$ be continuous on the closed interval $[a, b]$.
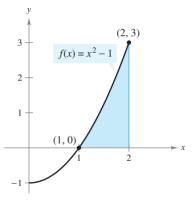
**1.** $\displaystyle\int_a^b k f(x)\, dx = k \int_a^b f(x)\, dx,\quad k$ is a constant.

**2.** $\displaystyle\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

**3.** $\displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx,\quad a < c < b$

**4.** $\displaystyle\int_a^a f(x)\, dx = 0$

**5.** $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

FIGURE 5.9    Area $= \int_1^2 (x^2 - 1)\, dx$

✓ **CHECKPOINT 2**

Find the area of the region bounded by the $x$-axis and the graph of $f(x) = x^2 + 1$, $2 \le x \le 3$. ∎

**Example 2**    Finding Area by the Fundamental Theorem

Find the area of the region bounded by the $x$-axis and the graph of

$$f(x) = x^2 - 1, \quad 1 \le x \le 2.$$

**SOLUTION**    Note that $f(x) \ge 0$ on the interval $1 \le x \le 2$, as shown in Figure 5.9. So, you can represent the area of the region by a definite integral. To find the area, use the Fundamental Theorem of Calculus.

$$
\begin{aligned}
\text{Area} &= \int_1^2 (x^2 - 1)\, dx && \text{\color{magenta}Definition of definite integral}\\[4pt]
&= \left(\frac{x^3}{3} - x\right)\Big|_1^2 && \text{\color{magenta}Find antiderivative.}\\[4pt]
&= \left(\frac{2^3}{3} - 2\right) - \left(\frac{1^3}{3} - 1\right) && \text{\color{magenta}Apply Fundamental Theorem.}\\[4pt]
&= \frac{4}{3} && \text{\color{magenta}Simplify.}
\end{aligned}
$$

So, the area of the region is $\frac{4}{3}$ square units.

**STUDY TIP**

It is easy to make errors in signs when evaluating definite integrals. To avoid such errors, enclose the values of the antiderivative at the upper and lower limits of integration in separate sets of parentheses, as shown above.
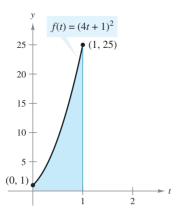


FIGURE 5.10

✓ **CHECKPOINT 3**

Evaluate $\int_0^1 (2t + 3)^3\, dt.$ ∎

**Example 3**    Evaluating a Definite Integral

Evaluate the definite integral

$$\int_0^1 (4t + 1)^2\, dt$$

and sketch the region whose area is represented by the integral.

**SOLUTION**

$$
\begin{aligned}
\int_0^1 (4t + 1)^2\, dt &= \frac{1}{4}\int_0^1 (4t + 1)^2 (4)\, dt && \text{\color{magenta}Multiply and divide by 4.}\\[4pt]
&= \frac{1}{4}\left[\frac{(4t + 1)^3}{3}\right]\Big|_0^1 && \text{\color{magenta}Find antiderivative.}\\[4pt]
&= \frac{1}{4}\left[\left(\frac{5^3}{3}\right) - \left(\frac{1}{3}\right)\right] && \text{\color{magenta}Apply Fundamental Theorem.}\\[4pt]
&= \frac{31}{3} && \text{\color{magenta}Simplify.}
\end{aligned}
$$

The region is shown in Figure 5.10.

## Example 4    Evaluating Definite Integrals

Evaluate each definite integral.

**a.** $\displaystyle\int_0^3 e^{2x}\,dx$    **b.** $\displaystyle\int_1^2 \frac{1}{x}\,dx$    **c.** $\displaystyle\int_1^4 -3\sqrt{x}\,dx$

**SOLUTION**

**a.** $\displaystyle\int_0^3 e^{2x}\,dx = \frac{1}{2}e^{2x}\Big]_0^3 = \frac{1}{2}(e^6 - e^0) \approx 201.21$

**b.** $\displaystyle\int_1^2 \frac{1}{x}\,dx = \ln x\Big]_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.69$

**c.** $\displaystyle\int_1^4 -3\sqrt{x}\,dx = -3\int_1^4 x^{1/2}\,dx$    <span style="color:magenta">Rewrite with rational exponent.</span>

$$= -3\left[\frac{x^{3/2}}{3/2}\right]_1^4 \qquad \text{Find antiderivative.}$$

$$= -2x^{3/2}\Big]_1^4$$

$$= -2(4^{3/2} - 1^{3/2}) \qquad \text{Apply Fundamental Theorem.}$$

$$= -2(8 - 1)$$

$$= -14 \qquad \text{Simplify.}$$

---

magenta">✓</span> **CHECKPOINT 4**

Evaluate each definite integral.

**a.** $\displaystyle\int_0^1 e^{4x}\,dx$

**b.** $\displaystyle\int_2^5 -\frac{1}{x}\,dx$ ■

---

**STUDY TIP**

In Example 4(c), note that the value of a definite integral can be negative.

---



$y = |2x - 1|$
$(2, 3)$
$(0, 1)$
$y = -(2x - 1)$    $y = 2x - 1$
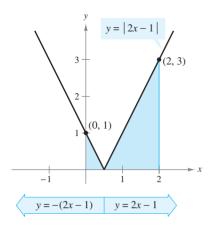
**FIGURE 5.11**

<span style="color:magenta">✓</span> **CHECKPOINT 5**

Evaluate $\displaystyle\int_0^5 |x - 2|\,dx.$ ■

## Example 5    Interpreting Absolute Value

Evaluate $\displaystyle\int_0^2 |2x - 1|\,dx.$

**SOLUTION**    The region represented by the definite integral is shown in Figure 5.11. From the definition of absolute value, you can write

$$|2x - 1| = \begin{cases} -(2x - 1), & x < \frac{1}{2} \\ 2x - 1, & x \geq \frac{1}{2} \end{cases}.$$

Using Property 3 of definite integrals, you can rewrite the integral as two definite integrals.

$$\int_0^2 |2x - 1|\,dx = \int_0^{1/2} -(2x - 1)\,dx + \int_{1/2}^2 (2x - 1)\,dx$$

$$= \left[-x^2 + x\right]_0^{1/2} + \left[x^2 - x\right]_{1/2}^2$$

$$= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0 + 0) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{5}{2}$$

## Marginal Analysis

You have already studied *marginal analysis* in the context of derivatives and differentials (Sections 2.3 and 3.8). There, you were given a cost, revenue, or profit function, and you used the derivative to approximate the additional cost, revenue, or profit obtained by selling one additional unit. In this section, you will examine the reverse process. That is, you will be given the marginal cost, marginal revenue, or marginal profit and will be asked to use a definite integral to find the exact increase or decrease in cost, revenue, or profit obtained by selling one or several additional units.

For instance, suppose you wanted to find the additional revenue obtained by increasing sales from $x_1$ to $x_2$ units. If you knew the revenue function $R$ you could simply subtract $R(x_1)$ from $R(x_2)$. If you didn't know the revenue function, but did know the marginal revenue function, you could still find the additional revenue by using a definite integral, as shown.

$$\int_{x_1}^{x_2} \frac{dR}{dx}\, dx = R(x_2) - R(x_1)$$

*Example 6*  **Analyzing a Profit Function**  (R)

The marginal profit for a product is modeled by $\dfrac{dP}{dx} = -0.0005x + 12.2$.

**a.** Find the change in profit when sales increase from 100 to 101 units.

**b.** Find the change in profit when sales increase from 100 to 110 units.

**SOLUTION**

**a.** The change in profit obtained by increasing sales from 100 to 101 units is

$$\int_{100}^{101} \frac{dP}{dx}\, dx = \int_{100}^{101} (-0.0005x + 12.2)\, dx$$

$$= \left[-0.00025x^2 + 12.2x\right]_{100}^{101}$$

$$\approx \$12.15.$$

**b.** The change in profit obtained by increasing sales from 100 to 110 units is

$$\int_{100}^{110} \frac{dP}{dx}\, dx = \int_{100}^{110} (-0.0005x + 12.2)\, dx$$

$$= \left[-0.00025x^2 + 12.2x\right]_{100}^{110}$$

$$\approx \$121.48$$

✓**CHECKPOINT 6**

The marginal profit for a product is modeled by

$$\frac{dP}{dx} = -0.0002x + 14.2.$$

**a.** Find the change in profit when sales increase from 100 to 101 units.

**b.** Find the change in profit when sales increase from 100 to 110 units. ∎

---

**TECHNOLOGY**

(T)  Symbolic integration utilities can be used to evaluate definite integrals as well as indefinite integrals. If you have access to such a program, try using it to evaluate several of the definite integrals in this section.

## Average Value

The *average value* of a function on a closed interval is defined below.

---

**Definition of the Average Value of a Function**

If $f$ is continuous on $[a, b]$, then the **average value** of $f$ on $[a, b]$ is

$$\text{Average value of } f \text{ on } [a, b] = \frac{1}{b - a} \int_a^b f(x) \, dx.$$

---

In Section 3.5, you studied the effects of production levels on cost using an average cost function. In the next example, you will study the effects of time on cost by using integration to find the average cost.

**Average Cost**

$c = 0.005t^2 + 0.01t + 13.15$

Average cost = $14.23

Cost per unit (in dollars)

Time (in months)

**FIGURE 5.12**

*Example 7*
**MAKE A DECISION    Finding the Average Cost    ®**

The cost per unit $c$ of producing CD players over a two-year period is modeled by

$$c = 0.005t^2 + 0.01t + 13.15, \quad 0 \le t \le 24$$

where $t$ is the time in months. Approximate the average cost per unit over the two-year period. Will the average cost per unit be less than $15?

**SOLUTION**    The average cost can be found by integrating $c$ over the interval $[0, 24]$.

$$\begin{aligned}
\text{Average cost per unit} &= \frac{1}{24} \int_0^{24} (0.005t^2 + 0.01t + 13.15) \, dt \\
&= \frac{1}{24} \left[ \frac{0.005t^3}{3} + \frac{0.01t^2}{2} + 13.15t \right]_0^{24} \\
&= \frac{1}{24}(341.52) \\
&= \$14.23 \qquad \text{(See Figure 5.12.)}
\end{aligned}$$

Yes, the average cost per unit will be less than $15.

To check the reasonableness of the average value found in Example 7, assume that one unit is produced each month, beginning with $t = 0$ and ending with $t = 24$. When $t = 0$, the cost is

$$c = 0.005(0)^2 + 0.01(0) + 13.15$$
$$= \$13.15.$$

Similarly, when $t = 1$, the cost is

$$c = 0.005(1)^2 + 0.01(1) + 13.15$$
$$\approx \$13.17.$$

Each month, the cost increases, and the average of the 25 costs is

$$\frac{13.15 + 13.17 + 13.19 + 13.23 + \cdots + 16.27}{25} \approx \$14.25.$$

✓ **CHECKPOINT 7**

Find the average cost per unit over a two-year period if the cost per unit $c$ of roller blades is given by $c = 0.005t^2 + 0.02t + 12.5$, for $0 \le t \le 24$, where $t$ is the time in months. ■

## Even and Odd Functions
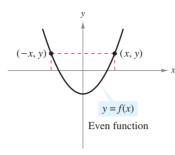
Several common functions have graphs that are symmetric with respect to the $y$-axis or the origin, as shown in Figure 5.13. If the graph of $f$ is symmetric with respect to the $y$-axis, as in Figure 5.13(a), then

$$f(-x) = f(x) \qquad \text{Even function}$$

and $f$ is called an **even** function. If the graph of $f$ is symmetric with respect to the origin, as in Figure 5.13(b), then

$$f(-x) = -f(x) \qquad \text{Odd function}$$

and $f$ is called an **odd** function.



(a) $y$-axis symmetry

(b) Origin symmetry

**FIGURE 5.13**

---

### Integration of Even and Odd Functions

**1.** If $f$ is an *even* function, then $\displaystyle\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$

**2.** If $f$ is an *odd* function, then $\displaystyle\int_{-a}^{a} f(x)\, dx = 0.$

---

*Example 8*  **Integrating Even and Odd Functions**

Evaluate each definite integral.

**a.** $\displaystyle\int_{-2}^{2} x^2\, dx$ **b.** $\displaystyle\int_{-2}^{2} x^3\, dx$

**SOLUTION**

**a.** Because $f(x) = x^2$ is even,

$$\int_{-2}^{2} x^2\, dx = 2\int_{0}^{2} x^2\, dx = 2\left[\frac{x^3}{3}\right]_{0}^{2} = 2\left(\frac{8}{3} - 0\right) = \frac{16}{3}.$$

**b.** Because $f(x) = x^3$ is odd,

$$\int_{-2}^{2} x^3\, dx = 0.$$

✓ **CHECKPOINT 8**

Evaluate each definite integral.

**a.** $\displaystyle\int_{-1}^{1} x^4\, dx$

**b.** $\displaystyle\int_{-1}^{1} x^5\, dx$

## Annuity

A sequence of equal payments made at regular time intervals over a period of time is called an **annuity.** Some examples of annuities are payroll savings plans, monthly home mortgage payments, and individual retirement accounts. The **amount of an annuity** is the sum of the payments plus the interest earned and can be found as shown below.

---

**Amount of an Annuity**

If $c$ represents a continuous income function in dollars per year (where $t$ is the time in years), $r$ represents the interest rate compounded continuously, and $T$ represents the term of the annuity in years, then the **amount of an annuity** is

$$\text{Amount of an annuity} = e^{rT} \int_0^T c(t)e^{-rt}\, dt.$$

---

### Example 9    Finding the Amount of an Annuity    ®

You deposit $2000 each year for 15 years in an individual retirement account (IRA) paying 5% interest. How much will you have in your IRA after 15 years?

**SOLUTION**    The income function for your deposit is $c(t) = 2000$. So, the amount of the annuity after 15 years will be

$$\text{Amount of an annuity} = e^{rT} \int_0^T c(t)e^{-rt}\, dt$$

$$= e^{(0.05)(15)} \int_0^{15} 2000e^{-0.05t}\, dt$$

$$= 2000e^{0.75}\left[ -\frac{e^{-0.05t}}{0.05} \right]_0^{15}$$

$$\approx \$44{,}680.00.$$

✓ **CHECKPOINT 9**

If you deposit $1000 in a savings account every year, paying 4% interest, how much will be in the account after 10 years? ■

---

**CONCEPT CHECK**

1. Complete the following: The indefinite integral $\int f(x)\, dx$ denotes a family of _____ , each of which is a(n) _____ of $f$, whereas the definite integral $\int_a^b f(x)\, dx$ is a _____ .

2. If $f$ is an odd function, then $\int_{-a}^{a} f(x)\, dx$ equals what?

3. State the Fundamental Theorem of Calculus.

4. What is an annuity?

**Skills Review 5.4**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 5.1–5.3.

In Exercises 1–4, find the indefinite integral.

**1.** $\int (3x + 7)\, dx$    **2.** $\int \left(x^{3/2} + 2\sqrt{x}\right) dx$    **3.** $\int \frac{1}{5x}\, dx$    **4.** $\int e^{-6x}\, dx$

In Exercises 5 and 6, evaluate the expression when $a = 5$ and $b = 3$.

**5.** $\left(\dfrac{a}{5} - a\right) - \left(\dfrac{b}{5} - b\right)$    **6.** $\left(6a - \dfrac{a^3}{3}\right) - \left(6b - \dfrac{b^3}{3}\right)$

In Exercises 7–10, integrate the marginal function.

**7.** $\dfrac{dC}{dx} = 0.02x^{3/2} + 29{,}500$    **8.** $\dfrac{dR}{dx} = 9000 + 2x$

**9.** $\dfrac{dP}{dx} = 25{,}000 - 0.01x$    **10.** $\dfrac{dC}{dx} = 0.03x^2 + 4600$

## Exercises 5.4

(T) In Exercises 1 and 2, use a graphing utility to graph the integrand. Use the graph to determine whether the definite integral is positive, negative, or zero.

**1.** $\int_0^3 \dfrac{5x}{x^2 + 1}\, dx$    **2.** $\int_{-2}^2 x\sqrt{x^2 + 1}\, dx$

In Exercises 3–12, sketch the region whose area is represented by the definite integral. Then use a geometric formula to evaluate the integral.

**3.** $\int_0^2 3\, dx$    **4.** $\int_0^3 4\, dx$

**5.** $\int_0^4 x\, dx$    **6.** $\int_0^4 \dfrac{x}{2}\, dx$

**7.** $\int_0^5 (x + 1)\, dx$    **8.** $\int_0^3 (2x + 1)\, dx$

**9.** $\int_{-2}^3 |x - 1|\, dx$    **10.** $\int_{-1}^4 |x - 2|\, dx$

**11.** $\int_{-3}^3 \sqrt{9 - x^2}\, dx$    **12.** $\int_0^2 \sqrt{4 - x^2}\, dx$

In Exercises 13 and 14, use the values $\int_0^5 f(x)\, dx = 6$ and $\int_0^5 g(x)\, dx = 2$ to evaluate the definite integral.

**13.** (a) $\int_0^5 [f(x) + g(x)]\, dx$    (b) $\int_0^5 [f(x) - g(x)]\, dx$

(c) $\int_0^5 -4f(x)\, dx$    (d) $\int_0^5 [f(x) - 3g(x)]\, dx$

**14.** (a) $\int_0^5 2g(x)\, dx$    (b) $\int_5^0 f(x)\, dx$
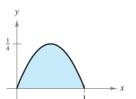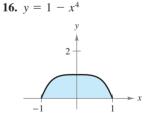
(c) $\int_5^5 f(x)\, dx$    (d) $\int_0^5 [f(x) - f(x)]\, dx$
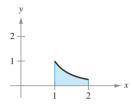
In Exercises 15–22, find the area of the region.

**15.** $y = x - x^2$    **16.** $y = 1 - x^4$



**17.** $y = \dfrac{1}{x^2}$    **18.** $y = \dfrac{2}{\sqrt{x}}$
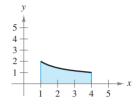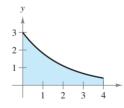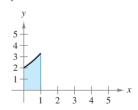


**19.** $y = 3e^{-x/2}$    **20.** $y = 2e^{x/2}$
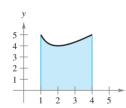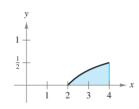
**21.** $y = \dfrac{x^2 + 4}{x}$

**22.** $y = \dfrac{x - 2}{x}$





In Exercises 23–46, evaluate the definite integral.

**23.** $\displaystyle\int_0^1 2x \, dx$

**24.** $\displaystyle\int_2^7 3v \, dv$

**25.** $\displaystyle\int_{-1}^0 (x - 2) \, dx$

**26.** $\displaystyle\int_2^5 (-3x + 4) \, dx$

**27.** $\displaystyle\int_{-1}^1 (2t - 1)^2 \, dt$

**28.** $\displaystyle\int_0^1 (1 - 2x)^2 \, dx$

**29.** $\displaystyle\int_0^3 (x - 2)^3 \, dx$

**30.** $\displaystyle\int_2^2 (x - 3)^4 \, dx$

**31.** $\displaystyle\int_{-1}^1 \left(\sqrt[3]{t} - 2\right) dt$

**32.** $\displaystyle\int_1^4 \sqrt{\dfrac{2}{x}} \, dx$

**33.** $\displaystyle\int_1^4 \dfrac{u - 2}{\sqrt{u}} \, du$

**34.** $\displaystyle\int_0^1 \dfrac{x - \sqrt{x}}{3} \, dx$

**35.** $\displaystyle\int_{-1}^0 (t^{1/3} - t^{2/3}) \, dt$

**36.** $\displaystyle\int_0^4 (x^{1/2} + x^{1/4}) \, dx$

**37.** $\displaystyle\int_0^4 \dfrac{1}{\sqrt{2x + 1}} \, dx$

**38.** $\displaystyle\int_0^2 \dfrac{x}{\sqrt{1 + 2x^2}} \, dx$

**39.** $\displaystyle\int_0^1 e^{-2x} \, dx$

**40.** $\displaystyle\int_1^2 e^{1-x} \, dx$

**41.** $\displaystyle\int_1^3 \dfrac{e^{3/x}}{x^2} \, dx$

**42.** $\displaystyle\int_{-1}^1 (e^x - e^{-x}) \, dx$

**43.** $\displaystyle\int_0^1 e^{2x}\sqrt{e^{2x} + 1} \, dx$

**44.** $\displaystyle\int_0^1 \dfrac{e^{-x}}{\sqrt{e^{-x} + 1}} \, dx$

**45.** $\displaystyle\int_0^2 \dfrac{x}{1 + 4x^2} \, dx$

**46.** $\displaystyle\int_0^1 \dfrac{e^{2x}}{e^{2x} + 1} \, dx$

In Exercises 47–50, evaluate the definite integral by the most convenient method. Explain your approach.

**47.** $\displaystyle\int_{-1}^1 |4x| \, dx$

**48.** $\displaystyle\int_0^3 |2x - 3| \, dx$

**49.** $\displaystyle\int_0^4 (2 - |x - 2|) \, dx$

**50.** $\displaystyle\int_{-4}^4 (4 - |x|) \, dx$

Ⓣ In Exercises 51–54, evaluate the definite integral by hand. Then use a symbolic integration utility to evaluate the definite integral. Briefly explain any differences in your results.

**51.** $\displaystyle\int_{-1}^2 \dfrac{x}{x^2 - 9} \, dx$

**52.** $\displaystyle\int_2^3 \dfrac{x + 1}{x^2 + 2x - 3} \, dx$

**53.** $\displaystyle\int_0^3 \dfrac{2e^x}{2 + e^x} \, dx$

**54.** $\displaystyle\int_1^2 \dfrac{(2 + \ln x)^3}{x} \, dx$

Ⓣ In Exercises 55–60, evaluate the definite integral by hand. Then use a graphing utility to graph the region whose area is represented by the integral.

**55.** $\displaystyle\int_1^3 (4x - 3) \, dx$

**56.** $\displaystyle\int_0^2 (x + 4) \, dx$

**57.** $\displaystyle\int_0^1 (x - x^3) \, dx$

**58.** $\displaystyle\int_0^2 (2 - x)\sqrt{x} \, dx$

**59.** $\displaystyle\int_2^4 \dfrac{3x^2}{x^3 - 1} \, dx$

**60.** $\displaystyle\int_0^{\ln 6} \dfrac{e^x}{2} \, dx$

In Exercises 61–64, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your results.

**61.** $y = 3x^2 + 1$, $\quad y = 0$, $\quad x = 0$, and $\quad x = 2$

**62.** $y = 1 + \sqrt{x}$, $\quad y = 0$, $\quad x = 0$, and $\quad x = 4$

**63.** $y = 4/x$ $\quad y = 0$, $\quad x = 1$, and $\quad x = 3$

**64.** $y = e^x$, $\quad y = 0$, $\quad x = 0$, and $\quad x = 2$

Ⓣ In Exercises 65–72, use a graphing utility to graph the function over the interval. Find the average value of the function over the interval. Then find all $x$-values in the interval for which the function is equal to its average value.

| Function | Interval |
|---|---|
| **65.** $f(x) = 4 - x^2$ | $[-2, 2]$ |
| **66.** $f(x) = x - 2\sqrt{x}$ | $[0, 4]$ |
| **67.** $f(x) = 2e^x$ | $[-1, 1]$ |
| **68.** $f(x) = e^{x/4}$ | $[0, 4]$ |
| **69.** $f(x) = x\sqrt{4 - x^2}$ | $[0, 2]$ |
| **70.** $f(x) = \dfrac{1}{(x - 3)^2}$ | $[0, 2]$ |
| **71.** $f(x) = \dfrac{6x}{x^2 + 1}$ | $[0, 7]$ |
| **72.** $f(x) = \dfrac{4x}{x^2 + 1}$ | $[0, 1]$ |

In Exercises 73–76, state whether the function is even, odd, or neither.

**73.** $f(x) = 3x^4$

**74.** $g(x) = x^3 - 2x$

**75.** $g(t) = 2t^5 - 3t^2$

**76.** $f(t) = 5t^4 + 1$

**77.** Use the value $\displaystyle\int_0^1 x^2 \, dx = \dfrac{1}{3}$ to evaluate each definite integral. Explain your reasoning.

(a) $\displaystyle\int_{-1}^0 x^2 \, dx$  (b) $\displaystyle\int_{-1}^1 x^2 \, dx$  (c) $\displaystyle\int_0^1 -x^2 \, dx$

**78.** Use the value $\int_0^2 x^3 \, dx = 4$ to evaluate each definite integral. Explain your reasoning.

(a) $\int_{-2}^{0} x^3 \, dx$  (b) $\int_{-2}^{2} x^3 \, dx$  (c) $\int_0^2 3x^3 \, dx$

**Marginal Analysis** In Exercises 79–84, find the change in cost C, revenue R, or profit P, for the given marginal. In each case, assume that the number of units x increases by 3 from the specified value of x.

| Marginal | Number of Units, x |
|---|---|
| **79.** $\dfrac{dC}{dx} = 2.25$ | $x = 100$ |
| **80.** $\dfrac{dC}{dx} = \dfrac{20{,}000}{x^2}$ | $x = 10$ |
| **81.** $\dfrac{dR}{dx} = 48 - 3x$ | $x = 12$ |
| **82.** $\dfrac{dR}{dx} = 75\left(20 + \dfrac{900}{x}\right)$ | $x = 500$ |
| **83.** $\dfrac{dP}{dx} = \dfrac{400 - x}{150}$ | $x = 200$ |
| **84.** $\dfrac{dP}{dx} = 12.5\left(40 - 3\sqrt{x}\right)$ | $x = 125$ |

**Annuity** In Exercises 85–88, find the amount of an annuity with income function $c(t)$, interest rate r, and term T.

**85.** $c(t) = \$250$,  $r = 8\%$,  $T = 6$ years

**86.** $c(t) = \$500$,  $r = 7\%$,  $T = 4$ years

**87.** $c(t) = \$1500$,  $r = 2\%$,  $T = 10$ years

**88.** $c(t) = \$2000$,  $r = 3\%$,  $T = 15$ years

**Capital Accumulation** In Exercises 89–92, you are given the rate of investment $dI/dt$. Find the capital accumulation over a five-year period by evaluating the definite integral

$$\text{Capital accumulation} = \int_0^5 \frac{dI}{dt} \, dt$$

where t is the time in years.

**89.** $\dfrac{dI}{dt} = 500$  **90.** $\dfrac{dI}{dt} = 100t$

**91.** $\dfrac{dI}{dt} = 500\sqrt{t + 1}$  **92.** $\dfrac{dI}{dt} = \dfrac{12{,}000t}{(t^2 + 2)^2}$

**93. Cost** The total cost of purchasing and maintaining a piece of equipment for x years can be modeled by

$$C = 5000\left(25 + 3\int_0^x t^{1/4} \, dt\right).$$

Find the total cost after (a) 1 year, (b) 5 years, and (c) 10 years.

**94. Depreciation** A company purchases a new machine for which the rate of depreciation can be modeled by

$$\frac{dV}{dt} = 10{,}000(t - 6), \quad 0 \le t \le 5$$

where V is the value of the machine after t years. Set up and evaluate the definite integral that yields the total loss of value of the machine over the first 3 years.

**95. Compound Interest** A deposit of $2250 is made in a savings account at an annual interest rate of 6%, compounded continuously. Find the average balance in the account during the first 5 years.

**96. Mortgage Debt** The rate of change of mortgage debt outstanding for one- to four-family homes in the United States from 1998 through 2005 can be modeled by

$$\frac{dM}{dt} = 5.142t^2 - 283{,}426.2e^{-t}$$

where M is the mortgage debt outstanding (in billions of dollars) and t is the year, with $t = 8$ corresponding to 1998. In 1998, the mortgage debt outstanding in the United States was $4259 billion. *(Source: Board of Governors of the Federal Reserve System)*

(a) Write a model for the debt as a function of t.

(b) What was the average mortgage debt outstanding for 1998 through 2005?

**97. Biology** In the North Sea, cod fish are in danger of becoming extinct because a large proportion of the catch is being taken before the cod can reach breeding age. The fishing quotas set in the United Kingdom from the years 1999 through 2006 can be approximated by the equation

$$y = -0.7020t^3 + 29.802t^2 - 422.77t + 2032.9$$

where y is the total catch weight (in thousands of kilograms) and t is the year, with $t = 9$ corresponding to 1999. Determine the average recommended quota during the years 1995 through 2006. *(Source: International Council for Exploration of the Sea)*

**98. Blood Flow** The velocity v of the flow of blood at a distance r from the center of an artery of radius R can be modeled by

$$v = k(R^2 - r^2), \quad k > 0$$

where k is a constant. Find the average velocity along a radius of the artery. (Use 0 and R as the limits of integration.)

(T) In Exercises 99–102, use a symbolic integration utility to evaluate the definite integral.

**99.** $\displaystyle\int_3^6 \frac{x}{3\sqrt{x^2 - 8}} \, dx$  **100.** $\displaystyle\int_{1/2}^{1} (x + 1)\sqrt{1 - x} \, dx$

**101.** $\displaystyle\int_2^5 \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx$  **102.** $\displaystyle\int_0^1 x^3(x^3 + 1)^3 \, dx$

# The Area of a Region Bounded by Two Graphs

■ Find the areas of regions bounded by two graphs.
■ Find consumer and producer surpluses.
■ Use the areas of regions bounded by two graphs to solve real-life problems.

## Area of a Region Bounded by Two Graphs

With a few modifications, you can extend the use of definite integrals from finding the area of a region *under a graph* to finding the area of a region *bounded by two graphs.* To see how this is done, consider the region bounded by the graphs of $f$, $g$, $x = a$, and $x = b$, as shown in Figure 5.14. If the graphs of both $f$ and $g$ lie above the $x$-axis, then you can interpret the area of the region between the graphs as the area of the region under the graph of $g$ subtracted from the area of the region under the graph of $f$, as shown in Figure 5.14.



(Area between $f$ and $g$) $=$ (Area of region under $f$) $-$ (Area of region under $g$)

$$\int_a^b [f(x) - g(x)]\, dx \quad = \quad \int_a^b f(x)\, dx \quad - \quad \int_a^b g(x)\, dx$$
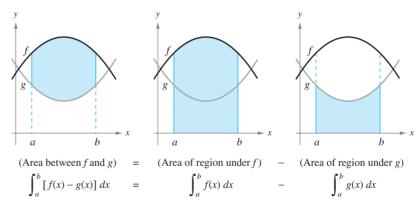
**FIGURE 5.14**

Although Figure 5.14 depicts the graphs of $f$ and $g$ lying above the $x$-axis, this is not necessary, and the same integrand $[f(x) - g(x)]$ can be used as long as both functions are continuous and $g(x) \leq f(x)$ on the interval $[a, b]$.

### Area of a Region Bounded by Two Graphs

If $f$ and $g$ are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all $x$ in the interval, then the area of the region bounded by the graphs of $f$, $g$, $x = a$, and $x = b$ is given by

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

### *DISCOVERY*

Sketch the graph of $f(x) = x^3 - 4x$ and shade in the regions bounded by the graph of $f$ and the $x$-axis. Write the appropriate integral(s) for this area.
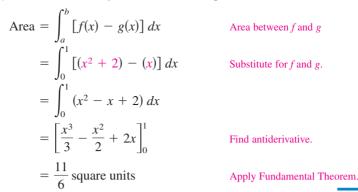
*Example 1*  **Finding the Area Bounded by Two Graphs**

Find the area of the region bounded by the graphs of

$$y = x^2 + 2 \quad \text{and} \quad y = x$$

for $0 \le x \le 1$.

**SOLUTION**  Begin by sketching the graphs of both functions, as shown in Figure 5.15. From the figure, you can see that $x \le x^2 + 2$ for all $x$ in $[0, 1]$. So, you can let $f(x) = x^2 + 2$ and $g(x) = x$. Then compute the area as shown.

$$\text{Area} = \int_a^b [f(x) - g(x)] \, dx \qquad \text{Area between } f \text{ and } g$$

$$= \int_0^1 [(x^2 + 2) - (x)] \, dx \qquad \text{Substitute for } f \text{ and } g.$$

$$= \int_0^1 (x^2 - x + 2) \, dx$$

$$= \left[ \frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_0^1 \qquad \text{Find antiderivative.}$$

$$= \frac{11}{6} \text{ square units} \qquad \text{Apply Fundamental Theorem.}$$
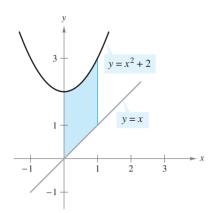
**FIGURE 5.15**

**✓ CHECKPOINT 1**

Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x$ for $0 \le x \le 2$. Sketch the region bounded by the graphs. ■
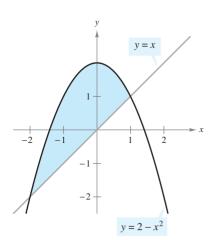
*Example 2*  **Finding the Area Between Intersecting Graphs**

Find the area of the region bounded by the graphs of

$$y = 2 - x^2 \quad \text{and} \quad y = x.$$

**SOLUTION**  In this problem, the values of $a$ and $b$ are not given and you must compute them by finding the points of intersection of the two graphs. To do this, equate the two functions and solve for $x$. When you do this, you will obtain $x = -2$ and $x = 1$. In Figure 5.16, you can see that the graph of $f(x) = 2 - x^2$ lies above the graph of $g(x) = x$ for all $x$ in the interval $[-2, 1]$.

$$\text{Area} = \int_a^b [f(x) - g(x)] \, dx \qquad \text{Area between } f \text{ and } g$$

$$= \int_{-2}^1 [(2 - x^2) - (x)] \, dx \qquad \text{Substitute for } f \text{ and } g.$$

$$= \int_{-2}^1 (-x^2 - x + 2) \, dx$$

$$= \left[ -\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 \qquad \text{Find antiderivative.}$$

$$= \frac{9}{2} \text{ square units} \qquad \text{Apply Fundamental Theorem.}$$

**FIGURE 5.16**

**✓ CHECKPOINT 2**

Find the area of the region bounded by the graphs of $y = 3 - x^2$ and $y = 2x$. ■

$y = x^2 - 3x - 4$
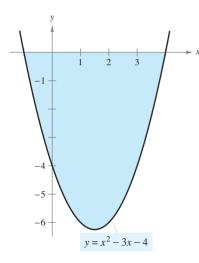
**FIGURE 5.17**

**STUDY TIP**

When finding the area of a region bounded by two graphs, be sure to use the integrand $[f(x) - g(x)]$. Be sure you realize that you cannot interchange $f(x)$ and $g(x)$. For instance, when solving Example 3, if you subtract $f(x)$ from $g(x)$, you will obtain an answer of $-\frac{125}{6}$, which is not correct.

*Example 3*  **Finding an Area Below the *x*-Axis**

Find the area of the region bounded by the graph of

$$y = x^2 - 3x - 4$$

and the *x*-axis.

**SOLUTION**  Begin by finding the *x*-intercepts of the graph. To do this, set the function equal to zero and solve for *x*.

$$x^2 - 3x - 4 = 0 \qquad \text{Set function equal to 0.}$$
$$(x - 4)(x + 1) = 0 \qquad \text{Factor.}$$
$$x = 4, \, x = -1 \qquad \text{Solve for } x.$$

From Figure 5.17, you can see that $x^2 - 3x - 4 \le 0$ for all $x$ in the interval $[-1, 4]$. So, you can let $f(x) = 0$ and $g(x) = x^2 - 3x - 4$, and compute the area as shown.
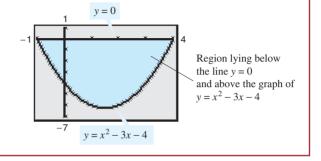
$$
\begin{aligned}
\text{Area} &= \int_a^b \left[ f(x) - g(x) \right] dx && \text{Area between } f \text{ and } g \\
&= \int_{-1}^4 \left[ (0) - (x^2 - 3x - 4) \right] dx && \text{Substitute for } f \text{ and } g. \\
&= \int_{-1}^4 (-x^2 + 3x + 4) \, dx \\
&= \left[ -\frac{x^3}{3} + \frac{3x^2}{2} + 4x \right]_{-1}^4 && \text{Find antiderivative.} \\
&= \frac{125}{6} \text{ square units} && \text{Apply Fundamental Theorem.}
\end{aligned}
$$

✓**CHECKPOINT 3**

Find the area of the region bounded by the graph of $y = x^2 - x - 2$ and the *x*-axis.  ■

---

**TECHNOLOGY**

Ⓣ Most graphing utilities can display regions that are bounded by two graphs. For instance, to graph the region in Example 3, set the viewing window to $-1 \le x \le 4$ and $-7 \le y \le 1$. Consult your user's manual for specific keystrokes on how to shade the graph. You should obtain the graph shown at the right.*



Region lying below the line $y = 0$ and above the graph of $y = x^2 - 3x - 4$

---

*Specific calculator keystroke instructions for operations in this and other technology boxes can be found at *college.hmco.com/info/larsonapplied*.

Sometimes two graphs intersect at more than two points. To determine the area of the region bounded by two such graphs, you must find *all* points of intersection and check to see which graph is above the other in each interval determined by the points.

**Example 4** **Using Multiple Points of Intersection**

Find the area of the region bounded by the graphs of

$$f(x) = 3x^3 - x^2 - 10x \quad \text{and} \quad g(x) = -x^2 + 2x.$$

**SOLUTION**    To find the points of intersection of the two graphs, set the functions equal to each other and solve for $x$.

| | |
|---|---|
| $f(x) = g(x)$ | Set $f(x)$ equal to $g(x)$. |
| $3x^3 - x^2 - 10x = -x^2 + 2x$ | Substitute for $f(x)$ and $g(x)$. |
| $3x^3 - 12x = 0$ | Write in general form. |
| $3x(x^2 - 4) = 0$ | |
| $3x(x - 2)(x + 2) = 0$ | Factor. |
| $x = 0, x = 2, x = -2$ | Solve for $x$. |
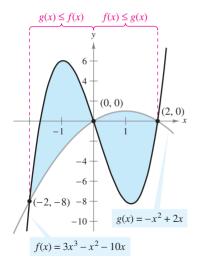
These three points of intersection determine two intervals of integration: $[-2, 0]$ and $[0, 2]$. In Figure 5.18, you can see that $g(x) \le f(x)$ in the interval $[-2, 0]$, and that $f(x) \le g(x)$ in the interval $[0, 2]$. So, you must use two integrals to determine the area of the region bounded by the graphs of $f$ and $g$: one for the interval $[-2, 0]$ and one for the interval $[0, 2]$.

$$\text{Area} = \int_{-2}^{0} [f(x) - g(x)]\, dx + \int_{0}^{2} [g(x) - f(x)]\, dx$$

$$= \int_{-2}^{0} (3x^3 - 12x)\, dx + \int_{0}^{2} (-3x^3 + 12x)\, dx$$

$$= \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[-\frac{3x^4}{4} + 6x^2\right]_{0}^{2}$$

$$= (0 - 0) - (12 - 24) + (-12 + 24) - (0 + 0)$$

$$= 24$$

So, the region has an area of 24 square units.



$g(x) \le f(x)$    $f(x) \le g(x)$

$(0, 0)$

$(2, 0)$

$(-2, -8)$

$g(x) = -x^2 + 2x$

$f(x) = 3x^3 - x^2 - 10x$

**FIGURE 5.18**

**✓ CHECKPOINT 4**

Find the area of the region bounded by the graphs of $f(x) = x^3 + 2x^2 - 3x$ and $g(x) = x^2 + 3x$. Sketch a graph of the region. ■

**STUDY TIP**

It is easy to make an error when calculating areas such as that in Example 4. To give yourself some idea about the reasonableness of your solution, you could make a careful sketch of the region on graph paper and then use the grid on the graph paper to approximate the area. Try doing this with the graph shown in Figure 5.18. Is your approximation close to 24 square units?
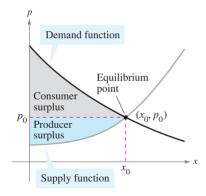
**FIGURE 5.19**

## Consumer Surplus and Producer Surplus

In Section 1.2, you learned that a demand function relates the price of a product to the consumer demand. You also learned that a supply function relates the price of a product to producers' willingness to supply the product. The point $(x_0, p_0)$ at which a demand function $p = D(x)$ and a supply function $p = S(x)$ intersect is the equilibrium point.

Economists call the area of the region bounded by the graph of the demand function, the horizontal line $p = p_0$, and the vertical line $x = 0$ the **consumer surplus.** Similarly, the area of the region bounded by the graph of the supply function, the horizontal line $p = p_0$, and the vertical line $x = 0$ is called the **producer surplus,** as shown in Figure 5.19.

**Example 5**    **Finding Surpluses**    (R)

The demand and supply functions for a product are modeled by

   *Demand:* $p = -0.36x + 9$   and   *Supply:* $p = 0.14x + 2$

where $x$ is the number of units (in millions). Find the consumer and producer surpluses for this product.

**SOLUTION**    By equating the demand and supply functions, you can determine that the point of equilibrium occurs when $x = 14$ (million) and the price is $3.96 per unit.

$$\text{Consumer surplus} = \int_0^{14} (\text{demand function} - \text{price})\, dx$$

$$= \int_0^{14} [(-0.36x + 9) - 3.96]\, dx$$

$$= \left[-0.18x^2 + 5.04x\right]_0^{14}$$

$$= 35.28$$

$$\text{Producer surplus} = \int_0^{14} (\text{price} - \text{supply function})\, dx$$

$$= \int_0^{14} [3.96 - (0.14x + 2)]\, dx$$

$$= \left[-0.07x^2 + 1.96x\right]_0^{14}$$

$$= 13.72$$

The consumer surplus and producer surplus are shown in Figure 5.20.



**FIGURE 5.20**

✓ **CHECKPOINT 5**

The demand and supply functions for a product are modeled by

   *Demand:* $p = -0.2x + 8$   and   *Supply:* $p = 0.1x + 2$

where $x$ is the number of units (in millions). Find the consumer and producer surpluses for this product. ■

## Application

In addition to consumer and producer surpluses, there are many other types of applications involving the area of a region bounded by two graphs. Example 6 shows one of these applications.

**Example 6**  **Modeling Petroleum Consumption**  Ⓡ

In the *Annual Energy Outlook*, the U.S. Energy Information Administration projected the consumption $C$ (in quadrillions of Btu per year) of petroleum to follow the model

$$C_1 = 0.004t^2 + 0.330t + 38.3, \quad 0 \le t \le 30$$

where $t = 0$ corresponds to 2000. If the actual consumption more closely followed the model

$$C_2 = 0.005t^2 + 0.301t + 38.2, \quad 0 \le t \le 30$$

how much petroleum would be saved?

**SOLUTION**  The petroleum saved can be represented as the area of the region between the graphs of $C_1$ and $C_2$, as shown in Figure 5.21.

$$
\begin{aligned}
\text{Petroleum saved} &= \int_0^{30} (C_1 - C_2) \, dt \\
&= \int_0^{30} (-0.001t^2 + 0.029t + 0.1) \, dt \\
&= \left[ -\frac{0.001}{3} t^3 + \frac{0.029}{2} t^2 + 0.1t \right]_0^{30} \\
&\approx 7.1
\end{aligned}
$$

So, about 7.1 quadrillion Btu of petroleum would be saved.

✓**CHECKPOINT 6**

The projected fuel cost $C$ (in millions of dollars per year) for a trucking company from 2008 through 2020 is $C_1 = 5.6 + 2.21t$, $8 \le t \le 20$, where $t = 8$ corresponds to 2008. If the company purchases more efficient truck engines, fuel cost is expected to decrease and to follow the model $C_2 = 4.7 + 2.04t$, $8 \le t \le 20$. How much can the company save with the more efficient engines? ■



**U.S. Petroleum Consumption**

Petroleum saved

$C_1$  $C_2$

Year (0 ↔ 2000)

**FIGURE 5.21**



In 2005, the United States consumed about 40.4 quadrillion Btu of petroleum.

---

**CONCEPT CHECK**

1. When finding the area of a region bounded by two graphs, you use the integrand $[f(x) - g(x)]$. Identify what $f$ and $g$ represent.

2. Consider the functions $f$ and $g$, where $f$ and $g$ are continuous on $[a, b]$ and $g(x) \le f(x)$ for all $x$ in the interval. How can you find the area of the region bounded by the graphs of $f$, $g$, $x = a$, and $x = b$?

3. Describe the characteristics of typical demand and supply functions.

4. Suppose that the demand and supply functions for a product do not intersect. What can you conclude?

| **Skills Review 5.5** | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.1 and 1.2. |
|---|---|

In Exercises 1–4, simplify the expression.

**1.** $(-x^2 + 4x + 3) - (x + 1)$

**2.** $(-2x^2 + 3x + 9) - (-x + 5)$

**3.** $(-x^3 + 3x^2 - 1) - (x^2 - 4x + 4)$

**4.** $(3x + 1) - (-x^3 + 9x + 2)$

In Exercises 5–10, find the points of intersection of the graphs.

**5.** $f(x) = x^2 - 4x + 4$, $g(x) = 4$

**6.** $f(x) = -3x^2$, $g(x) = 6 - 9x$

**7.** $f(x) = x^2$, $g(x) = -x + 6$

**8.** $f(x) = \frac{1}{2}x^3$, $g(x) = 2x$

**9.** $f(x) = x^2 - 3x$, $g(x) = 3x - 5$

**10.** $f(x) = e^x$, $g(x) = e$

## Exercises 5.5
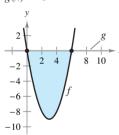
In Exercises 1–6, find the area of the region.

**1.** $f(x) = x^2 - 6x$
$g(x) = 0$
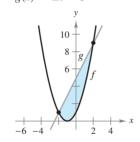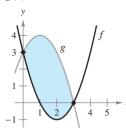


**2.** $f(x) = x^2 + 2x + 1$
$g(x) = 2x + 5$
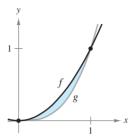


**3.** $f(x) = x^2 - 4x + 3$
$g(x) = -x^2 + 2x + 3$
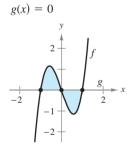


**4.** $f(x) = x^2$
$g(x) = x^3$



**5.** $f(x) = 3(x^3 - x)$
$g(x) = 0$
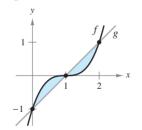


**6.** $f(x) = (x - 1)^3$
$g(x) = x - 1$



In Exercises 7–12, the integrand of the definite integral is a difference of two functions. Sketch the graph of each function and shade the region whose area is represented by the integral.

**7.** $\int_0^4 \left[(x + 1) - \frac{1}{2}x\right] dx$

**8.** $\int_{-1}^1 \left[(1 - x^2) - (x^2 - 1)\right] dx$

**9.** $\int_{-2}^2 \left[2x^2 - (x^4 - 2x^2)\right] dx$

**10.** $\int_{-4}^0 \left[(x - 6) - (x^2 + 5x - 6)\right] dx$

**11.** $\int_{-1}^2 \left[(y^2 + 2) - 1\right] dy$

**12.** $\int_{-2}^3 \left[(y + 6) - y^2\right] dy$

**Think About It**   In Exercises 13 and 14, determine which value best approximates the area of the region bounded by the graphs of $f$ and $g$. (Make your selection on the basis of a sketch of the region and not by performing any calculations.)

**13.** $f(x) = x + 1$,   $g(x) = (x - 1)^2$

(a) $-2$   (b) 2   (c) 10   (d) 4   (e) 8

**14.** $f(x) = 2 - \frac{1}{2}x$,   $g(x) = 2 - \sqrt{x}$

(a) 1   (b) 6   (c) $-3$   (d) 3   (e) 4

In Exercises 15–30, sketch the region bounded by the graphs of the functions and find the area of the region.

**15.** $y = \frac{1}{x^2}, y = 0, x = 1, x = 5$

**16.** $y = x^3 - 2x + 1, y = -2x, x = 1$

**17.** $f(x) = \sqrt[3]{x}, g(x) = x$

**18.** $f(x) = \sqrt{3x} + 1, g(x) = x + 1$

**19.** $y = x^2 - 4x + 3, y = 3 + 4x - x^2$

**20.** $y = 4 - x^2, y = x^2$

**21.** $y = xe^{-x^2}, y = 0, x = 0, x = 1$

**22.** $y = \dfrac{e^{1/x}}{x^2}, y = 0, x = 1, x = 3$

**23.** $y = \dfrac{8}{x}, y = x^2, y = 0, x = 1, x = 4$

**24.** $y = \dfrac{1}{x}, y = x^3, x = \dfrac{1}{2}, x = 1$

**25.** $f(x) = e^{0.5x}, g(x) = -\dfrac{1}{x}, x = 1, x = 2$

**26.** $f(x) = \dfrac{1}{x}, g(x) = -e^x, x = \dfrac{1}{2}, x = 1$

**27.** $f(y) = y^2, g(y) = y + 2$

**28.** $f(y) = y(2 - y), g(y) = -y$

**29.** $f(y) = \sqrt{y}, y = 9, x = 0$

**30.** $f(y) = y^2 + 1, g(y) = 4 - 2y$

Ⓣ In Exercises 31–34, use a graphing utility to graph the region bounded by the graphs of the functions. Write the definite integrals that represent the area of the region. (*Hint:* Multiple integrals may be necessary.)

**31.** $f(x) = 2x, \ g(x) = 4 - 2x, \ h(x) = 0$

**32.** $f(x) = x(x^2 - 3x + 3), g(x) = x^2$

**33.** $y = \dfrac{4}{x}, y = x, x = 1, x = 4$

**34.** $y = x^3 - 4x^2 + 1, y = x - 3$

Ⓣ In Exercises 35–38, use a graphing utility to graph the region bounded by the graphs of the functions, and find the area of the region.

**35.** $f(x) = x^2 - 4x, g(x) = 0$

**36.** $f(x) = 3 - 2x - x^2, g(x) = 0$

**37.** $f(x) = x^2 + 2x + 1, g(x) = x + 1$

**38.** $f(x) = -x^2 + 4x + 2, g(x) = x + 2$

In Exercises 39 and 40, use integration to find the area of the triangular region having the given vertices.

**39.** $(0, 0), (4, 0), (4, 4)$

**40.** $(0, 0), (4, 0), (6, 4)$

**Consumer and Producer Surpluses** In Exercises 41–44, find the consumer and producer surpluses.

| Demand Function | Supply Function |
|---|---|
| **41.** $p_1(x) = 50 - 0.5x$ | $p_2(x) = 0.125x$ |
| **42.** $p_1(x) = 300 - x$ | $p_2(x) = 100 + x$ |
| **43.** $p_1(x) = 200 - 0.4x$ | $p_2(x) = 100 + 1.6x$ |
| **44.** $p_1(x) = 975 - 23x$ | $p_2(x) = 42x$ |
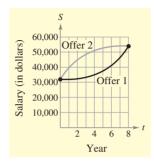
**45.** *MAKE A DECISION: JOB OFFERS* A college graduate has two job offers. The starting salary for each is \$32,000, and after 8 years of service each will pay \$54,000. The salary increase for each offer is shown in the figure. From a strictly monetary viewpoint, which is the better offer? Explain.
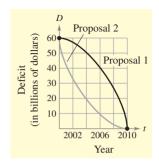


Figure for 45                          Figure for 46

**46.** *MAKE A DECISION: BUDGET DEFICITS* A state legislature is debating two proposals for eliminating the annual budget deficits by the year 2010. The rate of decrease of the deficits for each proposal is shown in the figure. From the viewpoint of minimizing the cumulative state deficit, which is the better proposal? Explain.

**Revenue** In Exercises 47 and 48, two models, $R_1$ and $R_2$, are given for revenue (in billions of dollars per year) for a large corporation. Both models are estimates of revenues for 2007 through 2011, with $t = 7$ corresponding to 2007. Which model is projecting the greater revenue? How much more total revenue does that model project over the five-year period?

**47.** $R_1 = 7.21 + 0.58t, R_2 = 7.21 + 0.45t$

**48.** $R_1 = 7.21 + 0.26t + 0.02t^2, R_2 = 7.21 + 0.1t + 0.01t^2$

**49. Fuel Cost** The projected fuel cost $C$ (in millions of dollars per year) for an airline company from 2007 through 2013 is $C_1 = 568.5 + 7.15t$, where $t = 7$ corresponds to 2007. If the company purchases more efficient airplane engines, fuel cost is expected to decrease and to follow the model $C_2 = 525.6 + 6.43t$. How much can the company save with the more efficient engines? Explain your reasoning.

**50. Health**  An epidemic was spreading such that $t$ weeks after its outbreak it had infected

$$N_1(t) = 0.1t^2 + 0.5t + 150, \quad 0 \le t \le 50$$

people. Twenty-five weeks after the outbreak, a vaccine was developed and administered to the public. At that point, the number of people infected was governed by the model

$$N_2(t) = -0.2t^2 + 6t + 200.$$

Approximate the number of people that the vaccine prevented from becoming ill during the epidemic.

**51. Consumer Trends**  For the years 1996 through 2004, the per capita consumption of fresh pineapples (in pounds per year) in the United States can be modeled by

$$C(t) = \begin{cases} -0.046t^2 + 1.07t - 2.9, & 6 \le t \le 10 \\ -0.164t^2 + 4.53t - 26.8, & 10 < t \le 14 \end{cases}$$

where $t$ is the year, with $t = 6$ corresponding to 1996. *(Source: U.S. Department of Agriculture)*

(T) (a) Use a graphing utility to graph this model.

(b) Suppose the fresh pineapple consumption from 2001 through 2004 had continued to follow the model for 1996 through 2000. How many more or fewer pounds of fresh pineapples would have been consumed from 2001 through 2004?

**52. Consumer and Producer Surpluses**  Factory orders for an air conditioner are about 6000 units per week when the price is $331 and about 8000 units per week when the price is $303. The supply function is given by $p = 0.0275x$. Find the consumer and producer surpluses. (Assume the demand function is linear.)

**53. Consumer and Producer Surpluses**  Repeat Exercise 52 with a demand of about 6000 units per week when the price is $325 and about 8000 units per week when the price is $300. Find the consumer and producer surpluses. (Assume the demand function is linear.)

**54. Cost, Revenue, and Profit**  The revenue from a manufacturing process (in millions of dollars per year) is projected to follow the model $R = 100$ for 10 years. Over the same period of time, the cost (in millions of dollars per year) is projected to follow the model $C = 60 + 0.2t^2$, where $t$ is the time (in years). Approximate the profit over the 10-year period.

**55. Cost, Revenue, and Profit**  Repeat Exercise 54 for revenue and cost models given by $R = 100 + 0.08t$ and $C = 60 + 0.2t^2$.

**56. Lorenz Curve**  Economists use *Lorenz curves* to illustrate the distribution of income in a country. Letting $x$ represent the percent of families in a country and $y$ the percent of total income, the model $y = x$ would represent a country in which each family had the same income. The Lorenz curve, $y = f(x)$, represents the actual income distribution. The area between these two models, for

$0 \le x \le 100$, indicates the "income inequality" of a country. In 2005, the Lorenz curve for the United States could be modeled by

$$y = (0.00061x^2 + 0.0218x + 1.723)^2, \quad 0 \le x \le 100$$

where $x$ is measured from the poorest to the wealthiest families. Find the income inequality for the United States in 2005. *(Source: U.S. Census Bureau)*

(S) **57. Income Distribution**  Using the Lorenz curve in Exercise 56 and a spreadsheet, complete the table, which lists the percent of total income earned by each quintile in the United States in 2005.

| Quintile | Lowest | 2nd | 3rd | 4th | Highest |
|----------|--------|-----|-----|-----|---------|
| Percent  |        |     |     |     |         |

**58. Extended Application**  To work an extended application analyzing the receipts and expenditures for the Old-Age and Survivors Insurance Trust Fund (Social Security Trust Fund) from 1990 through 2005, visit this text's website at *college.hmco.com.*  *(Data Source: Social Security Administration)*

**Business Capsule**

Photo courtesy of Avis Yates Rivers

After losing her job as an account executive in 1985, Avis Yates Rivers used $2500 to start a word processing business from the basement of her home. In 1996, as a spin-off from her word processing business, Rivers established Technology Concepts Group. Today, this Somerset, New Jersey-based firm provides information technology management consulting, e-business solutions, and network and desktop support for corporate and government customers. Annual revenue is currently $1.1 million.

**59. Research Project**  Use your school's library, the Internet, or some other reference source to research a small company similar to that described above. Describe the impact of different factors, such as start-up capital and market conditions, on a company's revenue.

## Section 5.6

# The Definite Integral as the Limit of a Sum

- Use the Midpoint Rule to approximate definite integrals.
- Use a symbolic integration utility to approximate definite integrals.

## The Midpoint Rule

In Section 5.4, you learned that you cannot use the Fundamental Theorem of Calculus to evaluate a definite integral unless you can find an antiderivative of the integrand. In cases where this cannot be done, you can approximate the value of the integral using an approximation technique. One such technique is called the **Midpoint Rule.** (Two other techniques are discussed in Section 6.4.)

### Example 1    Approximating the Area of a Plane Region

Use the five rectangles in Figure 5.22 to approximate the area of the region bounded by the graph of $f(x) = -x^2 + 5$, the $x$-axis, and the lines $x = 0$ and $x = 2$.

**SOLUTION**    You can find the heights of the five rectangles by evaluating $f$ at the midpoint of each of the following intervals.

$$\left[0, \frac{2}{5}\right], \quad \left[\frac{2}{5}, \frac{4}{5}\right], \quad \left[\frac{4}{5}, \frac{6}{5}\right], \quad \left[\frac{6}{5}, \frac{8}{5}\right], \quad \left[\frac{8}{5}, \frac{10}{5}\right]$$

Evaluate $f$ at the midpoints of these intervals.

The width of each rectangle is $\frac{2}{5}$. So, the sum of the five areas is

$$
\begin{aligned}
\text{Area} &\approx \frac{2}{5}f\left(\frac{1}{5}\right) + \frac{2}{5}f\left(\frac{3}{5}\right) + \frac{2}{5}f\left(\frac{5}{5}\right) + \frac{2}{5}f\left(\frac{7}{5}\right) + \frac{2}{5}f\left(\frac{9}{5}\right) \\
&= \frac{2}{5}\left[f\left(\frac{1}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{5}{5}\right) + f\left(\frac{7}{5}\right) + f\left(\frac{9}{5}\right)\right] \\
&= \frac{2}{5}\left(\frac{124}{25} + \frac{116}{25} + \frac{100}{25} + \frac{76}{25} + \frac{44}{25}\right) \\
&= \frac{920}{125} \\
&= 7.36.
\end{aligned}
$$



$f(x) = -x^2 + 5$
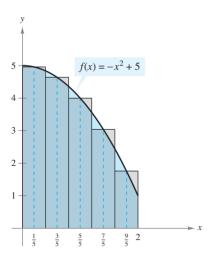
**FIGURE 5.22**

### ✓ CHECKPOINT 1

Use four rectangles to approximate the area of the region bounded by the graph of $f(x) = x^2 + 1$, the $x$-axis, $x = 0$ and $x = 2$. ■

For the region in Example 1, you can find the exact area with a definite integral. That is,

$$\text{Area} = \int_0^2 (-x^2 + 5) \, dx = \frac{22}{3} \approx 7.33.$$

The approximation procedure used in Example 1 is the **Midpoint Rule.** You can use the Midpoint Rule to approximate *any* definite integral—not just those representing area. The basic steps are summarized below.

**Guidelines for Using the Midpoint Rule**

To approximate the definite integral $\int_a^b f(x)\,dx$ with the Midpoint Rule, use the steps below.

1. Divide the interval $[a, b]$ into $n$ subintervals, each of width

$$\Delta x = \frac{b - a}{n}.$$

2. Find the midpoint of each subinterval.

$$\text{Midpoints} = \{x_1, x_2, x_3, \ldots, x_n\}$$

3. Evaluate $f$ at each midpoint and form the sum as shown.

$$\int_a^b f(x)\,dx \approx \frac{b - a}{n}\left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)\right]$$

An important characteristic of the Midpoint Rule is that the approximation tends to improve as $n$ increases. The table below shows the approximations for the area of the region described in Example 1 for various values of $n$. For example, for $n = 10$, the Midpoint Rule yields

$$\int_0^2 (-x^2 + 5)\,dx \approx \frac{2}{10}\left[f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + \cdots + f\left(\frac{19}{10}\right)\right]$$
$$= 7.34.$$

| $n$ | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|
| Approximation | 7.3600 | 7.3400 | 7.3363 | 7.3350 | 7.3344 | 7.3341 |

Note that as $n$ increases, the approximation gets closer and closer to the exact value of the integral, which was found to be

$$\frac{22}{3} \approx 7.3333.$$

**STUDY TIP**

In Example 1, the Midpoint Rule is used to approximate an integral whose exact value can be found with the Fundamental Theorem of Calculus. This was done to illustrate the accuracy of the rule. In practice, of course, you would use the Midpoint Rule to approximate the values of definite integrals for which you cannot find an antiderivative. Examples 2 and 3 illustrate such integrals.

## Example 2 Using the Midpoint Rule

Use the Midpoint Rule with $n = 5$ to approximate $\int_0^1 \frac{1}{x^2 + 1}\, dx$.

**SOLUTION** With $n = 5$, the interval $[0, 1]$ is divided into five subintervals.

$$\left[0, \frac{1}{5}\right], \quad \left[\frac{1}{5}, \frac{2}{5}\right], \quad \left[\frac{2}{5}, \frac{3}{5}\right], \quad \left[\frac{3}{5}, \frac{4}{5}\right], \quad \left[\frac{4}{5}, 1\right]$$

The midpoints of these intervals are $\frac{1}{10}, \frac{3}{10}, \frac{5}{10}, \frac{7}{10}$, and $\frac{9}{10}$. Because each subinterval has a width of $\Delta x = (1 - 0)/5 = \frac{1}{5}$, you can approximate the value of the definite integral as shown.

$$\int_0^1 \frac{1}{x^2 + 1}\, dx \approx \frac{1}{5}\left(\frac{1}{1.01} + \frac{1}{1.09} + \frac{1}{1.25} + \frac{1}{1.49} + \frac{1}{1.81}\right)$$
$$\approx 0.786$$

The region whose area is represented by the definite integral is shown in Figure 5.23. The actual area of this region is $\pi/4 \approx 0.785$. So, the approximation is off by only 0.001.



**FIGURE 5.23**

✓ **CHECKPOINT 2**

Use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of $f(x) = 1/(x^2 + 2)$, the x-axis, and the lines $x = 0$ and $x = 1$. ■

## Example 3 Using the Midpoint Rule

Use the Midpoint Rule with $n = 10$ to approximate $\int_1^3 \sqrt{x^2 + 1}\, dx$.

**SOLUTION** Begin by dividing the interval $[1, 3]$ into 10 subintervals. The midpoints of these intervals are

$$\frac{11}{10}, \quad \frac{13}{10}, \quad \frac{3}{2}, \quad \frac{17}{10}, \quad \frac{19}{10}, \quad \frac{21}{10}, \quad \frac{23}{10}, \quad \frac{5}{2}, \quad \frac{27}{10}, \quad \text{and} \quad \frac{29}{10}.$$

Because each subinterval has a width of $\Delta x = (3 - 1)/10 = \frac{1}{5}$, you can approximate the value of the definite integral as shown.

$$\int_1^3 \sqrt{x^2 + 1}\, dx \approx \frac{1}{5}\left[\sqrt{(1.1)^2 + 1} + \sqrt{(1.3)^2 + 1} + \cdots + \sqrt{(2.9)^2 + 1}\right]$$
$$\approx 4.504$$

The region whose area is represented by the definite integral is shown in Figure 5.24. Using techniques that are not within the scope of this course, it can be shown that the actual area is

$$\frac{1}{2}\left[3\sqrt{10} + \ln\left(3 + \sqrt{10}\right) - \sqrt{2} - \ln\left(1 + \sqrt{2}\right)\right] \approx 4.505.$$

So, the approximation is off by only 0.001.



**FIGURE 5.24**

### STUDY TIP

The Midpoint Rule is necessary for solving certain real-life problems, such as measuring irregular areas like bodies of water (see Exercise 38).

✓ **CHECKPOINT 3**

Use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of $f(x) = \sqrt{x^2 - 1}$, the x-axis, and the lines $x = 2$ and $x = 4$. ■

## The Definite Integral as the Limit of a Sum

Consider the closed interval $[a, b]$, divided into $n$ subintervals whose midpoints are $x_i$ and whose widths are $\Delta x = (b - a)/n$. In this section, you have seen that the midpoint approximation

$$\int_a^b f(x)\,dx \approx f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

$$= [f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)]\Delta x$$

becomes better and better as $n$ increases. In fact, the limit of this sum as $n$ approaches infinity is exactly equal to the definite integral. That is,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} [f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)]\Delta x.$$

It can be shown that this limit is valid as long as $x_i$ is *any* point in the *i*th interval.

### *Example 4*   **Approximating a Definite Integral** ⓣ

Use a computer, programmable calculator, or symbolic integration utility to approximate the definite integral

$$\int_0^1 e^{-x^2}dx.$$

**SOLUTION**   Using the program on page 404, with $n = 10, 20, 30, 40$, and 50, it appears that the value of the integral is approximately 0.7468. If you have access to a computer or calculator with a built-in program for approximating definite integrals, try using it to approximate this integral. When a computer with such a built-in program approximated the integral, it returned a value of 0.746824.

### ✓ **CHECKPOINT 4**

Use a computer, programmable calculator, or symbolic integration utility to approximate the definite integral

$$\int_0^1 e^{x^2}\,dx.$$  ▪

---

**CONCEPT CHECK**

1. Complete the following: In cases where the Fundamental Theorem of Calculus cannot be used to evaluate a definite integral, you can approximate the value of the integral using the _____ _____.

2. True or false: The Midpoint Rule can be used to approximate any definite integral.

3. In the Midpoint Rule, as the number of subintervals *n* increases, does the approximation of a definite integral become better or worse?

4. State the guidelines for using the Midpoint Rule.

In Exercises 1–6, find the midpoint of the interval.

**1.** $\left[0, \frac{1}{3}\right]$

**2.** $\left[\frac{1}{10}, \frac{2}{10}\right]$

**3.** $\left[\frac{3}{20}, \frac{4}{20}\right]$

**4.** $\left[1, \frac{7}{6}\right]$

**5.** $\left[2, \frac{31}{15}\right]$

**6.** $\left[\frac{26}{9}, 3\right]$

In Exercises 7–10, find the limit.

**7.** $\displaystyle\lim_{x \to \infty} \frac{2x^2 + 4x - 1}{3x^2 - 2x}$
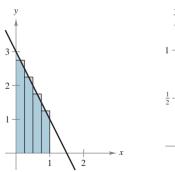
**8.** $\displaystyle\lim_{x \to \infty} \frac{4x + 5}{7x - 5}$

**9.** $\displaystyle\lim_{x \to \infty} \frac{x - 7}{x^2 + 1}$

**10.** $\displaystyle\lim_{x \to \infty} \frac{5x^3 + 1}{x^3 + x^2 + 4}$

# Exercises 5.6
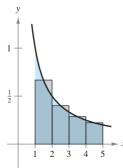
In Exercises 1–4, use the Midpoint Rule with $n = 4$ to approximate the area of the region. Compare your result with the exact area obtained with a definite integral.

**1.** $f(x) = -2x + 3, \quad [0, 1]$

**2.** $f(x) = \dfrac{1}{x}, \quad [1, 5]$



**3.** $f(x) = \sqrt{x}, \quad [0, 1]$
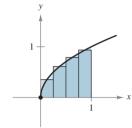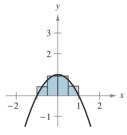
**4.** $f(x) = 1 - x^2, \quad [-1, 1]$



In Exercises 5–16, use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of $f$ and the $x$-axis over the interval. Compare your result with the exact area. Sketch the region.

| Function | Interval |
|---|---|
| **5.** $f(x) = 4 - x^2$ | $[0, 2]$ |
| **6.** $f(x) = 4x^2$ | $[0, 2]$ |
| **7.** $f(x) = x^2 + 3$ | $[-1, 1]$ |
| **8.** $f(x) = 4 - x^2$ | $[-2, 2]$ |
| **9.** $f(x) = 2x^2$ | $[1, 3]$ |
| **10.** $f(x) = 3x^2 + 1$ | $[-1, 3]$ |
| **11.** $f(x) = 2x - x^3$ | $[0, 1]$ |
| **12.** $f(x) = x^2 - x^3$ | $[0, 1]$ |
| **13.** $f(x) = x^2 - x^3$ | $[-1, 0]$ |
| **14.** $f(x) = x(1 - x)^2$ | $[0, 1]$ |
| **15.** $f(x) = x^2(3 - x)$ | $[0, 3]$ |
| **16.** $f(x) = x^2 + 4x$ | $[0, 4]$ |

(T) In Exercises 17–22, use a program similar to that on page 404 to approximate the area of the region. How large must $n$ be to obtain an approximation that is correct to within 0.01?

**17.** $\displaystyle\int_0^4 (2x^2 + 3)\, dx$

**18.** $\displaystyle\int_0^4 (2x^3 + 3)\, dx$

**19.** $\displaystyle\int_1^2 (2x^2 - x + 1)\, dx$

**20.** $\displaystyle\int_1^2 (x^3 - 1)\, dx$
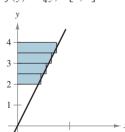
**21.** $\displaystyle\int_1^4 \frac{1}{x + 1}\, dx$
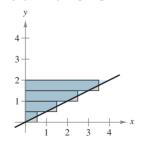
**22.** $\displaystyle\int_1^2 \sqrt{x + 2}\, dx$

In Exercises 23–26, use the Midpoint Rule with $n = 4$ to approximate the area of the region. Compare your result with the exact area obtained with a definite integral.
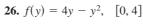
**23.** $f(y) = \frac{1}{4}y, \quad [2, 4]$
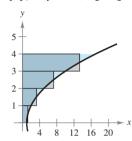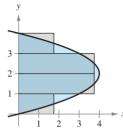


**24.** $f(y) = 2y, \quad [0, 2]$



**25.** $f(y) = y^2 + 1, \quad [0, 4]$



**26.** $f(y) = 4y - y^2, \quad [0, 4]$
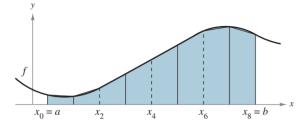


**Trapezoidal Rule**  In Exercises 27 and 28, use the Trapezoidal Rule with $n = 8$ to approximate the definite integral. Compare the result with the exact value and the approximation obtained with $n = 8$ and the Midpoint Rule. Which approximation technique appears to be better? Let $f$ be continuous on $[a, b]$ and let $n$ be the number of equal subintervals (see figure). Then the Trapezoidal Rule for approximating $\int_a^b f(x)\, dx$ is

$$\frac{b - a}{2n}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$



**27.** $\int_0^2 x^3\, dx$

**28.** $\int_1^3 \frac{1}{x^2}\, dx$

In Exercises 29–32, use the Trapezoidal Rule with $n = 4$ to approximate the definite integral.

**29.** $\int_0^2 \frac{1}{x + 1}\, dx$

**30.** $\int_0^4 \sqrt{1 + x^2}\, dx$

**31.** $\int_{-1}^1 \frac{1}{x^2 + 1}\, dx$

**32.** $\int_1^5 \frac{\sqrt{x - 1}}{x}\, dx$

Ⓣ In Exercises 33 and 34, use a computer or programmable calculator to approximate the definite integral using the Midpoint Rule and the Trapezoidal Rule for $n = 4$, 8, 12, 16, and 20.

**33.** $\int_0^4 \sqrt{2 + 3x^2}\, dx$

**34.** $\int_0^2 \frac{5}{x^3 + 1}\, dx$
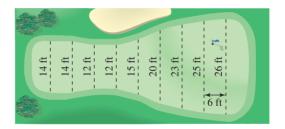
In Exercises 35 and 36, use the Trapezoidal Rule with $n = 10$ to approximate the area of the region bounded by the graphs of the equations.
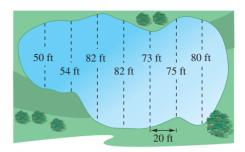
**35.** $y = \sqrt{\dfrac{x^3}{4 - x}}, \quad y = 0, \quad x = 3$

**36.** $y = x\sqrt{\dfrac{4 - x}{4 + x}}, \quad y = 0, \quad x = 4$

**37.** **Surface Area**  Estimate the surface area of the golf green shown in the figure using (a) the Midpoint Rule and (b) the Trapezoidal Rule.



**38.** **Surface Area**  To estimate the surface area of a pond, a surveyor takes several measurements, as shown in the figure. Estimate the surface area of the pond using (a) the Midpoint Rule and (b) the Trapezoidal Rule.



Ⓣ **39.** **Numerical Approximation**  Use the Midpoint Rule and the Trapezoidal Rule with $n = 4$ to approximate $\pi$ where

$$\pi = \int_0^1 \frac{4}{1 + x^2}\, dx.$$

Then use a graphing utility to evaluate the definite integral. Compare all of your results.

# Algebra Review

## "Unsimplifying" an Algebraic Expression

In algebra it is often helpful to write an expression in simplest form. In this chapter, you have seen that the reverse is often true in integration. That is, to fit an integrand to an integration formula, it often helps to "unsimplify" the expression. To do this, you use the same algebraic rules, but your goal is different. Here are some examples.

### Example 1   Rewriting an Algebraic Expression

Rewrite each algebraic expression as indicated in the example.

**a.** $\dfrac{x + 1}{\sqrt{x}}$        Example 6, page 358        **b.** $x(3 - 4x^2)^2$        Example 2, page 367

**c.** $7x^2\sqrt{x^3 + 1}$        Example 4, page 368        **d.** $5xe^{-x^2}$        Example 3, page 375

**SOLUTION**

**a.** $\dfrac{x + 1}{\sqrt{x}} = \dfrac{x}{\sqrt{x}} + \dfrac{1}{\sqrt{x}}$        Example 6, page 358
Rewrite as two fractions.

$\qquad = \dfrac{x^1}{x^{1/2}} + \dfrac{1}{x^{1/2}}$        Rewrite with rational exponents.

$\qquad = x^{1 - 1/2} + x^{-1/2}$        Properties of exponents

$\qquad = x^{1/2} + x^{-1/2}$        Simplify exponent.

**b.** $x(3 - 4x^2)^2 = \dfrac{-8}{-8}x(3 - 4x^2)^2$        Example 2, page 367
Multiply and divide by $-8$.

$\qquad = \left(-\dfrac{1}{8}\right)(-8)x(3 - 4x^2)^2$        Regroup.

$\qquad = \left(-\dfrac{1}{8}\right)(3 - 4x^2)^2(-8x)$        Regroup.

**c.** $7x^2\sqrt{x^3 + 1} = 7x^2(x^3 + 1)^{1/2}$        Example 4, page 368
Rewrite with rational exponent.

$\qquad = \dfrac{3}{3}(7x^2)(x^3 + 1)^{1/2}$        Multiply and divide by 3.

$\qquad = \dfrac{7}{3}(3x^2)(x^3 + 1)^{1/2}$        Regroup.

$\qquad = \dfrac{7}{3}(x^3 + 1)^{1/2}(3x^2)$        Regroup.

**d.** $5xe^{-x^2} = \dfrac{-2}{-2}(5x)e^{-x^2}$        Example 3, page 375
Multiply and divide by $-2$.

$\qquad = \left(-\dfrac{5}{2}\right)(-2x)e^{-x^2}$        Regroup.

$\qquad = \left(-\dfrac{5}{2}\right)e^{-x^2}(-2x)$        Regroup.

### *Example 2*  **Rewriting an Algebraic Expression**

Rewrite each algebraic expression.

**a.** $\dfrac{3x^2 + 2x - 1}{x^2}$     **b.** $\dfrac{1}{1 + e^{-x}}$

**c.** $\dfrac{x^2 + x + 1}{x - 1}$     **d.** $\dfrac{x^2 + 6x + 1}{x^2 + 1}$

**SOLUTION**

**a.** $\dfrac{3x^2 + 2x - 1}{x^2} = \dfrac{3x^2}{x^2} + \dfrac{2x}{x^2} - \dfrac{1}{x^2}$     Example 7(a), page 378
Rewrite as separate fractions.

$= 3 + \dfrac{2}{x} - x^{-2}$     Properties of exponents.

$= 3 + 2\left(\dfrac{1}{x}\right) - x^{-2}$     Regroup.

**b.** $\dfrac{1}{1 + e^{-x}} = \left(\dfrac{e^x}{e^x}\right)\dfrac{1}{1 + e^{-x}}$     Example 7(b), page 378
Multiply and divide by $e^x$.

$= \dfrac{e^x}{e^x + e^x(e^{-x})}$     Multiply.

$= \dfrac{e^x}{e^x + e^{x-x}}$     Property of exponents

$= \dfrac{e^x}{e^x + e^0}$     Simplify exponent.

$= \dfrac{e^x}{e^x + 1}$     $e^0 = 1$

**c.** $\dfrac{x^2 + x + 1}{x - 1} = x + 2 + \dfrac{3}{x - 1}$     Example 7(c), page 378
Use long division as shown below.

$$
\begin{array}{r}
x + 2 \phantom{00} \\
x - 1 \overline{)\, x^2 + x + 1} \\
\underline{x^2 - x} \phantom{0000} \\
2x + 1 \\
\underline{2x - 2} \\
3
\end{array}
$$

**d.** $\dfrac{x^2 + 6x + 1}{x^2 + 1} = 1 + \dfrac{6x}{x^2 + 1}$     Bottom of page 377.
Use long division as shown below.

$$
\begin{array}{r}
1 \phantom{0000000} \\
x^2 + 1 \overline{)\, x^2 + 6x + 1} \\
\underline{x^2 \phantom{0000} + 1} \\
6x
\end{array}
$$

# Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 413.
Answers to odd-numbered Review Exercises are given in the back of the text.*

| Section 5.1 | Review Exercises |
|---|---|

■ Use basic integration rules to find indefinite integrals.                    *1–10*

$$\int k\, dx = kx + C \qquad\qquad \int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$$

$$\int kf(x)\, dx = k\int f(x)\, dx \qquad\qquad \int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

■ Use initial conditions to find particular solutions of indefinite integrals.    *11–14*

■ Use antiderivatives to solve real-life problems.                             *15, 16*

## Section 5.2

■ Use the General Power Rule or integration by substitution to find indefinite integrals.    *17–24*

$$\int u^n \frac{du}{dx}\, dx = \int u^n\, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

■ Use the General Power Rule or integration by substitution to solve real-life problems.    *25, 26*

## Section 5.3

■ Use the Exponential and Log Rules to find indefinite integrals.             *27–32*

$$\int e^x\, dx = e^x + C \qquad\qquad \int \frac{1}{x}\, dx = \ln|x| + C$$

$$\int e^u \frac{du}{dx}\, dx = \int e^u\, du = e^u + C \qquad \int \frac{du/dx}{u}\, dx = \int \frac{1}{u}\, du = \ln|u| + C$$

■ Use a symbolic integration utility to find indefinite integrals.            *33, 34*

## Section 5.4

■ Find the areas of regions using a geometric formula.                         *35, 36*

■ Find the areas of regions bounded by the graph of a function and the *x*-axis.    *37–44*

■ Use properties of definite integrals.                                        *45, 46*

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\* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

## Section 5.4   (continued)                                          Review Exercises

■ Use the Fundamental Theorem of Calculus to evaluate definite integrals.        47–64

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a), \quad \text{where} \quad F'(x) = f(x)$$

■ Use definite integrals to solve marginal analysis problems.                    65, 66

■ Find average values of functions over closed intervals.                        67–70

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

■ Use average values to solve real-life problems.                                71–74

■ Find amounts of annuities.                                                     75, 76

■ Use properties of even and odd functions to help evaluate definite integrals.  77–80

   *Even function:* $f(-x) = f(x)$        *Odd function:* $f(-x) = -f(x)$

   If $f$ is an *even* function, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$

   If $f$ is an *odd* function, then $\int_{-a}^a f(x)\,dx = 0.$

## Section 5.5

■ Find areas of regions bounded by two (or more) graphs.                         81–90

$$A = \int_a^b [f(x) - g(x)]\,dx$$

■ Find consumer and producer surpluses.                                          91, 92

■ Use the areas of regions bounded by two graphs to solve real-life problems.    93–96

## Section 5.6

■ Use the Midpoint Rule to approximate values of definite integrals.             97–100

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)]$$

■ Use the Midpoint Rule to solve real-life problems.                             101, 102

## Study Strategies

■ **Indefinite and Definite Integrals**   When evaluating integrals, remember that an indefinite integral is a *family of antiderivatives*, each differing by a constant $C$, whereas a definite integral is a number.

■ **Checking Antiderivatives by Differentiating**   When finding an antiderivative, remember that you can check your result by differentiating. For example, you can check that the antiderivative

$$\int (3x^3 - 4x)\,dx = \frac{3}{4}x^4 - 2x^2 + C \qquad \text{is correct by differentiating to obtain} \qquad \frac{d}{dx}\left[\frac{3}{4}x^4 - 2x^2 + C\right] = 3x^3 - 4x.$$

Because the derivative is equal to the original integrand, you know that the antiderivative is correct.

■ **Grouping Symbols and the Fundamental Theorem**   When using the Fundamental Theorem of Calculus to evaluate a definite integral, you can avoid sign errors by using grouping symbols. Here is an example.

$$\int_1^3 (x^3 - 9x)\,dx = \left[\frac{x^4}{4} - \frac{9x^2}{2}\right]_1^3 = \left[\frac{3^4}{4} - \frac{9(3^2)}{2}\right] - \left[\frac{1^4}{4} - \frac{9(1^2)}{2}\right] = \frac{81}{4} - \frac{81}{2} - \frac{1}{4} + \frac{9}{2} = -16$$

# Review Exercises

In Exercises 1–10, find the indefinite integral.

**1.** $\int 16\, dx$

**2.** $\int \frac{3}{5}x\, dx$

**3.** $\int (2x^2 + 5x)\, dx$

**4.** $\int (5 - 6x^2)\, dx$

**5.** $\int \frac{2}{3\sqrt[3]{x}}\, dx$

**6.** $\int 6x^2\sqrt{x}\, dx$

**7.** $\int \left(\sqrt[3]{x^4} + 3x\right) dx$

**8.** $\int \left(\frac{4}{\sqrt{x}} + \sqrt{x}\right) dx$

**9.** $\int \frac{2x^4 - 1}{\sqrt{x}}\, dx$

**10.** $\int \frac{1 - 3x}{x^2}\, dx$

In Exercises 11–14, find the particular solution, $y = f(x)$, that satisfies the conditions.

**11.** $f'(x) = 3x + 1, \quad f(2) = 6$

**12.** $f'(x) = x^{-1/3} - 1, \quad f(8) = 4$

**13.** $f''(x) = 2x^2, \quad f'(3) = 10, \quad f(3) = 6$

**14.** $f''(x) = \dfrac{6}{\sqrt{x}} + 3, \quad f'(1) = 12, \quad f(4) = 56$

**15. Vertical Motion**  An object is projected upward from the ground with an initial velocity of 80 feet per second.

   (a) How long does it take the object to rise to its maximum height?

   (b) What is the maximum height?

   (c) When is the velocity of the object half of its initial velocity?

   (d) What is the height of the object when its velocity is one-half the initial velocity?

**16. Revenue**  The weekly revenue for a new product has been increasing. The rate of change of the revenue can be modeled by

$$\frac{dR}{dt} = 0.675t^{3/2}, \quad 0 \le t \le 225$$

where $t$ is the time (in weeks). When $t = 0, R = 0$.

   (a) Find a model for the revenue function.

   (b) When will the weekly revenue be $27,000?

In Exercises 17–24, find the indefinite integral.

**17.** $\int (1 + 5x)^2\, dx$

**18.** $\int (x - 6)^{4/3}\, dx$

**19.** $\int \frac{1}{\sqrt{5x - 1}}\, dx$

**20.** $\int \frac{4x}{\sqrt{1 - 3x^2}}\, dx$

**21.** $\int x(1 - 4x^2)\, dx$

**22.** $\int \frac{x^2}{(x^3 - 4)^2}\, dx$

**23.** $\int (x^4 - 2x)(2x^3 - 1)\, dx$

**24.** $\int \frac{\sqrt{x}}{(1 - x^{3/2})^3}\, dx$

**25. Production**  The output $P$ (in board-feet) of a small sawmill changes according to the model

$$\frac{dP}{dt} = 2t(0.001t^2 + 0.5)^{1/4}, \quad 0 \le t \le 40$$

where $t$ is measured in hours. Find the numbers of board-feet produced in (a) 6 hours and (b) 12 hours.

**26. Cost**  The marginal cost for a catering service to cater to $x$ people can be modeled by

$$\frac{dC}{dx} = \frac{5x}{\sqrt{x^2 + 1000}}.$$

When $x = 225$, the cost is $1136.06. Find the costs of catering to (a) 500 people and (b) 1000 people.

In Exercises 27–32, find the indefinite integral.

**27.** $\int 3e^{-3x}\, dx$

**28.** $\int (2t - 1)e^{t^2 - t}\, dt$

**29.** $\int (x - 1)e^{x^2 - 2x}\, dx$

**30.** $\int \frac{4}{6x - 1}\, dx$

**31.** $\int \frac{x^2}{1 - x^3}\, dx$

**32.** $\int \frac{x - 4}{x^2 - 8x}\, dx$

(T) In Exercises 33 and 34, use a symbolic integration utility to find the indefinite integral.

**33.** $\int \frac{(\sqrt{x} + 1)^2}{\sqrt{x}}\, dx$

**34.** $\int \frac{e^{5x}}{5 + e^{5x}}\, dx$

In Exercises 35 and 36, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral.
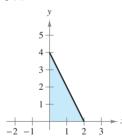
**35.** $\int_0^5 \left(5 - |x - 5|\right) dx$

**36.** $\int_{-4}^4 \sqrt{16 - x^2}\, dx$
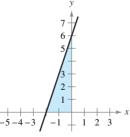
In Exercises 37–44, find the area of the region.

**37.** $f(x) = 4 - 2x$
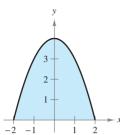


**38.** $f(x) = 3x + 6$
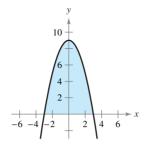


**39.** $f(x) = 4 - x^2$



**40.** $f(x) = 9 - x^2$



**41.** $f(y) = (y - 2)^2$
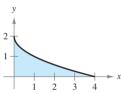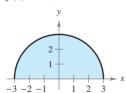


**42.** $f(x) = \sqrt{9 - x^2}$
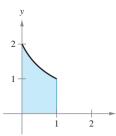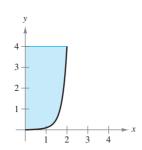


**43.** $f(x) = \dfrac{2}{x + 1}$



**44.** $f(x) = 2xe^{x^2 - 4}$



**45.** Given $\displaystyle\int_2^6 f(x)\, dx = 10$ and $\displaystyle\int_2^6 g(x)\, dx = 3$, evaluate the definite integral.

(a) $\displaystyle\int_2^6 [f(x) + g(x)]\, dx$

(b) $\displaystyle\int_2^6 [f(x) - g(x)]\, dx$

(c) $\displaystyle\int_2^6 [2f(x) - 3g(x)]\, dx$

(d) $\displaystyle\int_2^6 5f(x)\, dx$

**46.** Given $\displaystyle\int_0^3 f(x)\, dx = 4$ and $\displaystyle\int_3^6 f(x)\, dx = -1$, evaluate the definite integral.

(a) $\displaystyle\int_0^6 f(x)\, dx$

(b) $\displaystyle\int_6^3 f(x)\, dx$

(c) $\displaystyle\int_4^4 f(x)\, dx$

(d) $\displaystyle\int_3^6 -10 f(x)\, dx$

In Exercises 47–60, use the Fundamental Theorem of Calculus to evaluate the definite integral.

**47.** $\displaystyle\int_0^4 (2 + x)\, dx$

**48.** $\displaystyle\int_{-1}^1 (t^2 + 2)\, dt$

**49.** $\displaystyle\int_4^9 x\sqrt{x}\, dx$

**50.** $\displaystyle\int_1^4 2x\sqrt{x}\, dx$

**51.** $\displaystyle\int_{-1}^1 (4t^3 - 2t)\, dt$

**52.** $\displaystyle\int_{-2}^2 (x^4 + 2x^2 - 5)\, dx$

**53.** $\displaystyle\int_0^3 \frac{1}{\sqrt{1 + x}}\, dx$

**54.** $\displaystyle\int_3^6 \frac{x}{3\sqrt{x^2 - 8}}\, dx$

**55.** $\displaystyle\int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx$

**56.** $\displaystyle\int_0^1 x^2(x^3 + 1)^3\, dx$

**57.** $\displaystyle\int_1^3 \frac{(3 + \ln x)}{x}\, dx$

**58.** $\displaystyle\int_0^{\ln 5} e^{x/5}\, dx$

**59.** $\displaystyle\int_{-1}^1 3xe^{x^2 - 1}\, dx$

**60.** $\displaystyle\int_1^3 \frac{1}{x(\ln x + 2)^2}\, dx$

In Exercises 61–64, sketch the graph of the region whose area is given by the integral, and find the area.

**61.** $\displaystyle\int_1^3 (2x - 1)\, dx$

**62.** $\displaystyle\int_0^2 (x + 4)\, dx$

**63.** $\displaystyle\int_3^4 (x^2 - 9)\, dx$

**64.** $\displaystyle\int_{-1}^2 (-x^2 + x + 2)\, dx$

**65. Cost**  The marginal cost of serving an additional typical client at a law firm can be modeled by

$$\frac{dC}{dx} = 675 + 0.5x$$

where $x$ is the number of clients. How does the cost $C$ change when $x$ increases from 50 to 51 clients?

**66. Profit** The marginal profit obtained by selling $x$ dollars of automobile insurance can be modeled by

$$\frac{dP}{dx} = 0.4\left(1 - \frac{5000}{x}\right), \quad x \geq 5000.$$

Find the change in the profit when $x$ increases from $75,000 to $100,000.

In Exercises 67–70, find the average value of the function on the closed interval. Then find all $x$-values in the interval for which the function is equal to its average value.

**67.** $f(x) = \dfrac{1}{\sqrt{x}}, \quad [4, 9]$    **68.** $f(x) = \dfrac{20 \ln x}{x}, \quad [2, 10]$

**69.** $f(x) = e^{5-x}, \quad [2, 5]$    **70.** $f(x) = x^3, \quad [0, 2]$

**71. Compound Interest** An interest-bearing checking account yields 4% interest compounded continuously. If you deposit $500 in such an account, and never write checks, what will the average value of the account be over a period of 2 years? Explain your reasoning.

**72. Consumer Awareness** Suppose the price $p$ of gasoline can be modeled by

$$p = 0.0782t^2 - 0.352t + 1.75$$

where $t = 1$ corresponds to January 1, 2001. Find the cost of gasoline for an automobile that is driven 15,000 miles per year and gets 33 miles per gallon from 2001 through 2006. *(Source: U.S. Department of Energy)*

**73. Consumer Trends** The rates of change of lean and extra lean beef prices (in dollars per pound) in the United States from 1999 through 2006 can be modeled by

$$\frac{dB}{dt} = -0.0391t + 0.6108$$

where $t$ is the year, with $t = 9$ corresponding to 1999. The price of 1 pound of lean and extra lean beef in 2006 was $2.95. *(Source: U.S. Bureau of Labor Statistics)*

(a) Find the price function in terms of the year.

(b) If the price of beef per pound continues to change at this rate, in what year does the model predict the price per pound of lean and extra lean beef will surpass $3.25? Explain your reasoning.

**74. Medical Science** The volume $V$ (in liters) of air in the lungs during a five-second respiratory cycle is approximated by the model

$$V = 0.1729t + 0.1522t^2 - 0.0374t^3$$

where $t$ is time in seconds.

(T) (a) Use a graphing utility to graph the equation on the interval $[0, 5]$.

(b) Determine the intervals on which the function is increasing and decreasing.

(c) Determine the maximum volume during the respiratory cycle.

(d) Determine the average volume of air in the lungs during one cycle.

(e) Briefly explain your results for parts (a) through (d).

**Annuity** In Exercises 75 and 76, find the amount of an annuity with income function $c(t)$, interest rate $r$, and term $T$.

**75.** $c(t) = \$3000, r = 6\%, T = 5$ years

**76.** $c(t) = \$1200, r = 7\%, T = 8$ years

In Exercises 77–80, explain how the given value can be used to evaluate the second integral.

**77.** $\displaystyle\int_0^2 6x^5 \, dx = 64, \quad \int_{-2}^2 6x^5 \, dx$

**78.** $\displaystyle\int_0^3 (x^4 + x^2) \, dx = 57.6, \quad \int_{-3}^3 (x^4 + x^2) \, dx$

**79.** $\displaystyle\int_1^2 \frac{4}{x^2} \, dx = 2, \quad \int_{-2}^{-1} \frac{4}{x^2} \, dx$

**80.** $\displaystyle\int_0^1 (x^3 - x) \, dx = -\frac{1}{4}, \quad \int_{-1}^0 (x^3 - x) \, dx$

In Exercises 81–88, sketch the region bounded by the graphs of the equations. Then find the area of the region.

**81.** $y = \dfrac{1}{x^2}, y = 0, x = 1, x = 5$

**82.** $y = \dfrac{1}{x^2}, y = 4, x = 5$

**83.** $y = x, y = x^3$

**84.** $y = 1 - \dfrac{1}{2}x, \; y = x - 2, \; y = 1$

**85.** $y = \dfrac{4}{\sqrt{x + 1}}, \; y = 0, \; x = 0, \; x = 8$

**86.** $y = \sqrt{x}(x - 1), \; y = 0$

**87.** $y = (x - 3)^2, \; y = 8 - (x - 3)^2$

**88.** $y = 4 - x, \; y = x^2 - 5x + 8, \; x = 0$

(T) In Exercises 89 and 90, use a graphing utility to graph the region bounded by the graphs of the equations. Then find the area of the region.

**89.** $y = x, \; y = 2 - x^2$

**90.** $y = x, \; y = x^5$

**Consumer and Producer Surpluses** In Exercises 91 and 92, find the consumer surplus and producer surplus for the demand and supply functions.

**91.** Demand function: $p_2(x) = 500 - x$

Supply function: $p_1(x) = 1.25x + 162.5$

**92.** Demand function: $p_2(x) = \sqrt{100{,}000 - 0.15x^2}$

Supply function: $p_1(x) = \sqrt{0.01x^2 + 36{,}000}$

**93. Sales** The sales $S$ (in millions of dollars per year) for Avon from 1996 through 2001 can be modeled by

$$S = 12.73t^2 + 4379.7, \quad 6 \le t \le 11$$

where $t = 6$ corresponds to 1996. The sales for Avon from 2002 through 2005 can be modeled by

$$S = 24.12t^2 + 2748.7, \quad 11 < t \le 15.$$

If sales for Avon had followed the first model from 1996 through 2005, how much more or less sales would there have been for Avon? *(Source: Avon Products, Inc.)*

**94. Revenue** The revenues (in millions of dollars per year) for Telephone & Data Systems, U.S. Cellular, and IDT from 2001 through 2005 can be modeled by

$R = -35.643t^2 + 561.68t + 2047.0$    Telephone & Data Systems

$R = -23.307t^2 + 433.37t + 1463.4$    U.S. Cellular

$R = -1.321t^2 + 323.96t + 899.2$    IDT

where $1 \le t \le 5$ corresponds to the five-year period from 2001 through 2005. *(Source: Telephone & Data Systems Inc., U.S. Cellular Corp., and IDT Corp.)*

(a) From 2001 through 2005, how much more was Telephone & Data Systems' revenue than U.S. Cellular's revenue?

(b) From 2001 through 2005, how much more was U.S. Cellular's revenue than IDT's revenue?

**95. Revenue** The revenues (in millions of dollars per year) for The Men's Wearhouse from 1996 through 1999 can be modeled by
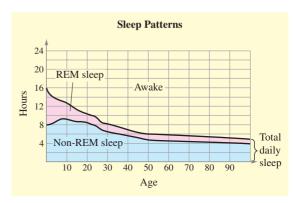
$$R = 67.800t^2 - 792.36t + 2811.5, \quad 6 \le t \le 9$$

where $t = 6$ corresponds to 1996. From 2000 through 2005, the revenues can be modeled by

$$R = 30.738t^2 - 686.29t + 5113.9, \quad 9 < t \le 15.$$

If sales for The Men's Wearhouse had followed the first model from 1996 through 2005, how much more or less revenues would there have been for The Men's Wearhouse? *(Source: The Men's Wearhouse, Inc.)*

**Ⓑ 96. Psychology: Sleep Patterns** The graph shows three areas, representing awake time, REM (rapid eye movement) sleep time, and non-REM sleep time, over a typical individual's lifetime. Make generalizations about the amount of total sleep, non-REM sleep, and REM sleep an individual

gets as he or she gets older. If you wanted to estimate mathematically the amount of non-REM sleep an individual gets between birth and age 50, how would you do so? How would you mathematically estimate the amount of REM sleep an individual gets during this interval? *(Source: Adapted from Bernstein/Clarke-Stewart/Roy/Wickens, Psychology, Seventh Edition)*



**Ⓣ** In Exercises 97–100, use the Midpoint Rule with $n = 4$ to approximate the definite integral. Then use a programmable calculator or computer to approximate the definite integral with $n = 20$. Compare the two approximations.
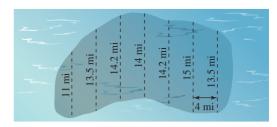
**97.** $\displaystyle \int_0^2 (x^2 + 1)^2 \, dx$

**98.** $\displaystyle \int_{-1}^1 \sqrt{1 - x^2} \, dx$

**99.** $\displaystyle \int_0^1 \frac{1}{x^2 + 1} \, dx$

**100.** $\displaystyle \int_{-1}^1 e^{3 - x^2} \, dx$

**101. Surface Area** Use the Midpoint Rule to estimate the surface area of the oil spill shown in the figure.



**102. Velocity and Acceleration** The table lists the velocity $v$ (in feet per second) of an accelerating car over a 20-second interval. Approximate the distance in feet that the car travels during the 20 seconds using (a) the Midpoint Rule and (b) the Trapezoidal Rule. (The distance is given by $s = \int_0^{20} v \, dt$.)

| Time, $t$ | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| Velocity, $v$ | 0.0 | 29.3 | 51.3 | 66.0 | 73.3 |

## Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–6, find the indefinite integral.

**1.** $\displaystyle\int (9x^2 - 4x + 13)\, dx$     **2.** $\displaystyle\int (x + 1)^2\, dx$     **3.** $\displaystyle\int 4x^3 \sqrt{x^4 - 7}\, dx$

**4.** $\displaystyle\int \frac{5x - 6}{\sqrt{x}}\, dx$     **5.** $\displaystyle\int 15e^{3x}\, dx$     **6.** $\displaystyle\int \frac{3x^2 - 11}{x^3 - 11x}\, dx$

In Exercises 7 and 8, find the particular solution $y = f(x)$ that satisfies the differential equation and initial condition.

**7.** $f'(x) = e^x + 1;\, f(0) = 1$     **8.** $f'(x) = \dfrac{1}{x};\, f(-1) = 2$

In Exercises 9–14, evaluate the definite integral.

**9.** $\displaystyle\int_0^1 16x\, dx$     **10.** $\displaystyle\int_{-3}^3 (3 - 2x)\, dx$     **11.** $\displaystyle\int_{-1}^1 (x^3 + x^2)\, dx$

**12.** $\displaystyle\int_{-1}^2 \frac{2x}{\sqrt{x^2 + 1}}\, dx$     **13.** $\displaystyle\int_0^3 e^{4x}\, dx$     **14.** $\displaystyle\int_{-2}^3 \frac{1}{x + 3}\, dx$

**15.** The rate of change in sales for PetSmart, Inc. from 1998 through 2005 can be modeled by

$$\frac{dS}{dt} = 15.7e^{0.23t}$$

where $S$ is the sales (in millions of dollars) and $t = 8$ corresponds to 1998. In 1998, the sales for PetSmart were \$2109.3 million.   *(Source: PetSmart, Inc.)*

(a) Write a model for the sales as a function of $t$.

(b) What were the average sales for 1998 through 2005?

(T) In Exercises 16 and 17, use a graphing utility to graph the region bounded by the graphs of the functions. Then find the area of the region.

**16.** $f(x) = 6,\ g(x) = x^2 - x - 6$     **17.** $f(x) = \sqrt[3]{x},\ g(x) = x^2$

**18.** The demand and supply functions for a product are modeled by

Demand: $p_1(x) = -0.625x + 10$     and     Supply: $p_2(x) = 0.25x + 3$

where $x$ is the number of units (in millions). Find the consumer and producer surpluses for this product.

In Exercises 19 and 20, use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of $f$ and the $x$-axis over the interval. Compare your result with the exact area. Sketch the region.

**19.** $f(x) = 3x^2,\, [0, 1]$

**20.** $f(x) = x^2 + 1,\, [-1, 1]$

# 6

# Techniques of Integration



Integration can be used to find the amount of lumber used per year for residential upkeep and improvements. (See Section 6.4, Exercise 51.)

## Applications

Integration has many real-life applications. The applications listed below represent a sample of the applications in this chapter.

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## Section 6.1

# Integration by Parts and Present Value

■ Use integration by parts to find indefinite and definite integrals.
■ Find the present value of future income.

## Integration by Parts

In this section, you will study an integration technique called **integration by parts.** This technique is particularly useful for integrands involving the products of algebraic and exponential or logarithmic functions, such as

$$\int x^2 e^x \, dx \quad \text{and} \quad \int x \ln x \, dx.$$

Integration by parts is based on the Product Rule for differentiation.

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{Product Rule}$$

$$uv = \int u\frac{dv}{dx} \, dx + \int v\frac{du}{dx} \, dx \qquad \text{Integrate each side.}$$

$$uv = \int u \, dv + \int v \, du \qquad \text{Write in differential form.}$$

$$\int u \, dv = uv - \int v \, du \qquad \text{Rewrite.}$$

---

**Integration by Parts**

Let $u$ and $v$ be differentiable functions of $x$.

$$\int u \, dv = uv - \int v \, du$$

---

Note that the formula for integration by parts expresses the original integral in terms of another integral. Depending on the choices for $u$ and $dv$, it may be easier to evaluate the second integral than the original one.

**STUDY TIP**

When using integration by parts, note that you can first choose $dv$ or first choose $u$. After you choose, however, the choice of the other factor is determined—it must be the remaining portion of the integrand. Also note that $dv$ *must* contain the differential $dx$ of the original integral.

**Guidelines for Integration by Parts**

1. Let $dv$ be the most complicated portion of the integrand that fits a basic integration formula. Let $u$ be the remaining factor.

2. Let $u$ be the portion of the integrand whose derivative is a function simpler than $u$. Let $dv$ be the remaining factor.

### Example 1    Integration by Parts

Find $\displaystyle\int xe^x \, dx$.

**SOLUTION**    To apply integration by parts, you must rewrite the original integral in the form $\int u \, dv$. That is, you must break $xe^x \, dx$ into two factors—one "part" representing $u$ and the other "part" representing $dv$. There are several ways to do this.

$$\int \underbrace{(x)}_{u}\underbrace{(e^x \, dx)}_{dv} \qquad \int \underbrace{(e^x)}_{u}\underbrace{(x \, dx)}_{dv} \qquad \int \underbrace{(1)}_{u}\underbrace{(xe^x \, dx)}_{dv} \qquad \int \underbrace{(xe^x)}_{u}\underbrace{(dx)}_{dv}$$

Following the guidelines, you should choose the first option because $dv = e^x \, dx$ is the most complicated portion of the integrand that fits a basic integration formula *and* because the derivative of $u = x$ is simpler than $x$.

$$dv = e^x \, dx \qquad \Longrightarrow \qquad v = \int dv = \int e^x \, dx = e^x$$

$$u = x \qquad \Longrightarrow \qquad du = dx$$

With these substitutions, you can apply the integration by parts formula as shown.

$$\int xe^x \, dx = xe^x - \int e^x \, dx \qquad \textcolor{red}{\int u \, dv = uv - \int v \, du}$$

$$= xe^x - e^x + C \qquad \textcolor{red}{\text{Integrate } \int e^x \, dx.}$$

#### ✓ CHECKPOINT 1

Find $\displaystyle\int xe^{2x} \, dx$.  ■

---

**STUDY TIP**

In Example 1, notice that you do not need to include a constant of integration when solving $v = \int e^x \, dx = e^x$. To see why this is true, try replacing $e^x$ by $e^x + C_1$ in the solution.

$$\int xe^x \, dx = x(e^x + C_1) - \int (e^x + C_1) \, dx$$
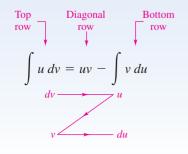
After integrating, you can see that the terms involving $C_1$ subtract out.

---

**TECHNOLOGY**

Ⓣ    If you have access to a symbolic integration utility, try using it to solve several of the exercises in this section. Note that the form of the integral may be slightly different from what you obtain when solving the exercise by hand.

## STUDY TIP

To remember the integration by parts formula, you might like to use the "Z" pattern below. The top row represents the original integral, the diagonal row represents $uv$, and the bottom row represents the new integral.

Top row    Diagonal row    Bottom row

$$\int u \, dv = uv - \int v \, du$$

$dv \longrightarrow u$

$v \longrightarrow du$

### Example 2  Integration by Parts

Find $\int x^2 \ln x \, dx$.

**SOLUTION**  For this integral, $x^2$ is more easily integrated than $\ln x$. Furthermore, the derivative of $\ln x$ is simpler than $\ln x$. So, you should choose $dv = x^2 \, dx$.

$$dv = x^2 \, dx \quad \Longrightarrow \quad v = \int dv = \int x^2 \, dx = \frac{x^3}{3}$$

$$u = \ln x \quad \Longrightarrow \quad du = \frac{1}{x} \, dx$$

Using these substitutions, apply the integration by parts formula as shown.

$$\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right) dx \qquad \int u \, dv = uv - \int v \, du$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 \, dx \qquad \text{Simplify.}$$

$$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C \qquad \text{Integrate.}$$

### ✓ CHECKPOINT 2

Find $\int x \ln x \, dx$. ■

### Example 3  Integrating by Parts with a Single Factor

Find $\int \ln x \, dx$.

**SOLUTION**  This integral is unusual because it has only one factor. In such cases, you should choose $dv = dx$ and choose $u$ to be the single factor.

$$dv = dx \quad \Longrightarrow \quad v = \int dv = \int dx = x$$

$$u = \ln x \quad \Longrightarrow \quad du = \frac{1}{x} \, dx$$

### ✓ CHECKPOINT 3

Differentiate $y = x \ln x - x + C$ to show that it is the antiderivative of $\ln x$. ■

Using these substitutions, apply the integration by parts formula as shown.

$$\int \ln x \, dx = x \ln x - \int (x)\left(\frac{1}{x}\right) dx \qquad \int u \, dv = uv - \int v \, du$$

$$= x \ln x - \int dx \qquad \text{Simplify.}$$

$$= x \ln x - x + C \qquad \text{Integrate.}$$

*Example 4*    **Using Integration by Parts Repeatedly**

Find $\int x^2 e^x \, dx$.

**SOLUTION**    Using the guidelines, notice that the derivative of $x^2$ becomes simpler, whereas the derivative of $e^x$ does not. So, you should let $u = x^2$ and let $dv = e^x \, dx$.

$$dv = e^x \, dx \qquad \Longrightarrow \qquad v = \int dv = \int e^x \, dx = e^x$$

$$u = x^2 \qquad \Longrightarrow \qquad du = 2x \, dx$$

Using these substitutions, apply the integration by parts formula as shown.

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx \qquad \text{\color{magenta}First application of integration by parts}$$

To evaluate the new integral on the right, apply integration by parts a second time, using the substitutions below.

$$dv = e^x \, dx \qquad \Longrightarrow \qquad v = \int dv = \int e^x \, dx = e^x$$

$$u = 2x \qquad \Longrightarrow \qquad du = 2 \, dx$$

Using these substitutions, apply the integration by parts formula as shown.

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx \qquad \text{\color{magenta}First application of integration by parts}$$

$$= x^2 e^x - \left( 2x e^x - \int 2 e^x \, dx \right) \qquad \text{\color{magenta}Second application of integration by parts}$$

$$= x^2 e^x - 2x e^x + 2 e^x + C \qquad \text{\color{magenta}Integrate.}$$

$$= e^x (x^2 - 2x + 2) + C \qquad \text{\color{magenta}Simplify.}$$

You can confirm this result by differentiating.

✓ **CHECKPOINT 4**

Find $\int x^3 e^x \, dx$. ■

**STUDY TIP**

Remember that you can check an indefinite integral by differentiating. For instance, in Example 4, try differentiating the antiderivative
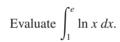
$$e^x(x^2 - 2x + 2) + C$$

to check that you obtain the original integrand, $x^2 e^x$.

When making repeated applications of integration by parts, be careful not to interchange the substitutions in successive applications. For instance, in Example 4, the first substitutions were $dv = e^x \, dx$ and $u = x^2$. If in the second application you had switched to $dv = 2x \, dx$ and $u = e^x$, you would have reversed the previous integration and returned to the *original* integral.

$$\int x^2 e^x \, dx = x^2 e^x - \left( x^2 e^x - \int x^2 e^x \, dx \right)$$

$$= \int x^2 e^x \, dx$$

*Example 5* **Evaluating a Definite Integral**

Evaluate $\displaystyle\int_1^e \ln x \, dx$.

**SOLUTION** Integration by parts was used to find the antiderivative of $\ln x$ in Example 3. Using this result, you can evaluate the definite integral as shown.

$$\int_1^e \ln x \, dx = \left[ x \ln x - x \right]_1^e \qquad \text{Use result of Example 3.}$$

$$= (e \ln e - e) - (1 \ln 1 - 1) \qquad \text{Apply Fundamental Theorem.}$$

$$= (e - e) - (0 - 1)$$

$$= 1 \qquad \text{Simplify.}$$

The area represented by this definite integral is shown in Figure 6.1.
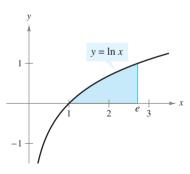
**FIGURE 6.1**

✓**CHECKPOINT 5**

Evaluate $\displaystyle\int_0^1 x^2 e^x \, dx$. ■

Before starting the exercises in this section, remember that it is not enough to know *how* to use the various integration techniques. You also must know *when* to use them. Integration is first and foremost a problem of recognition—recognizing which formula or technique to apply to obtain an antiderivative. Often, a slight alteration of an integrand will necessitate the use of a different integration technique. Here are some examples.

| *Integral* | *Technique* | *Antiderivative* |
|---|---|---|
| $\displaystyle\int x \ln x \, dx$ | Integration by parts | $\dfrac{x^2}{2} \ln x - \dfrac{x^2}{4} + C$ |
| $\displaystyle\int \dfrac{\ln x}{x} \, dx$ | Power Rule: $\displaystyle\int u^n \dfrac{du}{dx} \, dx$ | $\dfrac{(\ln x)^2}{2} + C$ |
| $\displaystyle\int \dfrac{1}{x \ln x} \, dx$ | Log Rule: $\displaystyle\int \dfrac{1}{u} \dfrac{du}{dx} \, dx$ | $\ln|\ln x| + C$ |

As you gain experience with integration by parts, your skill in determining $u$ and $dv$ will improve. The summary below gives suggestions for choosing $u$ and $dv$.

**Summary of Common Uses of Integration by Parts**

1. $\displaystyle\int x^n e^{ax} \, dx$        Let $u = x^n$ and $dv = e^{ax} \, dx$. (Examples 1 and 4)

2. $\displaystyle\int x^n \ln x \, dx$        Let $u = \ln x$ and $dv = x^n \, dx$. (Examples 2 and 3)

## Present Value

Recall from Section 4.2 that the present value of a future payment is the amount that would have to be deposited today to produce the future payment. What is the present value of a future payment of $1000 one year from now? Because of inflation, $1000 today buys more than $1000 will buy a year from now. The definition below considers only the effect of inflation.

**STUDY TIP**

According to this definition, if the rate of inflation were 4%, then the present value of $1000 one year from now is just $980.26.

**Present Value**

If $c$ represents a continuous income function in dollars per year and the annual rate of inflation is $r$, then the actual total income over $t_1$ years is

$$\text{Actual income over } t_1 \text{ years} = \int_0^{t_1} c(t)\, dt$$

and its **present value** is

$$\text{Present value} = \int_0^{t_1} c(t)e^{-rt}\, dt.$$

Ignoring inflation, the equation for present value also applies to an interest-bearing account where the annual interest rate $r$ is compounded continuously and $c$ is an income function in dollars per year.

---

**Example 6**  **Finding Present Value**  ®

You have just won a state lottery for $1,000,000. You will be paid an annuity of $50,000 a year for 20 years. Assuming an annual inflation rate of 6%, what is the present value of this income?

**SOLUTION**    The income function for your winnings is given by $c(t) = 50,000$. So,

$$\text{Actual income} = \int_0^{20} 50,000\, dt = \left[\, 50,000t \,\right]_0^{20} = \$1,000,000.$$

Because you do not receive this entire amount now, its present value is

$$\text{Present value} = \int_0^{20} 50,000e^{-0.06t}\, dt = \left[\frac{50,000}{-0.06}e^{-0.06t}\right]_0^{20} \approx \$582,338.$$

This present value represents the amount that the state must deposit now to cover your payments over the next 20 years. This shows why state lotteries are so profitable—for the states!



AP/Wide World Photos

On February 18, 2006, a group of eight coworkers at a meat processing plant in Nebraska won the largest lottery jackpot in the world. They chose to receive a lump sum payment of $177.3 million instead of an annuity that would have paid $365 million over a 29-year period. The odds of winning the PowerBall jackpot are about 1 in 146.1 million.

✓**CHECKPOINT 6**

Find the present value of the income from the lottery ticket in Example 6 if the inflation rate is 7%. ■

**Expected Income**

(a)

**Present Value of Expected Income**

(b)

**FIGURE 6.2**

### Example 7
**MAKE A DECISION** **Finding Present Value** (R)

A company expects its income during the next 5 years to be given by

$$c(t) = 100,000t, \quad 0 \le t \le 5. \qquad \text{See Figure 6.2(a).}$$

Assuming an annual inflation rate of 5%, can the company claim that the present value of this income is at least $1 million?

**SOLUTION** The present value is

$$\text{Present value} = \int_0^5 100,000te^{-0.05t}\, dt = 100,000\int_0^5 te^{-0.05t}\, dt.$$

Using integration by parts, let $dv = e^{-0.05t}\, dt$.

$$dv = e^{-0.05t}\, dt \quad \Longrightarrow \quad v = \int dv = \int e^{-0.05t}\, dt = -20e^{-0.05t}$$

$$u = t \quad \Longrightarrow \quad du = dt$$

This implies that

$$\int te^{-0.05t}\, dt = -20te^{-0.05t} + 20\int e^{-0.05t}\, dt$$

$$= -20te^{-0.05t} - 400e^{-0.05t}$$

$$= -20e^{-0.05t}(t + 20).$$

So, the present value is

$$\text{Present value} = 100,000\int_0^5 te^{-0.05t}\, dt \qquad \text{See Figure 6.2(b).}$$

$$= 100,000\left[ -20e^{-0.05t}(t + 20) \right]_0^5$$

$$\approx \$1,059,961.$$

Yes, the company can claim that the present value of its expected income during the next 5 years is at least $1 million.

✓ **CHECKPOINT 7**

A company expects its income during the next 10 years to be given by $c(t) = 20,000t$, for $0 \le t \le 10$. Assuming an annual inflation rate of 5%, what is the present value of this income? ■

**CONCEPT CHECK**

1. Integration by parts is based on what differentiation rule?
2. Write the formula for integration by parts.
3. State the guidelines for integration by parts.
4. Without integrating, which formula or technique of integration would you use to find $\int xe^{4x}\, dx$? Explain your reasoning.

**Skills Review 6.1**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 4.3, 4.5, and 5.5.

In Exercises 1–6, find $f'(x)$.

**1.** $f(x) = \ln(x + 1)$

**2.** $f(x) = \ln(x^2 - 1)$

**3.** $f(x) = e^{x^3}$

**4.** $f(x) = e^{-x^2}$

**5.** $f(x) = x^2 e^x$

**6.** $f(x) = xe^{-2x}$

In Exercises 7–10, find the area between the graphs of $f$ and $g$.

**7.** $f(x) = -x^2 + 4$, $g(x) = x^2 - 4$

**8.** $f(x) = -x^2 + 2$, $g(x) = 1$

**9.** $f(x) = 4x$, $g(x) = x^2 - 5$

**10.** $f(x) = x^3 - 3x^2 + 2$, $g(x) = x - 1$

## Exercises 6.1

In Exercises 1–4, identify $u$ and $dv$ for finding the integral using integration by parts. (Do not evaluate the integral.)

**1.** $\displaystyle\int xe^{3x}\, dx$

**2.** $\displaystyle\int x^2 e^{3x}\,dx$

**3.** $\displaystyle\int x \ln 2x\, dx$

**4.** $\displaystyle\int \ln 4x\, dx$

In Exercises 5–10, use integration by parts to find the indefinite integral.

**5.** $\displaystyle\int xe^{3x}\, dx$

**6.** $\displaystyle\int xe^{-x}\, dx$

**7.** $\displaystyle\int x^2 e^{-x}\, dx$

**8.** $\displaystyle\int x^2 e^{2x}\, dx$

**9.** $\displaystyle\int \ln 2x\, dx$

**10.** $\displaystyle\int \ln x^2\, dx$

In Exercises 11–38, find the indefinite integral. (*Hint:* Integration by parts is not required for all the integrals.)

**11.** $\displaystyle\int e^{4x}\, dx$

**12.** $\displaystyle\int e^{-2x}\, dx$

**13.** $\displaystyle\int xe^{4x}\, dx$

**14.** $\displaystyle\int xe^{-2x}\, dx$

**15.** $\displaystyle\int xe^{x^2}\, dx$

**16.** $\displaystyle\int x^2 e^{x^3}\, dx$

**17.** $\displaystyle\int \frac{x}{e^x}\, dx$

**18.** $\displaystyle\int \frac{2x}{e^x}\, dx$

**19.** $\displaystyle\int 2x^2 e^x\, dx$

**20.** $\displaystyle\int \frac{1}{2} x^3 e^x\, dx$

**21.** $\displaystyle\int t \ln(t + 1)\, dt$

**22.** $\displaystyle\int x^3 \ln x\, dx$

**23.** $\displaystyle\int (x - 1)e^x\, dx$

**24.** $\displaystyle\int x^4 \ln x\, dx$

**25.** $\displaystyle\int \frac{e^{1/t}}{t^2}\, dt$

**26.** $\displaystyle\int \frac{1}{x(\ln x)^3}\, dx$

**27.** $\displaystyle\int x(\ln x)^2\, dx$

**28.** $\displaystyle\int \ln 3x\, dx$

**29.** $\displaystyle\int \frac{(\ln x)^2}{x}\, dx$

**30.** $\displaystyle\int \frac{1}{x \ln x}\, dx$

**31.** $\displaystyle\int \frac{\ln x}{x^2}\, dx$

**32.** $\displaystyle\int \frac{\ln 2x}{x^2}\, dx$

**33.** $\displaystyle\int x\sqrt{x - 1}\, dx$

**34.** $\displaystyle\int \frac{x}{\sqrt{x - 1}}\, dx$

**35.** $\displaystyle\int x(x + 1)^2\, dx$

**36.** $\displaystyle\int \frac{x}{\sqrt{2 + 3x}}\, dx$

**37.** $\displaystyle\int \frac{xe^{2x}}{(2x + 1)^2}\, dx$

**38.** $\displaystyle\int \frac{x^3 e^{x^2}}{(x^2 + 1)^2}\, dx$

In Exercises 39–46, evaluate the definite integral.

**39.** $\int_1^2 x^2 e^x \, dx$

**40.** $\int_0^2 \frac{x^2}{e^x} \, dx$

**41.** $\int_0^4 \frac{x}{e^{x/2}} \, dx$

**42.** $\int_1^2 x^2 \ln x \, dx$

**43.** $\int_1^e x^5 \ln x \, dx$

**44.** $\int_1^e 2x \ln x \, dx$

**45.** $\int_{-1}^0 \ln(x + 2) \, dx$

**46.** $\int_0^1 \ln(1 + 2x) \, dx$

(T) In Exercises 47–50, find the area of the region bounded by the graphs of the equations. Then use a graphing utility to graph the region and verify your answer.

**47.** $y = x^3 e^x$, $y = 0$, $x = 0$, $x = 2$

**48.** $y = (x^2 - 1)e^x$, $y = 0$, $x = -1$, $x = 1$

**49.** $y = x^2 \ln x$, $y = 0$, $x = 1$, $x = e$

**50.** $y = \frac{\ln x}{x^2}$, $y = 0$, $x = 1$, $x = e$

In Exercises 51 and 52, use integration by parts to verify the formula.

**51.** $\int x^n \ln x \, dx = \frac{x^{n+1}}{(n+1)^2}[-1 + (n+1)\ln x] + C,$

$n \neq -1$

**52.** $\int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax} \, dx, \quad n > 0$

In Exercises 53–56, use the results of Exercises 51 and 52 to find the indefinite integral.

**53.** $\int x^2 e^{5x} \, dx$

**54.** $\int x e^{-3x} \, dx$
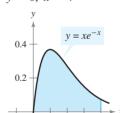
**55.** $\int x^{-2} \ln x \, dx$

**56.** $\int x^{1/2} \ln x \, dx$

In Exercises 57–60, find the area of the region bounded by the graphs of the given equations.

**57.** $y = xe^{-x}$,

$y = 0$, $x = 4$

**58.** $y = \frac{1}{9}xe^{-x/3}$,

$y = 0$, $x = 0$, $x = 3$



$y = xe^{-x}$



$y = \frac{1}{9}xe^{-x/3}$

**59.** $y = x \ln x$,

$y = 0$, $x = e$

**60.** $y = x^{-3} \ln x$,

$y = 0$, $x = e$



$y = x \ln x$



$y = x^{-3} \ln x$
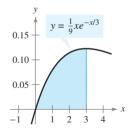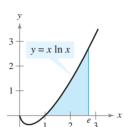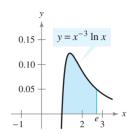
(T) In Exercises 61–64, use a symbolic integration utility to evaluate the integral.

**61.** $\int_0^2 t^3 e^{-4t} \, dt$

**62.** $\int_1^4 \ln x(x^2 + 4) \, dx$

**63.** $\int_0^5 x^4(25 - x^2)^{3/2} \, dx$

**64.** $\int_1^e x^9 \ln x \, dx$

**65. Demand** A manufacturing company forecasts that the demand $x$ (in units per year) for its product over the next 10 years can be modeled by $x = 500(20 + te^{-0.1t})$ for $0 \leq t \leq 10$, where $t$ is the time in years.

(T) (a) Use a graphing utility to decide whether the company is forecasting an increase or a decrease in demand over the decade.

(b) According to the model, what is the total demand over the next 10 years?

(c) Find the average annual demand during the 10-year period.

66. **Capital Campaign** The board of trustees of a college is planning a five-year capital gifts campaign to raise money for the college. The goal is to have an annual gift income $I$ that is modeled by $I = 2000(375 + 68te^{-0.2t})$ for $0 \le t \le 5$, where $t$ is the time in years.

(T) (a) Use a graphing utility to decide whether the board of trustees expects the gift income to increase or decrease over the five-year period.

(b) Find the expected total gift income over the five-year period.

(c) Determine the average annual gift income over the five-year period. Compare the result with the income given when $t = 3$.

67. **Memory Model** A model for the ability $M$ of a child to memorize, measured on a scale from 0 to 10, is

$M = 1 + 1.6t \ln t, \quad 0 < t \le 4$

where $t$ is the child's age in years. Find the average value of this model between

(a) the child's first and second birthdays.

(b) the child's third and fourth birthdays.

68. **Revenue** A company sells a seasonal product. The revenue $R$ (in dollars per year) generated by sales of the product can be modeled by

$R = 410.5t^2e^{-t/30} + 25,000, \quad 0 \le t \le 365$

where $t$ is the time in days.

(a) Find the average daily receipts during the first quarter, which is given by $0 \le t \le 90$.

(b) Find the average daily receipts during the fourth quarter, which is given by $274 \le t \le 365$.

(c) Find the total daily receipts during the year.

**Present Value** In Exercises 69–74, find the present value of the income $c$ (measured in dollars) over $t_1$ years at the given annual inflation rate $r$.

69. $c = 5000, \ r = 4\%, \ t_1 = 4$ years

70. $c = 450, \ r = 4\%, \ t_1 = 10$ years

71. $c = 100,000 + 4000t, \ r = 5\%, \ t_1 = 10$ years

72. $c = 30,000 + 500t, \ r = 7\%, \ t_1 = 6$ years

73. $c = 1000 + 50e^{t/2}, \ r = 6\%, \ t_1 = 4$ years

74. $c = 5000 + 25te^{t/10}, \ r = 6\%, \ t_1 = 10$ years

75. **Present Value** A company expects its income $c$ during the next 4 years to be modeled by

$c = 150,000 + 75,000t.$

(a) Find the actual income for the business over the 4 years.

(b) Assuming an annual inflation rate of 4%, what is the present value of this income?

76. **Present Value** A professional athlete signs a three-year contract in which the earnings can be modeled by

$c = 300,000 + 125,000t.$

(a) Find the actual value of the athlete's contract.

(b) Assuming an annual inflation rate of 3%, what is the present value of the contract?

**Future Value** In Exercises 77 and 78, find the future value of the income (in dollars) given by $f(t)$ over $t_1$ years at the annual interest rate of $r$. If the function $f$ represents a continuous investment over a period of $t_1$ years at an annual interest rate of $r$ (compounded continuously), then the future value of the investment is given by

$$\text{Future value} = e^{rt_1} \int_0^{t_1} f(t)e^{-rt} \, dt.$$

77. $f(t) = 3000, \ r = 8\%, \ t_1 = 10$ years

78. $f(t) = 3000e^{0.05t}, \ r = 10\%, \ t_1 = 5$ years

(B) 79. **Finance: Future Value** Use the equation from Exercises 77 and 78 to calculate the following. *(Source: Adapted from Garman/Forgue, Personal Finance, Eighth Edition)*

(a) The future value of $1200 saved each year for 10 years earning 7% interest.

(b) A person who wishes to invest $1200 each year finds one investment choice that is expected to pay 9% interest per year and another, riskier choice that may pay 10% interest per year. What is the difference in return (future value) if the investment is made for 15 years?

80. **MAKE A DECISION: COLLEGE TUITION FUND** In 2006, the total cost of attending Pennsylvania State University for 1 year was estimated to be $20,924. Assume your grandparents had continuously invested in a college fund according to the model

$f(t) = 400t$

for 18 years, at an annual interest rate of 10%. Will the fund have grown enough to allow you to cover 4 years of expenses at Pennsylvania State University? *(Source: Pennsylvania State University)*

(T) 81. Use a program similar to the Midpoint Rule program on page 404 with $n = 10$ to approximate

$$\int_1^4 \frac{4}{\sqrt{x} + \sqrt[3]{x}} \, dx.$$

(T) 82. Use a program similar to the Midpoint Rule program on page 404 with $n = 12$ to approximate the area of the region bounded by the graphs of

$$y = \frac{10}{\sqrt{x}e^x}, \ y = 0, \ x = 1, \text{ and } x = 4.$$

## Section 6.2

# Partial Fractions and Logistic Growth

- Use partial fractions to find indefinite integrals.
- Use logistic growth functions to model real-life situations.

## Partial Fractions

In Sections 5.2 and 6.1, you studied integration by substitution and by parts. In this section you will study a third technique called **partial fractions.** This technique involves the decomposition of a rational function into the sum of two or more simple rational functions. For instance, suppose you know that

$$\frac{x + 7}{x^2 - x - 6} = \frac{2}{x - 3} - \frac{1}{x + 2}.$$

Knowing the "partial fractions" on the right side would allow you to integrate the left side as shown.

$$\int \frac{x + 7}{x^2 - x - 6}\, dx = \int \left(\frac{2}{x - 3} - \frac{1}{x + 2}\right) dx$$

$$= 2 \int \frac{1}{x - 3}\, dx - \int \frac{1}{x + 2}\, dx$$

$$= 2 \ln|x - 3| - \ln|x + 2| + C$$

This method depends on the ability to factor the denominator of the original rational function *and* on finding the partial fraction decomposition of the function.

### STUDY TIP

Recall that finding the partial fraction decomposition of a rational function is a *precalculus* topic. Explain how you could verify that

$$\frac{1}{x - 1} + \frac{2}{x + 2}$$

is the partial fraction decomposition of

$$\frac{3x}{x^2 + x - 2}.$$

### Partial Fractions

To find the partial fraction decomposition of the *proper* rational function $p(x)/q(x)$, factor $q(x)$ and write an equation that has the form

$$\frac{p(x)}{q(x)} = (\text{sum of partial fractions}).$$

For each *distinct* linear factor $ax + b$, the right side should include a term of the form

$$\frac{A}{ax + b}.$$

For each *repeated* linear factor $(ax + b)^n$, the right side should include $n$ terms of the form

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdot \cdot \cdot + \frac{A_n}{(ax + b)^n}.$$

### STUDY TIP

A rational function $p(x)/q(x)$ is *proper* if the degree of the numerator is less than the degree of the denominator.

*Example 1*    **Finding a Partial Fraction Decomposition**

Write the partial fraction decomposition for

$$\frac{x + 7}{x^2 - x - 6}.$$

**SOLUTION**    Begin by factoring the denominator as $x^2 - x - 6 = (x - 3)(x + 2)$. Then, write the partial fraction decomposition as

$$\frac{x + 7}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2}.$$

To solve this equation for $A$ and $B$, multiply each side of the equation by the least common denominator $(x - 3)(x + 2)$. This produces the **basic equation** as shown.

$$x + 7 = A(x + 2) + B(x - 3) \qquad \text{Basic equation}$$

Because this equation is true for all $x$, you can substitute any convenient values of $x$ into the equation. The $x$-values that are especially convenient are the ones that make particular factors equal to 0.

> *To solve for B, substitute $x = -2$:*
>
> | | |
> |---|---|
> | $x + 7 = A(x + 2) + B(x - 3)$ | Write basic equation. |
> | $-2 + 7 = A(-2 + 2) + B(-2 - 3)$ | Substitute $-2$ for $x$. |
> | $5 = A(0) + B(-5)$ | Simplify. |
> | $-1 = B$ | Solve for $B$. |
>
> *To solve for A, substitute $x = 3$:*
>
> | | |
> |---|---|
> | $x + 7 = A(x + 2) + B(x - 3)$ | Write basic equation. |
> | $3 + 7 = A(3 + 2) + B(3 - 3)$ | Substitute 3 for $x$. |
> | $10 = A(5) + B(0)$ | Simplify. |
> | $2 = A$ | Solve for $A$. |

Now that you have solved the basic equation for $A$ and $B$, you can write the partial fraction decomposition as

$$\frac{x + 7}{x^2 - x - 6} = \frac{2}{x - 3} - \frac{1}{x + 2}$$

as indicated at the beginning of this section.

> **Algebra Review**
>
> You can check the result in Example 1 by subtracting the partial fractions to obtain the original fraction, as shown in Example 1(a) in the *Chapter 6 Algebra Review*, on page 470.

✓ **CHECKPOINT 1**

Write the partial fraction decomposition for $\dfrac{x + 8}{x^2 + 7x + 12}$. ■

> **STUDY TIP**
>
> Be sure you see that the substitutions for $x$ in Example 1 are chosen for their convenience in solving for $A$ and $B$. The value $x = -2$ is selected because it eliminates the term $A(x + 2)$, and the value $x = 3$ is chosen because it eliminates the term $B(x - 3)$.

*Example 2* **Integrating with Repeated Factors**

Find $\int \dfrac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} \, dx.$

**SOLUTION** Begin by factoring the denominator as $x(x + 1)^2$. Then, write the partial fraction decomposition as

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.$$

To solve this equation for $A$, $B$, and $C$, multiply each side of the equation by the least common denominator $x(x + 1)^2$.

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx \qquad \text{Basic equation}$$

Now, solve for $A$ and $C$ by substituting $x = -1$ and $x = 0$ into the basic equation.

*Substitute $x = -1$:*

$$5(-1)^2 + 20(-1) + 6 = A(-1 + 1)^2 + B(-1)(-1 + 1) + C(-1)$$
$$-9 = A(0) + B(0) - C$$
$$9 = C \qquad \text{Solve for } C.$$

*Substitute $x = 0$:*

$$5(0)^2 + 20(0) + 6 = A(0 + 1)^2 + B(0)(0 + 1) + C(0)$$
$$6 = A(1) + B(0) + C(0)$$
$$6 = A \qquad \text{Solve for } A.$$

At this point, you have exhausted the convenient choices for $x$ and have yet to solve for $B$. When this happens, you can use *any* other $x$-value along with the known values of $A$ and $C$.

*Substitute $x = 1$, $A = 6$, and $C = 9$:*

$$5(1)^2 + 20(1) + 6 = (6)(1 + 1)^2 + B(1)(1 + 1) + (9)(1)$$
$$31 = 6(4) + B(2) + 9(1)$$
$$-1 = B \qquad \text{Solve for } B.$$

Now that you have solved for $A$, $B$, and $C$, you can use the partial fraction decomposition to integrate.

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} \, dx = \int \left( \frac{6}{x} - \frac{1}{x + 1} + \frac{9}{(x + 1)^2} \right) dx$$

$$= 6 \ln|x| - \ln|x + 1| + 9\frac{(x + 1)^{-1}}{-1} + C$$

$$= \ln \left| \frac{x^6}{x + 1} \right| - \frac{9}{x + 1} + C$$

✓**CHECKPOINT 2**

Find $\int \dfrac{3x^2 + 7x + 4}{x^3 + 4x^2 + 4x} \, dx.$ ∎

You can use the partial fraction decomposition technique outlined in Examples 1 and 2 only with a *proper* rational function—that is, a rational function whose numerator is of lower degree than its denominator. If the numerator is of equal or greater degree, you must divide first. For instance, the rational function

$$\frac{x^3}{x^2 + 1}$$

is improper because the degree of the numerator is greater than the degree of the denominator. Before applying partial fractions to this function, you should divide the denominator into the numerator to obtain

$$\frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1}.$$

### Example 3   Integrating an Improper Rational Function

Find $\displaystyle\int \frac{x^5 + x - 1}{x^4 - x^3}\, dx$.

**SOLUTION**   This rational function is improper—its numerator has a degree greater than that of its denominator. So, you should begin by dividing the denominator into the numerator to obtain

$$\frac{x^5 + x - 1}{x^4 - x^3} = x + 1 + \frac{x^3 + x - 1}{x^4 - x^3}.$$

Now, applying partial fraction decomposition produces

$$\frac{x^3 + x - 1}{x^3(x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x - 1}.$$

Multiplying both sides by the least common denominator $x^3(x - 1)$ produces the basic equation.

$$x^3 + x - 1 = Ax^2(x - 1) + Bx(x - 1) + C(x - 1) + Dx^3 \qquad \text{Basic equation}$$

Using techniques similar to those in the first two examples, you can solve for $A$, $B$, $C$, and $D$ to obtain

$$A = 0, \quad B = 0, \quad C = 1, \quad \text{and} \quad D = 1.$$

So, you can integrate as shown.

$$\int \frac{x^5 + x - 1}{x^4 - x^3}\, dx = \int \left( x + 1 + \frac{x^3 + x - 1}{x^4 - x^3} \right) dx$$

$$= \int \left( x + 1 + \frac{1}{x^3} + \frac{1}{x - 1} \right) dx$$

$$= \frac{x^2}{2} + x - \frac{1}{2x^2} + \ln|x - 1| + C$$

> ### Algebra Review
>
> You can check the partial fraction decomposition in Example 3 by combining the partial fractions to obtain the original fraction, as shown in Example 2(a) in the *Chapter 6 Algebra Review*, on page 471.

✓ **CHECKPOINT 3**

Find $\displaystyle\int \frac{x^4 - x^3 + 2x^2 + x + 1}{x^3 + x^2}$. ∎

## Logistic Growth Function



**FIGURE 6.3**

In Section 4.6, you saw that exponential growth occurs in situations for which the rate of growth is proportional to the quantity present at any given time. That is, if $y$ is the quantity at time $t$, then $dy/dt = ky$. The general solution of this differential equation is $y = Ce^{kt}$. Exponential growth is unlimited. As long as $C$ and $k$ are positive, the value of $Ce^{kt}$ can be made arbitrarily large by choosing sufficiently large values of $t$.

In many real-life situations, however, the growth of a quantity is limited and cannot increase beyond a certain size $L$, as shown in Figure 6.3. This upper limit $L$ is called the **carrying capacity,** which is the maximum population $y(t)$ that can be sustained or supported as time $t$ increases. A model that is often used for this type of growth is the **logistic differential equation**

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Logistic differential equation}$$

where $k$ and $L$ are positive constants. A population that satisfies this equation does not grow without bound, but approaches $L$ as $t$ increases. The general solution of this differential equation is called the *logistic growth model* and is derived in Example 4.

### STUDY TIP

The graph of

$$y = \frac{L}{1 + be^{-kt}}$$

is called the *logistic curve*, as shown in Figure 6.3.

### Algebra Review

For help with the algebra used to solve for $y$ in Example 4, see Example 2(c) in the *Chapter 6 Algebra Review*, on page 471.

### ✓ CHECKPOINT 4

Show that if

$$y = \frac{1}{1 + be^{-kt}}, \text{ then}$$

$$\frac{dy}{dt} = ky(1 - y).$$

[*Hint:* First find $ky(1 - y)$ in terms of $t$, then find $dy/dt$ and show that they are equivalent.] ■

### Example 4  Deriving the Logistic Growth Model

Solve the equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right)$.

**SOLUTION**

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Write differential equation.}$$

$$\frac{1}{y(1 - y/L)}\, dy = k\, dt \qquad \text{Write in differential form.}$$

$$\int \frac{1}{y(1 - y/L)}\, dy = \int k\, dt \qquad \text{Integrate each side.}$$

$$\int \left(\frac{1}{y} + \frac{1}{L - y}\right) dy = \int k\, dt \qquad \text{Rewrite left side using partial fractions.}$$

$$\ln|y| - \ln|L - y| = kt + C \qquad \text{Find antiderivative of each side.}$$

$$\ln\left|\frac{L - y}{y}\right| = -kt - C \qquad \text{Multiply each side by } -1 \text{ and simplify.}$$

$$\left|\frac{L - y}{y}\right| = e^{-kt - C} = e^{-C}e^{-kt} \qquad \text{Exponentiate each side.}$$

$$\frac{L - y}{y} = be^{-kt} \qquad \text{Let } \pm e^{-C} = b.$$

Solving this equation for $y$ produces the **logistic growth model** $y = \dfrac{L}{1 + be^{-kt}}$.
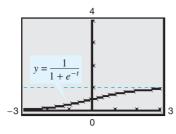
**Example 5**    **Comparing Logistic Growth Functions**    Ⓣ
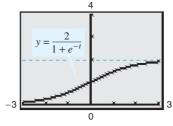
Use a graphing utility to investigate the effects of the values of $L$, $b$, and $k$ on the graph of

$$y = \frac{L}{1 + be^{-kt}}.$$    Logistic growth function $(L > 0, b > 0, k > 0)$

**SOLUTION**    The value of $L$ determines the horizontal asymptote of the graph to the right. In other words, as $t$ increases without bound, the graph approaches a limit of $L$ (see Figure 6.4).
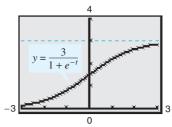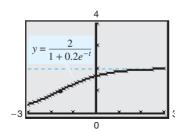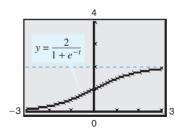


**FIGURE 6.4**

The value of $b$ determines the point of inflection of the graph. When $b = 1$, the point of inflection occurs when $t = 0$. If $b > 1$, the point of inflection is to the right of the $y$-axis. If $0 < b < 1$, the point of inflection is to the left of the $y$-axis (see Figure 6.5).
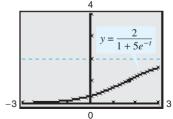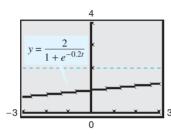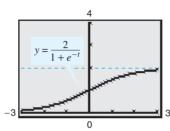


**FIGURE 6.5**

The value of $k$ determines the rate of growth of the graph. For fixed values of $b$ and $L$, larger values of $k$ correspond to higher rates of growth (see Figure 6.6).
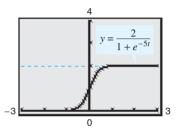


**FIGURE 6.6**

✓**CHECKPOINT 5**

Find the horizontal asymptote of the graph of $y = \dfrac{4}{1 + 5e^{-6t}}$. ∎

*Example 6* **Modeling a Population** Ⓡ

The state game commission releases 100 deer into a game preserve. During the first 5 years, the population increases to 432 deer. The commission believes that the population can be modeled by logistic growth with a limit of 2000 deer. Write a logistic growth model for this population. Then use the model to create a table showing the size of the deer population over the next 30 years.

**SOLUTION** Let $y$ represent the number of deer in year $t$. Assuming a logistic growth model means that the rate of change in the population is proportional to both $y$ and $(1 - y/2000)$. That is

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{2000}\right), \quad 100 \le y \le 2000.$$

The solution of this equation is

$$y = \frac{2000}{1 + be^{-kt}}.$$

Using the fact that $y = 100$ when $t = 0$, you can solve for $b$.

$$100 = \frac{2000}{1 + be^{-k(0)}} \implies b = 19$$

Then, using the fact that $y = 432$ when $t = 5$, you can solve for $k$.

$$432 = \frac{2000}{1 + 19e^{-k(5)}} \implies k \approx 0.33106$$

So, the logistic growth model for the population is

$$y = \frac{2000}{1 + 19e^{-0.33106t}}. \qquad \text{Logistic growth model}$$

The population, in five-year intervals, is shown in the table.

| Time, $t$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|
| Population, $y$ | 100 | 432 | 1181 | 1766 | 1951 | 1990 | 1998 |

Daniel J. Cox/Getty Images

**✓ CHECKPOINT 6**

Write the logistic growth model for the population of deer in Example 6 if the game preserve could contain a limit of 4000 deer.

■

**CONCEPT CHECK**

1. Complete the following: The technique of partial fractions involves the decomposition of a _____ function into the _____ of two or more simple _____ functions.

2. What is a proper rational function?

3. Before applying partial fractions to an improper rational function, what should you do?

4. Describe what the value of $L$ represents in the logistic growth function $y = \dfrac{L}{1 + be^{-kt}}.$

In Exercises 1–8, factor the expression.

**1.** $x^2 - 16$

**2.** $x^2 - 25$

**3.** $x^2 - x - 12$

**4.** $x^2 + x - 6$

**5.** $x^3 - x^2 - 2x$

**6.** $x^3 - 4x^2 + 4x$

**7.** $x^3 - 4x^2 + 5x - 2$

**8.** $x^3 - 5x^2 + 7x - 3$

In Exercises 9–14, rewrite the improper rational expression as the sum of a proper rational expression and a polynomial.

**9.** $\dfrac{x^2 - 2x + 1}{x - 2}$

**10.** $\dfrac{2x^2 - 4x + 1}{x - 1}$

**11.** $\dfrac{x^3 - 3x^2 + 2}{x - 2}$

**12.** $\dfrac{x^3 + 2x - 1}{x + 1}$

**13.** $\dfrac{x^3 + 4x^2 + 5x + 2}{x^2 - 1}$

**14.** $\dfrac{x^3 + 3x^2 - 4}{x^2 - 1}$

## Exercises 6.2

In Exercises 1–12, write the partial fraction decomposition for the expression.

**1.** $\dfrac{2(x + 20)}{x^2 - 25}$

**2.** $\dfrac{3x + 11}{x^2 - 2x - 3}$

**3.** $\dfrac{8x + 3}{x^2 - 3x}$

**4.** $\dfrac{10x + 3}{x^2 + x}$

**5.** $\dfrac{4x - 13}{x^2 - 3x - 10}$

**6.** $\dfrac{7x + 5}{6(2x^2 + 3x + 1)}$

**7.** $\dfrac{3x^2 - 2x - 5}{x^3 + x^2}$

**8.** $\dfrac{3x^2 - x + 1}{x(x + 1)^2}$

**9.** $\dfrac{x + 1}{3(x - 2)^2}$

**10.** $\dfrac{3x - 4}{(x - 5)^2}$

**11.** $\dfrac{8x^2 + 15x + 9}{(x + 1)^3}$

**12.** $\dfrac{6x^2 - 5x}{(x + 2)^3}$

In Exercises 13–32, use partial fractions to find the indefinite integral.

**13.** $\displaystyle\int \dfrac{1}{x^2 - 1}\,dx$

**14.** $\displaystyle\int \dfrac{4}{x^2 - 4}\,dx$

**15.** $\displaystyle\int \dfrac{-2}{x^2 - 16}\,dx$

**16.** $\displaystyle\int \dfrac{-4}{x^2 - 4}\,dx$

**17.** $\displaystyle\int \dfrac{1}{2x^2 - x}\,dx$

**18.** $\displaystyle\int \dfrac{2}{x^2 - 2x}\,dx$

**19.** $\displaystyle\int \dfrac{10}{x^2 - 10x}\,dx$

**20.** $\displaystyle\int \dfrac{5}{x^2 + x - 6}\,dx$

**21.** $\displaystyle\int \dfrac{3}{x^2 + x - 2}\,dx$

**22.** $\displaystyle\int \dfrac{1}{4x^2 - 9}\,dx$

**23.** $\displaystyle\int \dfrac{5 - x}{2x^2 + x - 1}\,dx$

**24.** $\displaystyle\int \dfrac{x + 1}{x^2 + 4x + 3}\,dx$

**25.** $\displaystyle\int \dfrac{x^2 - 4x - 4}{x^3 - 4x}\,dx$

**26.** $\displaystyle\int \dfrac{x^2 + 12x + 12}{x^3 - 4x}\,dx$

**27.** $\displaystyle\int \dfrac{x + 2}{x^2 - 4x}\,dx$

**28.** $\displaystyle\int \dfrac{4x^2 + 2x - 1}{x^3 + x^2}\,dx$

**29.** $\displaystyle\int \dfrac{2x - 3}{(x - 1)^2}\,dx$

**30.** $\displaystyle\int \dfrac{x^4}{(x - 1)^3}\,dx$

**31.** $\displaystyle\int \dfrac{3x^2 + 3x + 1}{x(x^2 + 2x + 1)}\,dx$

**32.** $\displaystyle\int \dfrac{3x}{x^2 - 6x + 9}\,dx$

In Exercises 33–40, evaluate the definite integral.

**33.** $\displaystyle\int_4^5 \dfrac{1}{9 - x^2}\,dx$

**34.** $\displaystyle\int_0^1 \dfrac{3}{2x^2 + 5x + 2}\,dx$

**35.** $\displaystyle\int_1^5 \dfrac{x - 1}{x^2(x + 1)}\,dx$

**36.** $\displaystyle\int_0^1 \dfrac{x^2 - x}{x^2 + x + 1}\,dx$

**37.** $\displaystyle\int_0^1 \dfrac{x^3}{x^2 - 2}\,dx$
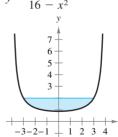
**38.** $\displaystyle\int_0^1 \dfrac{x^3 - 1}{x^2 - 4}\,dx$
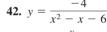
**39.** $\displaystyle\int_1^2 \dfrac{x^3 - 4x^2 - 3x + 3}{x^2 - 3x}\,dx$

**40.** $\displaystyle\int_2^4 \dfrac{x^4 - 4}{x^2 - 1}\,dx$
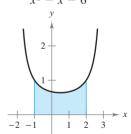
In Exercises 41–44, find the area of the shaded region.

**41.** $y = \dfrac{14}{16 - x^2}$



**42.** $y = \dfrac{-4}{x^2 - x - 6}$

**43.** $y = \dfrac{x+1}{x^2 - x}$

**44.** $y = \dfrac{x^2 + 2x - 1}{x^2 - 4}$
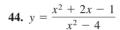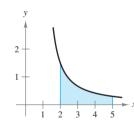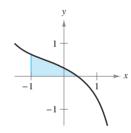
In Exercises 45 and 46, find the area of the region bounded by the graphs of the given equations.

**45.** $y = \dfrac{12}{x^2 + 5x + 6}, y = 0, x = 0, x = 1$

**46.** $y = \dfrac{-24}{x^2 - 16}, y = 0, x = 1, x = 3$

In Exercises 47–50, write the partial fraction decomposition for the rational expression. Check your result algebraically. Then assign a value to the constant *a* and use a graphing utility to check the result graphically.

**47.** $\dfrac{1}{a^2 - x^2}$

**48.** $\dfrac{1}{x(x + a)}$

**49.** $\dfrac{1}{x(a - x)}$

**50.** $\dfrac{1}{(x + 1)(a - x)}$

**51. Writing** What is the first step when integrating

$$\int \frac{x^2}{x - 5}\, dx?$$ Explain. (Do not integrate.)

**52. Writing** State the method you would use to evaluate each integral. Explain why you chose that method. (Do not integrate.)

(a) $\displaystyle\int \frac{2x + 1}{x^2 + x - 8}\, dx$

(b) $\displaystyle\int \frac{7x + 4}{x^2 + 2x - 8}\, dx$

Ⓣ **53. Biology** A conservation organization releases 100 animals of an endangered species into a game preserve. During the first 2 years, the population increases to 134 animals. The organization believes that the preserve has a capacity of 1000 animals and that the herd will grow according to a logistic growth model. That is, the size *y* of the herd will follow the equation

$$\int \frac{1}{y(1 - y/1000)}\, dy = \int k\, dt$$

where *t* is measured in years. Find this logistic curve. (To solve for the constant of integration *C* and the proportionality constant *k*, assume $y = 100$ when $t = 0$ and $y = 134$ when $t = 2$.) Use a graphing utility to graph your solution.

**54. Health: Epidemic** A single infected individual enters a community of 500 individuals susceptible to the disease. The disease spreads at a rate proportional to the product of the total number infected and the number of susceptible individuals not yet infected. A model for the time it takes for the disease to spread to *x* individuals is

$$t = 5010 \int \frac{1}{(x + 1)(500 - x)}\, dx$$

where *t* is the time in hours.

(a) Find the time it takes for 75% of the population to become infected (when $t = 0, x = 1$).

(b) Find the number of people infected after 100 hours.

**55. Marketing** After test-marketing a new menu item, a fast-food restaurant predicts that sales of the new item will grow according to the model

$$\frac{dS}{dt} = \frac{2t}{(t + 4)^2}$$

where *t* is the time in weeks and *S* is the sales (in thousands of dollars). Find the sales of the menu item at 10 weeks.

**56. Biology** One gram of a bacterial culture is present at time $t = 0$, and 10 grams is the upper limit of the culture's weight. The time required for the culture to grow to *y* grams is modeled by

$$kt = \int \frac{1}{y(1 - y/10)}\, dy$$
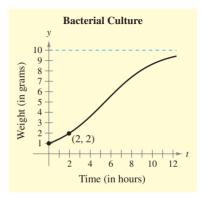
where *y* is the weight of the culture (in grams) and *t* is the time in hours.

(a) Verify that the weight of the culture at time *t* is modeled by

$$y = \frac{10}{1 + 9e^{-kt}}.$$

Use the fact that $y = 1$ when $t = 0$.

(b) Use the graph to determine the constant *k*.



**Bacterial Culture**

**57. Revenue**    The revenue $R$ (in millions of dollars per year) for Symantec Corporation from 1997 through 2005 can be modeled by

$$R = \frac{1340t^2 + 24{,}044t + 22{,}704}{-6t^2 + 94t + 100}$$

where $t = 7$ corresponds to 1997. Find the total revenue from 1997 through 2005. Then find the average revenue during this time period. *(Source: Symantec Corporation)*

**58. Environment**    The predicted cost $C$ (in hundreds of thousands of dollars) for a company to remove $p\%$ of a chemical from its waste water is shown in the table.

| $p$ | 0 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| $C$ | 0 | 0.7 | 1.0 | 1.3 | 1.7 |

| $p$ | 50 | 60 | 70 | 80 | 90 |
|---|---|---|---|---|---|
| $C$ | 2.0 | 2.7 | 3.6 | 5.5 | 11.2 |
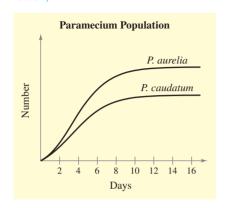
A model for the data is given by

$$C = \frac{124p}{(10 + p)(100 - p)}, \quad 0 \le p < 100.$$

Use the model to find the average cost for removing between 75% and 80% of the chemical.

**B 59. Biology: Population Growth**    The graph shows the logistic growth curves for two species of the single-celled *Paramecium* in a laboratory culture. During which time intervals is the rate of growth of each species increasing? During which time intervals is the rate of growth of each species decreasing? Which species has a higher limiting population under these conditions? *(Source: Adapted from Levine/Miller,* Biology: Discovering Life, *Second Edition)*

**Paramecium Population**

*P. aurelia*

*P. caudatum*

Number

Days
2  4  6  8  10  12  14  16

**60. Population Growth**    The population of the United States was 76 million people in 1900 and reached 300 million people in 2006. From 1900 through 2006, assume the population of the United States can be modeled by logistic growth with a limit of 839.1 million people. *(Source: U.S. Census Bureau)*

(a) Write a differential equation of the form

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

where $y$ represents the population of the United States (in millions of people) and $t$ represents the number of years since 1900.

(b) Find the logistic growth model $y = \dfrac{L}{1 + be^{-kt}}$ for this population.

(T) (c) Use a graphing utility to graph the model from part (b). Then estimate the year in which the population of the United States will reach 400 million people.

**Business Capsule**

**S**usie Wang and Ric Kostick graduated from the University of California at Berkeley with degrees in mathematics. In 1999, Wang used $10,000 to start Aqua Dessa Spa Therapy, a high-end cosmetics company that uses natural ingredients in their products. Now, the company run by Wang and Kostick has annual sales of over $10 million, operates under several brand names, including 100% Pure, and has a global customer base. Wang and Kostick attribute the success of their business to applying what they learned from their studies.

**61. Research Project**    Use your school's library, the Internet, or some other reference source to research the opportunity cost of attending graduate school for 2 years to receive a Masters of Business Administration (MBA) degree rather than working for 2 years with a bachelor's degree. Write a short paper describing these costs.

# Integration Tables

■ Use integration tables to find indefinite integrals.
■ Use reduction formulas to find indefinite integrals.

## Integration Tables

You have studied several integration techniques that can be used with the basic integration formulas. Certainly these techniques and formulas do not cover every possible method for finding an antiderivative, but they do cover most of the important ones.

In this section, you will expand the list of integration formulas to form a table of integrals. As you add new integration formulas to the basic list, two effects occur. On one hand, it becomes increasingly difficult to memorize, or even become familiar with, the entire list of formulas. On the other hand, with a longer list you need fewer techniques for fitting an integral to one of the formulas on the list. The procedure of integrating by means of a long list of formulas is called **integration by tables.** (The table in this section constitutes only a partial listing of integration formulas. Much longer lists exist, some of which contain several hundred formulas.)

Integration by tables should not be considered a trivial task. It requires considerable thought and insight, and it often requires substitution. Many people find a table of integrals to be a valuable supplement to the integration techniques discussed in this text. We encourage you to gain competence in the use of integration tables, as well as to continue to improve in the use of the various integration techniques. In doing so, you should find that a combination of techniques and tables is the most versatile approach to integration.

Each integration formula in the table on the next three pages can be developed using one or more of the techniques you have studied. You should try to verify several of the formulas. For instance, Formula 4

$$\int \frac{u}{(a + bu)^2}\, du = \frac{1}{b^2}\left(\frac{a}{a + bu} + \ln|a + bu|\right) + C \qquad \text{Formula 4}$$

can be verified using partial fractions, Formula 17

$$\int \frac{\sqrt{a + bu}}{u}\, du = 2\sqrt{a + bu} + a\int \frac{1}{u\sqrt{a + bu}}\, du \qquad \text{Formula 17}$$

can be verified using integration by parts, and Formula 37

$$\int \frac{1}{1 + e^u}\, du = u - \ln(1 + e^u) + C \qquad \text{Formula 37}$$

can be verified using substitution.

### STUDY TIP

A symbolic integration utility consists, in part, of a database of integration tables. The primary difference between using a symbolic integration utility and using a table of integrals is that with a symbolic integration utility the computer searches through the database to find a fit. With a table of integrals, *you* must do the searching.

In the table of integrals below and on the next two pages, the formulas have been grouped into eight different types according to the form of the integrand.

Forms involving $u^n$

Forms involving $a + bu$

Forms involving $\sqrt{a + bu}$

Forms involving $\sqrt{u^2 \pm a^2}$

Forms involving $u^2 - a^2$

Forms involving $\sqrt{a^2 - u^2}$

Forms involving $e^u$

Forms involving $\ln u$

## Table of Integrals

*Forms involving $u^n$*

**1.** $\displaystyle\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

**2.** $\displaystyle\int \frac{1}{u} \, du = \ln|u| + C$

*Forms involving $a + bu$*

**3.** $\displaystyle\int \frac{u}{a + bu} \, du = \frac{1}{b^2}(bu - a \ln|a + bu|) + C$

**4.** $\displaystyle\int \frac{u}{(a + bu)^2} \, du = \frac{1}{b^2}\left(\frac{a}{a + bu} + \ln|a + bu|\right) + C$

**5.** $\displaystyle\int \frac{u}{(a + bu)^n} \, du = \frac{1}{b^2}\left[\frac{-1}{(n-2)(a+bu)^{n-2}} + \frac{a}{(n-1)(a+bu)^{n-1}}\right] + C, \quad n \neq 1, 2$

**6.** $\displaystyle\int \frac{u^2}{a + bu} \, du = \frac{1}{b^3}\left[-\frac{bu}{2}(2a - bu) + a^2 \ln|a + bu|\right] + C$

**7.** $\displaystyle\int \frac{u^2}{(a + bu)^2} \, du = \frac{1}{b^3}\left(bu - \frac{a^2}{a + bu} - 2a \ln|a + bu|\right) + C$

**8.** $\displaystyle\int \frac{u^2}{(a + bu)^3} \, du = \frac{1}{b^3}\left[\frac{2a}{a + bu} - \frac{a^2}{2(a + bu)^2} + \ln|a + bu|\right] + C$

**9.** $\displaystyle\int \frac{u^2}{(a + bu)^n} \, du = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bu)^{n-3}} + \frac{2a}{(n-2)(a+bu)^{n-2}} - \frac{a^2}{(n-1)(a+bu)^{n-1}}\right] + C, \quad n \neq 1, 2, 3$

**10.** $\displaystyle\int \frac{1}{u(a + bu)} \, du = \frac{1}{a} \ln\left|\frac{u}{a + bu}\right| + C$

**11.** $\displaystyle\int \frac{1}{u(a + bu)^2} \, du = \frac{1}{a}\left(\frac{1}{a + bu} + \frac{1}{a} \ln\left|\frac{u}{a + bu}\right|\right) + C$

**12.** $\displaystyle\int \frac{1}{u^2(a + bu)} \, du = -\frac{1}{a}\left(\frac{1}{u} + \frac{b}{a} \ln\left|\frac{u}{a + bu}\right|\right) + C$

**13.** $\displaystyle\int \frac{1}{u^2(a + bu)^2} \, du = -\frac{1}{a^2}\left[\frac{a + 2bu}{u(a + bu)} + \frac{2b}{a} \ln\left|\frac{u}{a + bu}\right|\right] + C$

**Table of Integrals (*continued*)**

*Forms involving $\sqrt{a + bu}$*

14. $\displaystyle \int u^n \sqrt{a + bu}\, du = \frac{2}{b(2n + 3)}\left[ u^n (a + bu)^{3/2} - na \int u^{n-1}\sqrt{a + bu}\, du \right]$

15. $\displaystyle \int \frac{1}{u\sqrt{a + bu}}\, du = \frac{1}{\sqrt{a}} \ln\left| \frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}} \right| + C, \quad a > 0$

16. $\displaystyle \int \frac{1}{u^n \sqrt{a + bu}}\, du = \frac{-1}{a(n-1)}\left[ \frac{\sqrt{a + bu}}{u^{n-1}} + \frac{(2n - 3)b}{2} \int \frac{1}{u^{n-1}\sqrt{a + bu}}\, du \right], \quad n \neq 1$

17. $\displaystyle \int \frac{\sqrt{a + bu}}{u}\, du = 2\sqrt{a + bu} + a \int \frac{1}{u\sqrt{a + bu}}\, du$

18. $\displaystyle \int \frac{\sqrt{a + bu}}{u^n}\, du = \frac{-1}{a(n-1)}\left[ \frac{(a + bu)^{3/2}}{u^{n-1}} + \frac{(2n - 5)b}{2} \int \frac{\sqrt{a + bu}}{u^{n-1}}\, du \right], \quad n \neq 1$

19. $\displaystyle \int \frac{u}{\sqrt{a + bu}}\, du = -\frac{2(2a - bu)}{3b^2}\sqrt{a + bu} + C$

20. $\displaystyle \int \frac{u^n}{\sqrt{a + bu}}\, du = \frac{2}{(2n + 1)b}\left( u^n \sqrt{a + bu} - na \int \frac{u^{n-1}}{\sqrt{a + bu}}\, du \right)$

*Forms involving $\sqrt{u^2 \pm a^2}, \quad a > 0$*

21. $\displaystyle \int \sqrt{u^2 \pm a^2}\, du = \frac{1}{2}\left( u\sqrt{u^2 \pm a^2} \pm a^2 \ln\left| u + \sqrt{u^2 \pm a^2} \right| \right) + C$

22. $\displaystyle \int u^2 \sqrt{u^2 \pm a^2}\, du = \frac{1}{8}\left[ u(2u^2 \pm a^2)\sqrt{u^2 \pm a^2} - a^4 \ln\left| u + \sqrt{u^2 \pm a^2} \right| \right] + C$

23. $\displaystyle \int \frac{\sqrt{u^2 + a^2}}{u}\, du = \sqrt{u^2 + a^2} - a \ln\left| \frac{a + \sqrt{u^2 + a^2}}{u} \right| + C$

24. $\displaystyle \int \frac{\sqrt{u^2 \pm a^2}}{u^2}\, du = \frac{-\sqrt{u^2 \pm a^2}}{u} + \ln\left| u + \sqrt{u^2 \pm a^2} \right| + C$

25. $\displaystyle \int \frac{1}{\sqrt{u^2 \pm a^2}}\, du = \ln\left| u + \sqrt{u^2 \pm a^2} \right| + C$

26. $\displaystyle \int \frac{1}{u\sqrt{u^2 + a^2}}\, du = -\frac{1}{a} \ln\left| \frac{a + \sqrt{u^2 + a^2}}{u} \right| + C$

27. $\displaystyle \int \frac{u^2}{\sqrt{u^2 \pm a^2}}\, du = \frac{1}{2}\left( u\sqrt{u^2 \pm a^2} \mp a^2 \ln\left| u + \sqrt{u^2 \pm a^2} \right| \right) + C$

28. $\displaystyle \int \frac{1}{u^2 \sqrt{u^2 \pm a^2}}\, du = \mp \frac{\sqrt{u^2 \pm a^2}}{a^2 u} + C$

**Table of Integrals** (*continued*)

*Forms involving* $u^2 - a^2$, $\quad a > 0$

**29.** $\displaystyle \int \frac{1}{u^2 - a^2}\, du = -\int \frac{1}{a^2 - u^2}\, du = \frac{1}{2a} \ln\left|\frac{u - a}{u + a}\right| + C$

**30.** $\displaystyle \int \frac{1}{(u^2 - a^2)^n}\, du = \frac{-1}{2a^2(n-1)}\left[\frac{u}{(u^2 - a^2)^{n-1}} + (2n - 3)\int \frac{1}{(u^2 - a^2)^{n-1}}\, du\right], \quad n \neq 1$

*Forms involving* $\sqrt{a^2 - u^2}$, $\quad a > 0$

**31.** $\displaystyle \int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} - a \ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

**32.** $\displaystyle \int \frac{1}{u\sqrt{a^2 - u^2}}\, du = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

**33.** $\displaystyle \int \frac{1}{u^2\sqrt{a^2 - u^2}}\, du = \frac{-\sqrt{a^2 - u^2}}{a^2 u} + C$

*Forms involving* $e^u$

**34.** $\displaystyle \int e^u\, du = e^u + C$

**35.** $\displaystyle \int u e^u\, du = (u - 1)e^u + C$

**36.** $\displaystyle \int u^n e^u\, du = u^n e^u - n\int u^{n-1} e^u\, du$

**37.** $\displaystyle \int \frac{1}{1 + e^u}\, du = u - \ln(1 + e^u) + C$

**38.** $\displaystyle \int \frac{1}{1 + e^{nu}}\, du = u - \frac{1}{n} \ln(1 + e^{nu}) + C$

*Forms involving* $\ln u$

**39.** $\displaystyle \int \ln u\, du = u(-1 + \ln u) + C$

**40.** $\displaystyle \int u \ln u\, du = \frac{u^2}{4}(-1 + 2 \ln u) + C$

**41.** $\displaystyle \int u^n \ln u\, du = \frac{u^{n+1}}{(n+1)^2}[-1 + (n+1) \ln u] + C, \quad n \neq -1$

**42.** $\displaystyle \int (\ln u)^2\, du = u[2 - 2 \ln u + (\ln u)^2] + C$

**43.** $\displaystyle \int (\ln u)^n\, du = u(\ln u)^n - n\int (\ln u)^{n-1}\, du$

*Example 1* **Using Integration Tables**

Find $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$.

**SOLUTION** Because the expression inside the radical is linear, you should consider forms involving $\sqrt{a+bu}$, as in Formula 19.

$$\int \frac{u}{\sqrt{a+bu}}\,du = -\frac{2(2a-bu)}{3b^2}\sqrt{a+bu} + C \qquad \text{Formula 19}$$

Using this formula, let $a = -1$, $b = 1$, and $u = x$. Then $du = dx$, and you obtain

$$\int \frac{x}{\sqrt{x-1}}\,dx = -\frac{2(-2-x)}{3}\sqrt{x-1} + C \qquad \text{\color{magenta}{Substitute values of}}$$
$$\color{magenta}{a, b, \text{ and } u.}$$

$$= \frac{2}{3}(2+x)\sqrt{x-1} + C. \qquad \text{\color{magenta}{Simplify.}}$$

✓ **CHECKPOINT 1**

Use the integration table to find $\displaystyle\int \frac{x}{\sqrt{2+x}}\,dx$. ■

*Example 2* **Using Integration Tables**

Find $\displaystyle\int x\sqrt{x^4-9}\,dx$.

**SOLUTION** Because it is not clear which formula to use, you can begin by letting $u = x^2$ and $du = 2x\,dx$. With these substitutions, you can write the integral as shown.

$$\int x\sqrt{x^4-9}\,dx = \frac{1}{2}\int \sqrt{(x^2)^2-9}\,(2x)\,dx \qquad \text{\color{magenta}{Multiply and divide by 2.}}$$

$$= \frac{1}{2}\int \sqrt{u^2-9}\,du \qquad \text{\color{magenta}{Substitute } u \text{ and } du.}$$

Now, it appears that you can use Formula 21.

$$\int \sqrt{u^2-a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2-a^2} - a^2 \ln\left|u + \sqrt{u^2-a^2}\right|\right) + C$$

Letting $a = 3$, you obtain

$$\int x\sqrt{x^4-9}\,dx = \frac{1}{2}\int \sqrt{u^2-a^2}\,du$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(u\sqrt{u^2-a^2} - a^2 \ln\left|u + \sqrt{u^2-a^2}\right|\right)\right] + C$$

$$= \frac{1}{4}\left(x^2\sqrt{x^4-9} - 9 \ln\left|x^2 + \sqrt{x^4-9}\right|\right) + C.$$

✓ **CHECKPOINT 2**

Use the integration table to find

$$\int \frac{\sqrt{x^2+16}}{x}\,dx.$$ ■

*Example 3* **Using Integration Tables**

Find $\displaystyle\int \frac{1}{x\sqrt{x+1}}\,dx$.

**SOLUTION** Considering forms involving $\sqrt{a+bu}$, where $a=1$, $b=1$, and $u=x$, you can use Formula 15.

$$\int \frac{1}{u\sqrt{a+bu}}\,du = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad a > 0$$

So,

$$\int \frac{1}{x\sqrt{x+1}}\,dx = \int \frac{1}{u\sqrt{a+bu}}\,du = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C$$

$$= \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C.$$

**✓ CHECKPOINT 3**

Use the integration table to find

$$\int \frac{1}{x^2-4}\,dx.$$ ∎

*Example 4* **Using Integration Tables**

Evaluate $\displaystyle\int_0^2 \frac{x}{1+e^{-x^2}}\,dx$.

**SOLUTION** Of the forms involving $e^u$, Formula 37

$$\int \frac{1}{1+e^u}\,du = u - \ln(1+e^u) + C$$
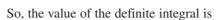
seems most appropriate. To use this formula, let $u = -x^2$ and $du = -2x\,dx$.

$$\int \frac{x}{1+e^{-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{1+e^{-x^2}}(-2x)\,dx = -\frac{1}{2}\int \frac{1}{1+e^u}\,du$$

$$= -\frac{1}{2}[u - \ln(1+e^u)] + C$$

$$= -\frac{1}{2}[-x^2 - \ln(1+e^{-x^2})] + C$$

$$= \frac{1}{2}[x^2 + \ln(1+e^{-x^2})] + C$$

So, the value of the definite integral is

$$\int_0^2 \frac{x}{1+e^{-x^2}}\,dx = \frac{1}{2}\left[x^2 + \ln(1+e^{-x^2})\right]_0^2 \approx 1.66$$

as shown in Figure 6.7.

$$y = \frac{x}{1+e^{-x^2}}$$
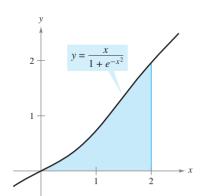
**FIGURE 6.7**

**✓ CHECKPOINT 4**

Use the integration table to evaluate $\displaystyle\int_0^1 \frac{x^2}{1+e^{x^3}}\,dx.$ ∎

## Reduction Formulas

Several of the formulas in the integration table have the form

$$\int f(x)\, dx = g(x) + \int h(x)\, dx$$

where the right side contains an integral. Such integration formulas are called **reduction formulas** because they reduce the original integral to the sum of a function and a simpler integral.

**Example 5**   **Using a Reduction Formula**

Find $\displaystyle\int x^2 e^x\, dx$.

**SOLUTION**   Using Formula 36

$$\int u^n e^u\, du = u^n e^u - n \int u^{n-1} e^u\, du$$

you can let $u = x$ and $n = 2$. Then $du = dx$, and you can write

$$\int x^2 e^x\, dx = x^2 e^x - 2 \int x e^x\, dx.$$

Then, using Formula 35

$$\int u e^u\, du = (u - 1)e^u + C$$

you can write

$$\int x^2 e^x\, dx = x^2 e^x - 2 \int x e^x\, dx$$
$$= x^2 e^x - 2(x - 1)e^x + C$$
$$= x^2 e^x - 2x e^x + 2e^x + C$$
$$= e^x(x^2 - 2x + 2) + C.$$

✓**CHECKPOINT 5**

Use the integration table to find the indefinite integral $\displaystyle\int (\ln x)^2\, dx$. ■

---

**TECHNOLOGY**

(T) You have now studied two ways to find the indefinite integral in Example 5. Example 5 uses an integration table, and Example 4 in Section 6.1 uses integration by parts. A third way would be to use a symbolic integration utility.

## Application

Researchers such as psychologists use definite integrals to represent the probability that an event will occur. For instance, a probability of 0.5 means that an event will occur about 50% of the time.

Integration can be used to find the probability that an event will occur. In such an application, the real-life situation is modeled by a *probability density function f*, and the probability that $x$ will lie between $a$ and $b$ is represented by

$$P(a \le x \le b) = \int_a^b f(x)\, dx.$$

The probability $P(a \le x \le b)$ must be a number between 0 and 1.

### Example 6    Finding a Probability   ®

A psychologist finds that the probability that a participant in a memory experiment will recall between $a$ and $b$ percent (in decimal form) of the material is

$$P(a \le x \le b) = \int_a^b \frac{1}{e-2} x^2 e^x\, dx, \quad 0 \le a \le b \le 1.$$

Find the probability that a randomly chosen participant will recall between 0% and 87.5% of the material.

**SOLUTION**

You can use the Constant Multiple Rule to rewrite the integral as

$$\frac{1}{e-2} \int_a^b x^2 e^x\, dx.$$

Note that the integrand is the same as the one in Example 5. Use the result of Example 5 to find the probability with $a = 0$ and $b = 0.875$.

$$\frac{1}{e-2} \int_0^{0.875} x^2 e^x\, dx = \frac{1}{e-2} \Big[ e^x (x^2 - 2x + 2) \Big]_0^{0.875} \approx 0.608$$

So, the probability is about 60.8%, as indicated in Figure 6.8.



**FIGURE 6.8**

Graph of $y = \dfrac{1}{e-2} x^2 e^x$ with Area ≈ 0.608

### ✓ CHECKPOINT 6

Use Example 6 to find the probability that a participant will recall between 0% and 62.5% of the material. ■

---

### CONCEPT CHECK

1. Which integration formula would you use to find $\displaystyle\int \frac{1}{e^x + 1}\, dx$? (Do not integrate.)

2. Which integration formula would you use to find $\displaystyle\int \sqrt{x^2 + 4}\, dx$? (Do not integrate.)

3. True or false: When using a table of integrals, you may have to make substitutions to rewrite your integral in the form in which it appears in the table.

4. Describe what is meant by a reduction formula. Give an example.

**Skills Review 6.3**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4, 6.1, and 6.2.

In Exercises 1–4, expand the expression.

**1.** $(x + 4)^2$

**2.** $(x - 1)^2$

**3.** $\left(x + \frac{1}{2}\right)^2$

**4.** $\left(x - \frac{1}{3}\right)^2$

In Exercises 5–8, write the partial fraction decomposition for the expression.

**5.** $\dfrac{4}{x(x + 2)}$

**6.** $\dfrac{3}{x(x - 4)}$

**7.** $\dfrac{x + 4}{x^2(x - 2)}$

**8.** $\dfrac{3x^2 + 4x - 8}{x(x - 2)(x + 1)}$

In Exercises 9 and 10, use integration by parts to find the indefinite integral.

**9.** $\displaystyle\int 2xe^x \, dx$

**10.** $\displaystyle\int 3x^2 \ln x \, dx$

---

# Exercises 6.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, use the indicated formula from the table of integrals in this section to find the indefinite integral.

**1.** $\displaystyle\int \frac{x}{(2 + 3x)^2} \, dx$,   Formula 4

**2.** $\displaystyle\int \frac{1}{x(2 + 3x)^2} \, dx$,   Formula 11

**3.** $\displaystyle\int \frac{x}{\sqrt{2 + 3x}} \, dx$,   Formula 19

**4.** $\displaystyle\int \frac{4}{x^2 - 9} \, dx$,   Formula 29

**5.** $\displaystyle\int \frac{2x}{\sqrt{x^4 - 9}} \, dx$,   Formula 25

**6.** $\displaystyle\int x^2\sqrt{x^2 + 9} \, dx$,   Formula 22

**7.** $\displaystyle\int x^3 e^{x^2} \, dx$,   Formula 35

**8.** $\displaystyle\int \frac{x}{1 + e^{x^2}} \, dx$,   Formula 37

In Exercises 9–36, use the table of integrals in this section to find the indefinite integral.

**9.** $\displaystyle\int \frac{1}{x(1 + x)} \, dx$

**10.** $\displaystyle\int \frac{1}{x(1 + x)^2} \, dx$

**11.** $\displaystyle\int \frac{1}{x\sqrt{x^2 + 9}} \, dx$

**12.** $\displaystyle\int \frac{1}{\sqrt{x^2 - 1}} \, dx$

**13.** $\displaystyle\int \frac{1}{x\sqrt{4 - x^2}} \, dx$

**14.** $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x^2} \, dx$

**15.** $\displaystyle\int x \ln x \, dx$

**16.** $\displaystyle\int (\ln 5x)^2 \, dx$

**17.** $\displaystyle\int \frac{6x}{1 + e^{3x^2}} \, dx$

**18.** $\displaystyle\int \frac{1}{1 + e^x} \, dx$

**19.** $\displaystyle\int x\sqrt{x^4 - 4} \, dx$

**20.** $\displaystyle\int \frac{x}{x^4 - 9} \, dx$

**21.** $\displaystyle\int \frac{t^2}{(2 + 3t)^3} \, dt$

**22.** $\displaystyle\int \frac{\sqrt{3 + 4t}}{t} \, dt$

**23.** $\displaystyle\int \frac{s}{s^2\sqrt{3 + s}} \, ds$

**24.** $\displaystyle\int \sqrt{3 + x^2} \, dx$

**25.** $\displaystyle\int \frac{x^2}{1 + x} \, dx$

**26.** $\displaystyle\int \frac{1}{1 + e^{2x}} \, dx$

**27.** $\displaystyle\int \frac{x^2}{(3 + 2x)^5} \, dx$

**28.** $\displaystyle\int \frac{1}{x^2\sqrt{x^2 - 4}} \, dx$

**29.** $\displaystyle\int \frac{1}{x^2\sqrt{1 - x^2}} \, dx$

**30.** $\displaystyle\int \frac{2x}{(1 - 3x)^2} \, dx$

**31.** $\displaystyle\int x^2 \ln x \, dx$

**32.** $\displaystyle\int xe^{x^2} \, dx$

**33.** $\displaystyle\int \frac{x^2}{(3x - 5)^2} \, dx$

**34.** $\displaystyle\int \frac{1}{2x^2(2x - 1)^2} \, dx$

**35.** $\displaystyle\int \frac{\ln x}{x(4 + 3 \ln x)} \, dx$

**36.** $\displaystyle\int (\ln x)^3 \, dx$

In Exercises 37–42, use the table of integrals to find the exact area of the region bounded by the graphs of the equations. Then use a graphing utility to graph the region and approximate the area.

**37.** $y = \dfrac{x}{\sqrt{x+1}}$, $y = 0$, $x = 8$

**38.** $y = \dfrac{2}{1 + e^{4x}}$, $y = 0$, $x = 0$, $x = 1$

**39.** $y = \dfrac{x}{1 + e^{x^2}}$, $y = 0$, $x = 2$

**40.** $y = \dfrac{-e^x}{1 - e^{2x}}$, $y = 0$, $x = 1$, $x = 2$

**41.** $y = x^2\sqrt{x^2 + 4}$, $y = 0$, $x = \sqrt{5}$

**42.** $y = x \ln x^2$, $y = 0$, $x = 4$

In Exercises 43–50, evaluate the definite integral.

**43.** $\displaystyle\int_0^1 \frac{x}{\sqrt{1+x}}\,dx$

**44.** $\displaystyle\int_0^5 \frac{x}{\sqrt{5+2x}}\,dx$

**45.** $\displaystyle\int_0^5 \frac{x}{(4+x)^2}\,dx$

**46.** $\displaystyle\int_2^4 \frac{x^2}{(3x-5)}\,dx$

**47.** $\displaystyle\int_0^4 \frac{6}{1+e^{0.5x}}\,dx$

**48.** $\displaystyle\int_2^4 \sqrt{3+x^2}\,dx$

**49.** $\displaystyle\int_1^4 x \ln x\,dx$

**50.** $\displaystyle\int_1^3 x^2 \ln x\,dx$

In Exercises 51–54, find the indefinite integral (a) using the integration table and (b) using the specified method.

| Integral | Method |
|---|---|
| **51.** $\displaystyle\int x^2 e^x\,dx$ | Integration by parts |
| **52.** $\displaystyle\int x^4 \ln x\,dx$ | Integration by parts |
| **53.** $\displaystyle\int \frac{1}{x^2(x+1)}\,dx$ | Partial fractions |
| **54.** $\displaystyle\int \frac{1}{x^2 - 75}\,dx$ | Partial fractions |
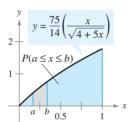
**55. Probability** The probability of recall in an experiment is modeled by

$$P(a \le x \le b) = \int_a^b \frac{75}{14}\left(\frac{x}{\sqrt{4+5x}}\right)dx, \quad 0 \le a \le b \le 1$$

where $x$ is the percent of recall (see figure).

(a) What is the probability of recalling between 40% and 80%?

(b) What is the probability of recalling between 0% and 50%?
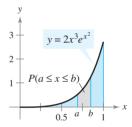


Figure for 55



Figure for 56

**56. Probability** The probability of finding between $a$ and $b$ percent iron in ore samples is modeled by

$$P(a \le x \le b) = \int_a^b 2x^3 e^{x^2}\,dx, \quad 0 \le a \le b \le 1$$

(see figure). Find the probabilities that a sample will contain between (a) 0% and 25% and (b) 50% and 100% iron.

**Population Growth** In Exercises 57 and 58, use a graphing utility to graph the growth function. Use the table of integrals to find the average value of the growth function over the interval, where $N$ is the size of a population and $t$ is the time in days.

**57.** $N = \dfrac{5000}{1 + e^{4.8 - 1.9t}}$, $[0, 2]$

**58.** $N = \dfrac{375}{1 + e^{4.20 - 0.25t}}$, $[21, 28]$

**59. Revenue** The revenue (in dollars per year) for a new product is modeled by

$$R = 10,000\left[1 - \frac{1}{(1 + 0.1t^2)^{1/2}}\right]$$

where $t$ is the time in years. Estimate the total revenue from sales of the product over its first 2 years on the market.

**60. Consumer and Producer Surpluses** Find the consumer surplus and the producer surplus for a product with the given demand and supply functions.

$$Demand: p = \frac{60}{\sqrt{x^2 + 81}}, \quad Supply: p = \frac{x}{3}$$

**61. Profit** The net profits $P$ (in billions of dollars per year) for The Hershey Company from 2002 through 2005 can be modeled by

$$P = \sqrt{0.00645t^2 + 0.1673}, \quad 2 \le t \le 5$$

where $t$ is time in years, with $t = 2$ corresponding to 2002. Find the average net profit over that time period. *(Source: The Hershey Co.)*

**62. Extended Application** To work an extended application analyzing the purchasing power of the dollar from 1983 through 2005, visit this text's website at *college.hmco.com.* *(Data Source: U.S. Bureau of Labor Statistics)*

## Mid-Chapter Quiz

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–6, use integration by parts to find the indefinite integral.

**1.** $\displaystyle\int xe^{5x}\,dx$

**2.** $\displaystyle\int \ln x^3\,dx$

**3.** $\displaystyle\int (x+1)\ln x\,dx$

**4.** $\displaystyle\int x\sqrt{x+3}\,dx$

**5.** $\displaystyle\int x\ln\sqrt{x}\,dx$

**6.** $\displaystyle\int x^2\,e^{-2x}\,dx$

**7.** A small business expects its income during the next 7 years to be given by

$$c(t) = 32{,}000t, \quad 0 \le t \le 7.$$

Assuming an annual inflation rate of 3.3%, can the business claim that the present value of its income during the next 7 years is at least $650,000?

In Exercises 8–10, use partial fractions to find the indefinite integral.

**8.** $\displaystyle\int \frac{10}{x^2-25}\,dx$

**9.** $\displaystyle\int \frac{x-14}{x^2+2x-8}\,dx$

**10.** $\displaystyle\int \frac{5x-1}{(x+1)^2}\,dx$

**11.** The population of a colony of bees can be modeled by logistic growth. The capacity of the colony's hive is 100,000 bees. One day in the early spring, there are 25,000 bees in the hive. Thirteen days later, the population of the hive increases to 28,000 bees. Write a logistic growth model for the colony.

In Exercises 12–17, use the table of integrals in Section 6.3 to find the indefinite integral.

**12.** $\displaystyle\int \frac{x}{1+2x}\,dx$

**13.** $\displaystyle\int \frac{1}{x(0.1+0.2x)}\,dx$

**14.** $\displaystyle\int \frac{\sqrt{x^2-16}}{x^2}\,dx$

**15.** $\displaystyle\int \frac{1}{x\sqrt{4+9x}}\,dx$

**16.** $\displaystyle\int \frac{2x}{1+e^{4x^2}}\,dx$

**17.** $\displaystyle\int 2x(x^2+1)e^{x^2+1}\,dx$

**18.** The number of Kohl's Corporation stores in the United States from 1999 through 2006 can be modeled by

$$N(t) = 75.0 + 1.07t^2\ln t, \quad 9 \le t \le 16$$

where $t$ is the year, with $t = 9$ corresponding to 1999. Find the average number of Kohl's stores in the U.S. from 1999 through 2006.   *(Source: Kohl's Corporation)*

In Exercises 19–24, evaluate the definite integral.

**19.** $\displaystyle\int_{-2}^{0} xe^{x/2}\,dx$

**20.** $\displaystyle\int_{1}^{e} (\ln x)^2\,dx$

**21.** $\displaystyle\int_{1}^{4} \frac{3x+1}{x(x+1)}\,dx$
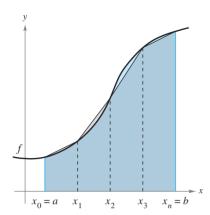
**22.** $\displaystyle\int_{4}^{5} \frac{120}{(x-3)(x+5)}\,dx$

**23.** $\displaystyle\int_{2}^{3} \frac{1}{x^2\sqrt{9-x^2}}\,dx$

**24.** $\displaystyle\int_{4}^{6} \frac{2x}{x^4-4}\,dx$

## Numerical Integration

■ Use the Trapezoidal Rule to approximate definite integrals.
■ Use Simpson's Rule to approximate definite integrals.
■ Analyze the sizes of the errors when approximating definite integrals with the Trapezoidal Rule and Simpson's Rule.

### Trapezoidal Rule

In Section 5.6, you studied one technique for approximating the value of a *definite* integral—the Midpoint Rule. In this section, you will study two other approximation techniques: the **Trapezoidal Rule** and **Simpson's Rule.**

To develop the Trapezoidal Rule, consider a function $f$ that is nonnegative and continuous on the closed interval $[a, b]$. To approximate the area represented by $\int_a^b f(x)dx$, partition the interval into $n$ subintervals, each of width

$$\Delta x = \frac{b - a}{n}. \qquad \text{Width of each subinterval}$$

Next, form $n$ trapezoids, as shown in Figure 6.9. As you can see in Figure 6.10, the area of the first trapezoid is

$$\text{Area of first trapezoid} = \left(\frac{b - a}{n}\right)\left[\frac{f(x_0) + f(x_1)}{2}\right].$$

The areas of the other trapezoids follow a similar pattern, and the sum of the $n$ areas is

$$\left(\frac{b - a}{n}\right)\left[\frac{f(x_0) + f(x_1)}{2} + \frac{f(x_1) + f(x_2)}{2} + \cdot\cdot\cdot + \frac{f(x_{n-1}) + f(x_n)}{2}\right]$$

$$= \left(\frac{b - a}{2n}\right)[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \cdot\cdot\cdot + f(x_{n-1}) + f(x_n)]$$

$$= \left(\frac{b - a}{2n}\right)[f(x_0) + 2f(x_1) + 2f(x_2) + \cdot\cdot\cdot + 2f(x_{n-1}) + f(x_n)].$$

Although this development assumes $f$ to be continuous *and* nonnegative on $[a, b]$, the resulting formula is valid as long as $f$ is continuous on $[a, b]$.



**FIGURE 6.9**    The area of the region can be approximated using four trapezoids.



**FIGURE 6.10**

### The Trapezoidal Rule

If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)dx \approx \left(\frac{b - a}{2n}\right)[f(x_0) + 2f(x_1) + \cdot\cdot\cdot + 2f(x_{n-1}) + f(x_n)].$$

**STUDY TIP**

The coefficients in the Trapezoidal Rule have the pattern

$$1 \quad 2 \quad 2 \quad 2 \ldots 2 \quad 2 \quad 1.$$
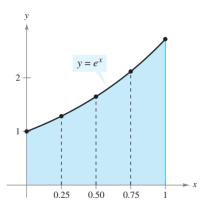
**FIGURE 6.11** Four Subintervals
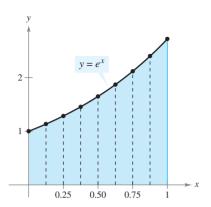


**FIGURE 6.12** Eight Subintervals

✓ **CHECKPOINT 1**

Use the Trapezoidal Rule with $n = 4$ to approximate

$$\int_0^1 e^{2x}\, dx.$$

■

*Example 1* **Using the Trapezoidal Rule**

Use the Trapezoidal Rule to approximate $\int_0^1 e^x\, dx$. Compare the results for $n = 4$ and $n = 8$.

**SOLUTION** When $n = 4$, the width of each subinterval is

$$\frac{1 - 0}{4} = \frac{1}{4}$$

and the endpoints of the subintervals are

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad \text{and} \quad x_4 = 1$$

as indicated in Figure 6.11. So, by the Trapezoidal Rule

$$\int_0^1 e^x\, dx = \frac{1}{8}(e^0 + 2e^{0.25} + 2e^{0.5} + 2e^{0.75} + e^1)$$

$$\approx 1.7272. \qquad \text{Approximation using } n = 4$$

When $n = 8$, the width of each subinterval is

$$\frac{1 - 0}{8} = \frac{1}{8}$$

and the endpoints of the subintervals are

$$x_0 = 0, \quad x_1 = \frac{1}{8}, \quad x_2 = \frac{1}{4}, \quad x_3 = \frac{3}{8}, \quad x_4 = \frac{1}{2}$$

$$x_5 = \frac{5}{8}, \quad x_6 = \frac{3}{4}, \quad x_7 = \frac{7}{8}, \quad \text{and} \quad x_8 = 1$$

as indicated in Figure 6.12. So, by the Trapezoidal Rule

$$\int_0^1 e^x\, dx = \frac{1}{16}(e^0 + 2e^{0.125} + 2e^{0.25} + \cdot \cdot \cdot + 2e^{0.875} + e^1)$$

$$\approx 1.7205. \qquad \text{Approximation using } n = 8$$

Of course, for *this particular* integral, you could have found an antiderivative and used the Fundamental Theorem of Calculus to find the exact value of the definite integral. The exact value is

$$\int_0^1 e^x\, dx = e - 1 \approx 1.718282. \qquad \text{Exact value}$$

**TECHNOLOGY**

Ⓣ A graphing utility can also evaluate a definite integral that does not have an elementary function as an antiderivative. Use the integration capabilities of a graphing utility to approximate the integral $\int_0^1 e^{x^2}\, dx$.*

There are two important points that should be made concerning the Trapezoidal Rule. First, the approximation tends to become more accurate as $n$ increases. For instance, in Example 1, if $n = 16$, the Trapezoidal Rule yields an approximation of 1.7188. Second, although you could have used the Fundamental Theorem of Calculus to evaluate the integral in Example 1, this theorem cannot be used to evaluate an integral as simple as $\int_0^1 e^{x^2}\, dx$, because $e^{x^2}$ has no elementary function as an antiderivative. Yet the Trapezoidal Rule can be easily applied to this integral.

*Specific calculator keystroke instructions for operations in this and other technology boxes can be found at *college.hmco.com/info/larsonapplied*.
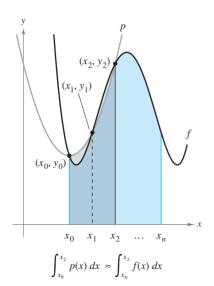
## Simpson's Rule

One way to view the Trapezoidal Rule is to say that on each subinterval, $f$ is approximated by a first-degree polynomial. In Simpson's Rule, $f$ is approximated by a second-degree polynomial on each subinterval.

To develop Simpson's Rule, partition the interval $[a, b]$ into an *even number* $n$ of subintervals, each of width

$$\Delta x = \frac{b - a}{n}.$$

On the subinterval $[x_0, x_2]$, approximate the function $f$ by the second-degree polynomial $p(x)$ that passes through the points

$$(x_0, f(x_0)), \quad (x_1, f(x_1)), \quad \text{and} \quad (x_2, f(x_2))$$

as shown in Figure 6.13. The Fundamental Theorem of Calculus can be used to show that



$$\int_{x_0}^{x_2} p(x)\, dx \approx \int_{x_0}^{x_2} f(x)\, dx$$

**FIGURE 6.13**

$$\int_{x_0}^{x_2} f(x)\, dx \approx \int_{x_0}^{x_2} p(x)\, dx$$

$$= \left( \frac{x_2 - x_0}{6} \right)\left[ p(x_0) + 4p\left( \frac{x_0 + x_2}{2} \right) + p(x_2) \right]$$

$$= \frac{2[(b - a)/n]}{6}\left[ p(x_0) + 4p(x_1) + p(x_2) \right]$$

$$= \left( \frac{b - a}{3n} \right)\left[ f(x_0) + 4f(x_1) + f(x_2) \right].$$

Repeating this process on the subintervals $[x_{i-2}, x_i]$ produces

$$\int_a^b f(x)\, dx \approx \left( \frac{b - a}{3n} \right)\left[ f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + \right.$$

$$\left. f(x_4) + \cdots + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right].$$

By grouping like terms, you can obtain the approximation shown below, which is known as Simpson's Rule. This rule is named after the English mathematician Thomas Simpson (1710–1761).

---

**Simpson's Rule (*n* Is Even)**

If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx \approx \left( \frac{b - a}{3n} \right)\left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \right.$$

$$\left. \cdots + 4f(x_{n-1}) + f(x_n) \right].$$

---

**STUDY TIP**

The coefficients in Simpson's Rule have the pattern

$$1 \quad 4 \quad 2 \quad 4 \quad 2 \quad 4 \ldots 4 \quad 2 \quad 4 \quad 1.$$

In Example 1, the Trapezoidal Rule was used to estimate the value of

$$\int_0^1 e^x \, dx.$$

The next example uses Simpson's Rule to approximate the same integral.

### Example 2 Using Simpson's Rule

Use Simpson's Rule to approximate

$$\int_0^1 e^x \, dx.$$

Compare the results for $n = 4$ and $n = 8$.

**SOLUTION** When $n = 4$, the width of each subinterval is $(1 - 0)/4 = \frac{1}{4}$ and the endpoints of the subintervals are

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad \text{and} \quad x_4 = 1$$

as indicated in Figure 6.14. So, by Simpson's Rule

$$\int_0^1 e^x dx = \frac{1}{12}(e^0 + 4e^{0.25} + 2e^{0.5} + 4e^{0.75} + e^1)$$

$$\approx 1.718319. \qquad \text{Approximation using } n = 4$$

When $n = 8$, the width of each subinterval is $(1 - 0)/8 = \frac{1}{8}$ and the endpoints of the subintervals are

$$x_0 = 0, \quad x_1 = \frac{1}{8}, \quad x_2 = \frac{1}{4}, \quad x_3 = \frac{3}{8}, \quad x_4 = \frac{1}{2}$$

$$x_5 = \frac{5}{8}, \quad x_6 = \frac{3}{4}, \quad x_7 = \frac{7}{8}, \quad \text{and} \quad x_8 = 1$$

as indicated in Figure 6.15. So, by Simpson's Rule

$$\int_0^1 e^x dx = \frac{1}{24}(e^0 + 4e^{0.125} + 2e^{0.25} + \cdots + 4e^{0.875} + e^1)$$

$$\approx 1.718284. \qquad \text{Approximation using } n = 8$$

Recall that the exact value of this integral is

$$\int_0^1 e^x dx = e - 1 \approx 1.718282. \qquad \text{Exact value}$$

So, with only eight subintervals, you obtained an approximation that is correct to the nearest 0.000002—an impressive result.

### ✓ CHECKPOINT 2

Use Simpson's Rule with $n = 4$ to approximate $\int_0^1 e^{2x} \, dx.$ ■
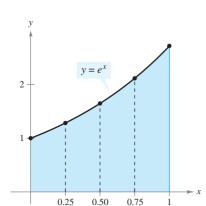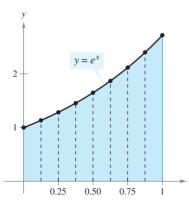


**FIGURE 6.14** Four Subintervals



**FIGURE 6.15** Eight Subintervals

**STUDY TIP**

Comparing the results of Examples 1 and 2, you can see that for a given value of $n$, Simpson's Rule tends to be more accurate than the Trapezoidal Rule.

## Error Analysis

In Examples 1 and 2, you were able to calculate the exact value of the integral and compare that value with the approximations to see how good they were. In practice, you need to have a different way of telling how good an approximation is: such a way is provided in the next result.

---

**Errors in the Trapezoidal Rule and Simpson's Rule**

The errors $E$ in approximating $\int_a^b f(x)\,dx$ are as shown.

*Trapezoidal Rule:* $|E| \leq \dfrac{(b-a)^3}{12n^2}\Big[\max|f''(x)|\Big], \quad a \leq x \leq b$

*Simpson's Rule:* $|E| \leq \dfrac{(b-a)^5}{180n^4}\Big[\max|f^{(4)}(x)|\Big], \quad a \leq x \leq b$

---

This result indicates that the errors generated by the Trapezoidal Rule and Simpson's Rule have upper bounds dependent on the extreme values of $f''(x)$ and $f^{(4)}(x)$ in the interval $[a, b]$. Furthermore, the bounds for the errors can be made arbitrarily small by *increasing n*. To determine what value of $n$ to choose, consider the steps below.

*Trapezoidal Rule*

**1.** Find $f''(x)$.

**2.** Find the maximum of $|f''(x)|$ on the interval $[a, b]$.

**3.** Set up the inequality

$$|E| \leq \frac{(b-a)^3}{12n^2}\Big[\max|f''(x)|\Big].$$

**4.** For an error less than $\epsilon$, solve for $n$ in the inequality

$$\frac{(b-a)^3}{12n^2}\Big[\max|f''(x)|\Big] < \epsilon.$$

**5.** Partition $[a, b]$ into $n$ subintervals and apply the Trapezoidal Rule.

*Simpson's Rule*

**1.** Find $f^{(4)}(x)$.

**2.** Find the maximum of $|f^{(4)}(x)|$ on the interval $[a, b]$.

**3.** Set up the inequality

$$|E| \leq \frac{(b-a)^5}{180n^4}\Big[\max|f^{(4)}(x)|\Big].$$

**4.** For an error less than $\epsilon$, solve for $n$ in the inequality

$$\frac{(b-a)^5}{180n^4}\Big[\max|f^{(4)}(x)|\Big] < \epsilon.$$

**5.** Partition $[a, b]$ into $n$ subintervals and apply Simpson's Rule.

*Example 3*   **Using the Trapezoidal Rule**

Use the Trapezoidal Rule to estimate the value of $\int_0^1 e^{-x^2}\,dx$ such that the approximation error is less than 0.01.

**SOLUTION**

1. Begin by finding the second derivative of $f(x) = e^{-x^2}$.

$$f(x) = e^{-x^2}$$
$$f'(x) = -2xe^{-x^2}$$
$$f''(x) = 4x^2 e^{-x^2} - 2e^{-x^2}$$
$$= 2e^{-x^2}(2x^2 - 1)$$

2. $f''$ has only one critical number in the interval $[0, 1]$, and the maximum value of $|f''(x)|$ on this interval is $|f''(0)| = 2$.

3. The error $E$ using the Trapezoidal Rule is bounded by

$$|E| \le \frac{(b-a)^3}{12n^2}(2) = \frac{1}{12n^2}(2) = \frac{1}{6n^2}.$$

4. To ensure that the approximation has an error of less than 0.01, you should choose $n$ such that

$$\frac{1}{6n^2} < 0.01.$$

Solving for $n$, you can determine that $n$ must be 5 or more.
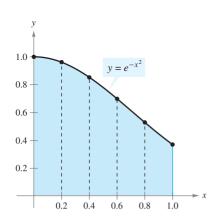
5. Partition $[0, 1]$ into five subintervals, as shown in Figure 6.16. Then apply the Trapezoidal Rule to obtain

$$\int_0^1 e^{-x^2}\,dx = \frac{1}{10}\left(\frac{1}{e^0} + \frac{2}{e^{0.04}} + \frac{2}{e^{0.16}} + \frac{2}{e^{0.36}} + \frac{2}{e^{0.64}} + \frac{1}{e^1}\right)$$
$$\approx 0.744.$$

So, with an error no larger than 0.01, you know that

$$0.734 \le \int_0^1 e^{-x^2}\,dx \le 0.754.$$

**FIGURE 6.16**

✓ **CHECKPOINT 3**

Use the Trapezoidal Rule to estimate the value of

$$\int_0^1 \sqrt{1 + x^2}\,dx$$

such that the approximation error is less than 0.01. ■

---

**CONCEPT CHECK**

1. For the Trapezoidal Rule, the number of subintervals $n$ can be odd or even. For Simpson's Rule, $n$ must be what?

2. As the number of subintervals $n$ increases, does an approximation given by the Trapezoidal Rule or Simpson's Rule tend to become less accurate or more accurate?

3. Write the formulas for (a) the Trapezoidal Rule and (b) Simpson's Rule.

4. The Trapezoidal Rule and Simpson's Rule yield approximations of a definite integral $\int_a^b f(x)\,dx$ based on polynomial approximations of $f$. What degree polynomial is used for each?

**Skills Review 6.4**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.1, 2.2, 2.6, 3.2, 4.3, and 4.5.

In Exercises 1–6, find the indicated derivative.

**1.** $f(x) = \dfrac{1}{x}$, $f''(x)$

**2.** $f(x) = \ln(2x + 1)$, $f^{(4)}(x)$

**3.** $f(x) = 2 \ln x$, $f^{(4)}(x)$

**4.** $f(x) = x^3 - 2x^2 + 7x - 12$, $f''(x)$

**5.** $f(x) = e^{2x}$, $f^{(4)}(x)$

**6.** $f(x) = e^{x^2}$, $f''(x)$

In Exercises 7 and 8, find the absolute maximum of $f$ on the interval.

**7.** $f(x) = -x^2 + 6x + 9$, $[0, 4]$

**8.** $f(x) = \dfrac{8}{x^3}$, $[1, 2]$

In Exercises 9 and 10, solve for $n$.

**9.** $\dfrac{1}{4n^2} < 0.001$
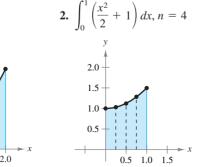
**10.** $\dfrac{1}{16n^4} < 0.0001$
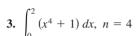
## Exercises 6.4
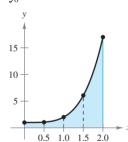
In Exercises 1–14, use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral for the indicated value of $n$. Compare these results with the exact value of the definite integral. Round your answers to four decimal places.

**1.** $\displaystyle\int_0^2 x^2\, dx$, $n = 4$

**2.** $\displaystyle\int_0^1 \left(\dfrac{x^2}{2} + 1\right) dx$, $n = 4$
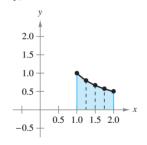


**3.** $\displaystyle\int_0^2 (x^4 + 1)\, dx$, $n = 4$

**4.** $\displaystyle\int_1^2 \dfrac{1}{x}\, dx$, $n = 4$



**5.** $\displaystyle\int_0^2 x^3\, dx$, $n = 8$

**6.** $\displaystyle\int_1^3 (4 - x^2)\, dx$, $n = 4$

**7.** $\displaystyle\int_1^2 \dfrac{1}{x}\, dx$, $n = 8$

**8.** $\displaystyle\int_1^2 \dfrac{1}{x^2}\, dx$, $n = 4$

**9.** $\displaystyle\int_0^4 \sqrt{x}\, dx$, $n = 8$

**10.** $\displaystyle\int_0^2 \sqrt{1 + x}\, dx$, $n = 4$

**11.** $\displaystyle\int_4^9 \sqrt{x}\, dx$, $n = 8$

**12.** $\displaystyle\int_0^8 \sqrt[3]{x}\, dx$, $n = 8$

**13.** $\displaystyle\int_0^1 \dfrac{1}{1 + x}\, dx$, $n = 4$

**14.** $\displaystyle\int_0^2 x\sqrt{x^2 + 1}\, dx$, $n = 4$

In Exercises 15–24, approximate the integral using (a) the Trapezoidal Rule and (b) Simpson's Rule for the indicated value of $n$. (Round your answers to three significant digits.)

**15.** $\displaystyle\int_0^1 \dfrac{1}{1 + x^2}\, dx$, $n = 4$

**16.** $\displaystyle\int_0^2 \dfrac{1}{\sqrt{1 + x^3}}\, dx$, $n = 4$

**17.** $\displaystyle\int_0^2 \sqrt{1 + x^3}\, dx$, $n = 4$

**18.** $\displaystyle\int_0^1 \sqrt{1 - x}\, dx$, $n = 4$

**19.** $\displaystyle\int_0^1 \sqrt{1 - x^2}\, dx$, $n = 4$

**20.** $\displaystyle\int_0^1 \sqrt{1 - x^2}\, dx$, $n = 8$

**21.** $\displaystyle\int_0^2 e^{-x^2}\, dx$, $n = 2$

**22.** $\displaystyle\int_0^2 e^{-x^2}\, dx$, $n = 4$

**23.** $\displaystyle\int_0^3 \dfrac{1}{2 - 2x + x^2}\, dx$, $n = 6$

**24.** $\displaystyle\int_0^3 \dfrac{x}{2 + x + x^2}\, dx$, $n = 6$

**Present Value** In Exercises 25 and 26, use a program similar to the Simpson's Rule program on page 454 with $n = 8$ to approximate the present value of the income $c(t)$ over $t_1$ years at the given annual interest rate $r$. Then use the integration capabilities of a graphing utility to approximate the present value. Compare the results. (Present value is defined in Section 6.1.)

**25.** $c(t) = 6000 + 200\sqrt{t}$, $r = 7\%$, $t_1 = 4$

**26.** $c(t) = 200{,}000 + 15{,}000\sqrt[3]{t}$, $r = 10\%$, $t_1 = 8$

**Marginal Analysis** In Exercises 27 and 28, use a program similar to the Simpson's Rule program on page 454 with $n = 4$ to approximate the change in revenue from the marginal revenue function $dR/dx$. In each case, assume that the number of units sold $x$ increases from 14 to 16.

**27.** $\dfrac{dR}{dx} = 5\sqrt{8000 - x^3}$     **28.** $\dfrac{dR}{dx} = 50\sqrt{x}\sqrt{20 - x}$

**Probability** In Exercises 29–32, use a program similar to the Simpson's Rule program on page 454 with $n = 6$ to approximate the indicated normal probability. The standard normal probability density function is $f(x) = \left(1/\sqrt{2\pi}\right)e^{-x^2/2}$. If $x$ is chosen at random from a population with this density, then the probability that $x$ lies in the interval $[a, b]$ is $P(a \le x \le b) = \int_a^b f(x)\,dx$.
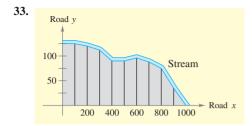
**29.** $P(0 \le x \le 1)$     **30.** $P(0 \le x \le 2)$

**31.** $P(0 \le x \le 4)$     **32.** $P(0 \le x \le 1.5)$

**Surveying** In Exercises 33 and 34, use a program similar to the Simpson's Rule program on page 454 to estimate the number of square feet of land in the lot, where $x$ and $y$ are measured in feet, as shown in the figures. In each case, the land is bounded by a stream and two straight roads.

**33.**



| x | 0 | 100 | 200 | 300 | 400 | 500 |
|---|---|-----|-----|-----|-----|-----|
| y | 125 | 125 | 120 | 112 | 90 | 90 |

| x | 600 | 700 | 800 | 900 | 1000 |
|---|-----|-----|-----|-----|------|
| y | 95 | 88 | 75 | 35 | 0 |

**34.**



| x | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|----|----|----|----|----|----|
| y | 75 | 81 | 84 | 76 | 67 | 68 | 69 |

| x | 70 | 80 | 90 | 100 | 110 | 120 |
|---|----|----|----|-----|-----|-----|
| y | 72 | 68 | 56 | 42 | 23 | 0 |

In Exercises 35–38, use the error formulas to find bounds for the error in approximating the integral using (a) the Trapezoidal Rule and (b) Simpson's Rule. (Let $n = 4$.)

**35.** $\displaystyle\int_0^2 x^3\,dx$     **36.** $\displaystyle\int_0^1 \frac{1}{x+1}\,dx$

**37.** $\displaystyle\int_0^1 e^{x^3}\,dx$     **38.** $\displaystyle\int_0^1 e^{-x^2}\,dx$

In Exercises 39–42, use the error formulas to find $n$ such that the error in the approximation of the definite integral is less than 0.0001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.

**39.** $\displaystyle\int_0^1 x^3\,dx$     **40.** $\displaystyle\int_1^3 \frac{1}{x}\,dx$

**41.** $\displaystyle\int_1^3 e^{2x}\,dx$     **42.** $\displaystyle\int_3^5 \ln x\,dx$

In Exercises 43–46, use a program similar to the Simpson's Rule program on page 454 to approximate the integral. Use $n = 100$.

**43.** $\displaystyle\int_1^4 x\sqrt{x+4}\,dx$     **44.** $\displaystyle\int_1^4 x^2\sqrt{x+4}\,dx$

**45.** $\displaystyle\int_2^5 10xe^{-x}\,dx$     **46.** $\displaystyle\int_2^5 10x^2 e^{-x}\,dx$

In Exercises 47 and 48, use a program similar to the Simpson's Rule program on page 454 with $n = 4$ to find the area of the region bounded by the graphs of the equations.
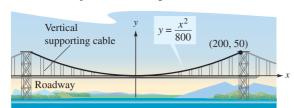
**47.** $y = x\sqrt[3]{x+4}$, $y = 0$, $x = 1$, $x = 5$

**48.** $y = \sqrt{2 + 3x^2}$, $y = 0$, $x = 1$, $x = 3$

In Exercises 49 and 50, use the definite integral below to find the required arc length. If $f$ has a continuous derivative, then the arc length of $f$ between the points $(a, f(a))$ and $(b, f(b))$ is

$$\int_b^a \sqrt{1 + [f'(x)]^2}\, dx.$$
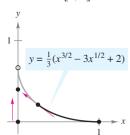
(T) **49. Arc Length** The suspension cable on a bridge that is 400 feet long is in the shape of a parabola whose equation is $y = x^2/800$ (see figure). Use a program similar to the Simpson's Rule program on page 454 with $n = 12$ to approximate the length of the cable. Compare this result with the length obtained by using the table of integrals in Section 6.3 to perform the integration.



**50. Arc Length** A fleeing hare leaves its burrow $(0, 0)$ and moves due north (up the $y$-axis). At the same time, a pursuing lynx leaves from 1 yard east of the burrow $(1, 0)$ and always moves toward the fleeing hare (see figure). If the lynx's speed is twice that of the hare's, the equation of the lynx's path is

$$y = \frac{1}{3}(x^{3/2} - 3x^{1/2} + 2).$$

Find the distance traveled by the lynx by integrating over the interval $[0, 1]$.



**51. Lumber Use** The table shows the amounts of lumber used for residential upkeep and improvements (in billions of board-feet per year) for the years 1997 through 2005. *(Source: U.S. Forest Service)*

| Year | 1997 | 1998 | 1999 | 2000 | 2001 |
|---|---|---|---|---|---|
| Amount | 15.1 | 14.7 | 15.1 | 16.4 | 17.0 |

| Year | 2002 | 2003 | 2004 | 2005 |
|---|---|---|---|---|
| Amount | 17.8 | 18.3 | 20.0 | 20.6 |

(a) Use Simpson's Rule to estimate the average number of board-feet (in billions) used per year over the time period.

(b) A model for the data is

$$L = 6.613 + 0.93t + 2095.7e^{-t}, \quad 7 \le t \le 15$$

where $L$ is the amount of lumber used and $t$ is the year, with $t = 7$ corresponding to 1997. Use integration to find the average number of board-feet (in billions) used per year over the time period.

(c) Compare the results of parts (a) and (b).

**52. Median Age** The table shows the median ages of the U.S. resident population for the years 1997 through 2005. *(Source: U.S. Census Bureau)*
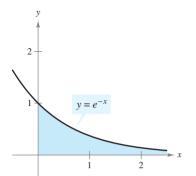
| Year | 1997 | 1998 | 1999 | 2000 | 2001 |
|---|---|---|---|---|---|
| Median age | 34.7 | 34.9 | 35.2 | 35.3 | 35.6 |

| Year | 2002 | 2003 | 2004 | 2005 |
|---|---|---|---|---|
| Median age | 35.7 | 35.9 | 36.0 | 36.2 |

(a) Use Simpson's Rule to estimate the average age over the time period.

(b) A model for the data is $A = 31.5 + 1.21\sqrt{t}$, $7 \le t \le 15$, where $A$ is the median age and $t$ is the year, with $t = 7$ corresponding to 1997. Use integration to find the average age over the time period.

(c) Compare the results of parts (a) and (b).

**53. Medicine** A body assimilates a 12-hour cold tablet at a rate modeled by $dC/dt = 8 - \ln(t^2 - 2t + 4)$, $0 \le t \le 12$, where $dC/dt$ is measured in milligrams per hour and $t$ is the time in hours. Use Simpson's Rule with $n = 8$ to estimate the total amount of the drug absorbed into the body during the 12 hours.

**54. Medicine** The concentration $M$ (in grams per liter) of a six-hour allergy medicine in a body is modeled by $M = 12 - 4\ln(t^2 - 4t + 6)$, $0 \le t \le 6$, where $t$ is the time in hours since the allergy medication was taken. Use Simpson's Rule with $n = 6$ to estimate the average level of concentration in the body over the six-hour period.

**55. Consumer Trends** The rate of change $S$ in the number of subscribers to a newly introduced magazine is modeled by $dS/dt = 1000t^2e^{-t}, 0 \le t \le 6$, where $t$ is the time in years. Use Simpson's Rule with $n = 12$ to estimate the total increase in the number of subscribers during the first 6 years.

**56.** Prove that Simpson's Rule is exact when used to approximate the integral of a cubic polynomial function, and demonstrate the result for $\int_0^1 x^3\, dx$, $n = 2$.

## Section 6.5

## Improper Integrals

- Recognize improper integrals.
- Evaluate improper integrals with infinite limits of integration.
- Evaluate improper integrals with infinite integrands.
- Use improper integrals to solve real-life problems.
- Find the present value of a perpetuity.
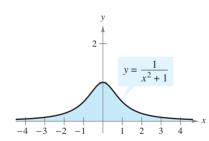
### Improper Integrals

The definition of the definite integral

$$\int_a^b f(x)\,dx$$

includes the requirements that the interval $[a, b]$ be finite and that $f$ be continuous on $[a, b]$. In this section, you will study integrals that do not satisfy these requirements because of one of the conditions below.

**1.** One or both of the limits of integration are infinite.

**2.** $f$ has an infinite discontinuity in the interval $[a, b]$.

Integrals having either of these characteristics are called **improper integrals.** For instance, the integrals

$$\int_0^\infty e^{-x}\,dx \quad \text{and} \quad \int_{-\infty}^\infty \frac{1}{x^2 + 1}\,dx$$
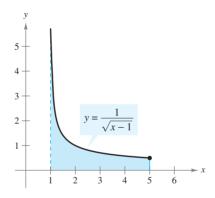
are improper because one or both limits of integration are infinite, as indicated in Figure 6.17. Similarly, the integrals

$$\int_1^5 \frac{1}{\sqrt{x - 1}}\,dx \quad \text{and} \quad \int_{-2}^2 \frac{1}{(x + 1)^2}\,dx$$

are improper because their integrands have an **infinite discontinuity**—that is, they approach infinity somewhere in the interval of integration, as indicated in Figure 6.18.

$y = e^{-x}$

$y = \dfrac{1}{x^2 + 1}$

**FIGURE 6.17**

$y = \dfrac{1}{\sqrt{x - 1}}$

$y = \dfrac{1}{(x + 1)^2}$

**FIGURE 6.18**
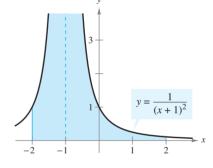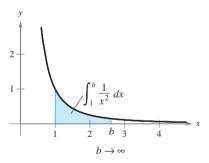
## Integrals with Infinite Limits of Integration

To see how to evaluate an improper integral, consider the integral shown in Figure 6.19. As long as $b$ is a real number that is greater than 1 (no matter how large), this is a definite integral whose value is

$$\int_1^b \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^b$$

$$= -\frac{1}{b} + 1$$

$$= 1 - \frac{1}{b}.$$

The curve shows $y$ versus $x$ with the region $\int_1^b \frac{1}{x^2}\,dx$ shaded, and $b \to \infty$.

**FIGURE 6.19**

The table shows the values of this integral for several values of $b$.

| $b$ | 2 | 5 | 10 | 100 | 1000 | 10,000 |
|---|---|---|---|---|---|---|
| $\int_1^b \frac{1}{x^2}\,dx = 1 - \frac{1}{b}$ | 0.5000 | 0.8000 | 0.9000 | 0.9900 | 0.9990 | 0.9999 |

From this table, it appears that the value of the integral is approaching a limit as $b$ increases without bound. This limit is denoted by the *improper integral* shown below.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left(1 - \frac{1}{b}\right)$$

$$= 1$$

---

**Improper Integrals (Infinite Limits of Integration)**

**1.** If $f$ is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

**2.** If $f$ is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

**3.** If $f$ is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where $c$ is any real number.

In the first two cases, if the limit exists, then the improper integral **converges;** otherwise, the improper integral **diverges.** In the third case, the integral on the left will diverge if either one of the integrals on the right diverges.

*Example 1* **Evaluating an Improper Integral**

Determine the convergence or divergence of $\int_1^\infty \frac{1}{x}\,dx$.

**SOLUTION** Begin by applying the definition of an improper integral.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \qquad \text{Definition of improper integral}$$

$$= \lim_{b\to\infty} \left[\ln x\right]_1^b \qquad \text{Find antiderivative.}$$

$$= \lim_{b\to\infty} (\ln b - 0) \qquad \text{Apply Fundamental Theorem.}$$

$$= \infty \qquad \text{Evaluate limit.}$$

Because the limit is infinite, the improper integral diverges.

**✓ CHECKPOINT 1**

Determine the convergence or divergence of each improper integral.

**a.** $\int_1^\infty \frac{1}{x^3}\,dx$    **b.** $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ ■
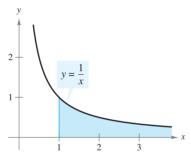
As you begin to work with improper integrals, you will find that integrals that appear to be similar can have very different values. For instance, consider the two improper integrals

$$\int_1^\infty \frac{1}{x}\,dx = \infty \qquad \text{Divergent integral}$$
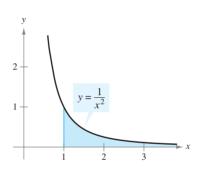
and

$$\int_1^\infty \frac{1}{x^2}\,dx = 1. \qquad \text{Convergent integral}$$

The first integral diverges and the second converges to 1. Graphically, this means that the areas shown in Figure 6.20 are very different. The region lying between the graph of $y = 1/x$ and the $x$-axis (for $x \geq 1$) has an *infinite* area, and the region lying between the graph of $y = 1/x^2$ and the $x$-axis (for $x \geq 1$) has a *finite* area.



Diverges (infinite area)          Converges (finite area)

**FIGURE 6.20**

**Example 2**    **Evaluating an Improper Integral**

Evaluate the improper integral.

$$\int_{-\infty}^{0} \frac{1}{(1 - 2x)^{3/2}} \, dx$$

**SOLUTION**    Begin by applying the definition of an improper integral.

$$\int_{-\infty}^{0} \frac{1}{(1 - 2x)^{3/2}} \, dx = \lim_{a \to -\infty} \int_{a}^{0} \frac{1}{(1 - 2x)^{3/2}} \, dx \quad \text{Definition of improper integral}$$

$$= \lim_{a \to -\infty} \left[ \frac{1}{\sqrt{1 - 2x}} \right]_{a}^{0} \quad \text{Find antiderivative.}$$

$$= \lim_{a \to -\infty} \left( 1 - \frac{1}{\sqrt{1 - 2a}} \right) \quad \text{Apply Fundamental Theorem.}$$

$$= 1 - 0 \quad \text{Evaluate limit.}$$

$$= 1 \quad \text{Simplify.}$$

So, the improper integral converges to 1. As shown in Figure 6.21, this implies that the region lying between the graph of $y = 1/(1 - 2x)^{3/2}$ and the $x$-axis (for $x \leq 0$) has an area of 1 square unit.

$y = \dfrac{1}{(1 - 2x)^{3/2}}$
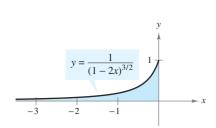
**FIGURE 6.21**

✓ **CHECKPOINT 2**

Evaluate the improper integral, if possible.

$$\int_{-\infty}^{0} \frac{1}{(x - 1)^2} \, dx$$
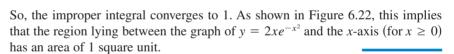
**Example 3**    **Evaluating an Improper Integral**

Evaluate the improper integral.

$$\int_{0}^{\infty} 2xe^{-x^2} \, dx$$

**SOLUTION**    Begin by applying the definition of an improper integral.

$$\int_{0}^{\infty} 2xe^{-x^2} \, dx = \lim_{b \to \infty} \int_{0}^{b} 2xe^{-x^2} \, dx \quad \text{Definition of improper integral}$$

$$= \lim_{b \to \infty} \left[ -e^{-x^2} \right]_{0}^{b} \quad \text{Find antiderivative.}$$

$$= \lim_{b \to \infty} \left( -e^{-b^2} + 1 \right) \quad \text{Apply Fundamental Theorem.}$$

$$= 0 + 1 \quad \text{Evaluate limit.}$$

$$= 1 \quad \text{Simplify.}$$

So, the improper integral converges to 1. As shown in Figure 6.22, this implies that the region lying between the graph of $y = 2xe^{-x^2}$ and the $x$-axis (for $x \geq 0$) has an area of 1 square unit.

$y = 2xe^{-x^2}$
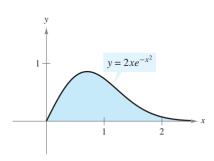
**FIGURE 6.22**

✓ **CHECKPOINT 3**

Evaluate the improper integral, if possible.

$$\int_{-\infty}^{0} e^{2x} \, dx$$

## Integrals with Infinite Integrands

> **Improper Integrals (Infinite Integrands)**
>
> 1. If $f$ is continuous on the interval $[a, b)$ and approaches infinity at $b$, then
> $$\int_a^b f(x)\ dx = \lim_{c \to b^-} \int_a^c f(x)\ dx.$$
> 2. If $f$ is continuous on the interval $(a, b]$ and approaches infinity at $a$, then
> $$\int_a^b f(x)\ dx = \lim_{c \to a^+} \int_c^b f(x)\ dx.$$
> 3. If $f$ is continuous on the interval $[a, b]$, except for some $c$ in $(a, b)$ at which $f$ approaches infinity, then
> $$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$
>
> In the first two cases, if the limit exists, then the improper integral **converges;** otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

$y = \dfrac{1}{\sqrt[3]{x - 1}}$

**FIGURE 6.23**

**Example 4** Evaluating an Improper Integral

Evaluate $\displaystyle \int_1^2 \frac{1}{\sqrt[3]{x - 1}}\ dx$.

**SOLUTION**

$$\int_1^2 \frac{1}{\sqrt[3]{x - 1}}\ dx = \lim_{c \to 1^+} \int_c^2 \frac{1}{\sqrt[3]{x - 1}}\ dx \qquad \text{Definition of improper integral}$$

$$= \lim_{c \to 1^+} \left[ \frac{3}{2}(x - 1)^{2/3} \right]_c^2 \qquad \text{Find antiderivative.}$$

$$= \lim_{c \to 1^+} \left[ \frac{3}{2} - \frac{3}{2}(c - 1)^{2/3} \right] \qquad \text{Apply Fundamental Theorem.}$$

$$= \frac{3}{2} - 0 \qquad \text{Evaluate limit.}$$

$$= \frac{3}{2} \qquad \text{Simplify.}$$

So, the integral converges to $\frac{3}{2}$. This implies that the region shown in Figure 6.23 has an area of $\frac{3}{2}$ square units.

✓**CHECKPOINT 4**

Evaluate $\displaystyle \int_1^2 \frac{1}{\sqrt{x - 1}}\ dx$. ◼
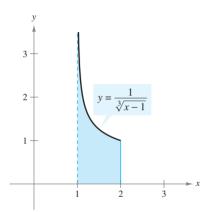
**TECHNOLOGY**

Ⓣ Use a graphing utility to verify the result of Example 4 by calculating each definite integral.

$$\int_{1.01}^2 \frac{1}{\sqrt[3]{x - 1}}\ dx$$

$$\int_{1.001}^2 \frac{1}{\sqrt[3]{x - 1}}\ dx$$

$$\int_{1.0001}^2 \frac{1}{\sqrt[3]{x - 1}}\ dx$$

**FIGURE 6.24**

*Example 5* **Evaluating an Improper Integral**

Evaluate $\int_1^2 \dfrac{2}{x^2 - 2x}\, dx$.

**SOLUTION**

$$\int_1^2 \frac{2}{x^2 - 2x}\, dx = \int_1^2 \left( \frac{1}{x - 2} - \frac{1}{x} \right) dx \qquad \text{Use partial fractions.}$$

$$= \lim_{c \to 2^-} \int_1^c \left( \frac{1}{x - 2} - \frac{1}{x} \right) dx \qquad \text{Definition of improper integral}$$

$$= \lim_{c \to 2^-} \left[ \ln|x - 2| - \ln|x| \right]_1^c \qquad \text{Find antiderivative.}$$

$$= -\infty \qquad \text{Evaluate limit.}$$

So, you can conclude that the integral diverges. This implies that the region shown in Figure 6.24 has an infinite area.

✓ **CHECKPOINT 5**

Evaluate $\int_1^3 \dfrac{3}{x^2 - 3x}\, dx$. ■

*Example 6* **Evaluating an Improper Integral**

Evaluate $\int_{-1}^2 \dfrac{1}{x^3}\, dx$.

**SOLUTION** This integral is improper because the integrand has an infinite discontinuity at the interior value $x = 0$, as shown in Figure 6.25. So, you can write

$$\int_{-1}^2 \frac{1}{x^3}\, dx = \int_{-1}^0 \frac{1}{x^3}\, dx + \int_0^2 \frac{1}{x^3}\, dx.$$

By applying the definition of an improper integral, you can show that each of these integrals diverges. So, the original improper integral also diverges.



**FIGURE 6.25**

✓ **CHECKPOINT 6**

Evaluate $\int_{-1}^1 \dfrac{1}{x^2}\, dx$. ■

**STUDY TIP**

Had you not recognized that the integral in Example 6 was improper, you would have obtained the incorrect result

$$\int_{-1}^2 \frac{1}{x^3}\, dx = \left[ -\frac{1}{2x^2} \right]_{-1}^2 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}. \qquad \text{Incorrect}$$

Improper integrals in which the integrand has an infinite discontinuity *between* the limits of integration are often overlooked, so keep alert for such possibilities. Even symbolic integrators can have trouble with this type of integral, and can give the same incorrect result.

## Application

In Section 4.3, you studied the graph of the *normal probability density function*
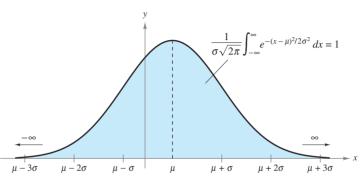
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}.$$

This function is used in statistics to represent a population that is normally distributed with a mean of $\mu$ and a standard deviation of $\sigma$. Specifically, if an outcome $x$ is chosen at random from the population, the probability that $x$ will have a value between $a$ and $b$ is

$$P(a \le x \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}\,dx.$$

As shown in Figure 6.26, the probability $P(-\infty < x < \infty)$ is

$$P(-\infty < x < \infty) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}\,dx = 1.$$



**FIGURE 6.26**

*Example 7* **Finding a Probability** ®

The mean height of American men (from 20 to 29 years old) is 70 inches, and the standard deviation is 3 inches. A 20- to 29-year-old man is chosen at random from the population. What is the probability that he is 6 feet tall or taller? *(Source: U.S. National Center for Health Statistics)*

**SOLUTION**  Using a mean of $\mu = 70$ and a standard deviation of $\sigma = 3$, the probability $P(72 \le x < \infty)$ is given by the improper integral

$$P(72 \le x < \infty) = \int_{72}^{\infty} \frac{1}{3\sqrt{2\pi}}e^{-(x-70)^2/18}\,dx.$$

Using a symbolic integration utility, you can approximate the value of this integral to be 0.252. So, the probability that the man is 6 feet tall or taller is about 25.2%.

Victor Baldizon/NBAE via Getty Images

Many professional basketball players are over $6\frac{1}{2}$ feet tall. If a man is chosen at random from the population, the probability that he is $6\frac{1}{2}$ feet tall or taller is less than half of one percent.

✓**CHECKPOINT 7**

Use Example 7 to find the probability that a 20- to 29-year-old man chosen at random from the population is 6 feet 6 inches tall or taller. ◼

## Present Value of a Perpetuity

Recall from Section 6.1 that for an interest-bearing account, the present value over $t_1$ years is

$$\text{Present value} = \int_0^{t_1} c(t)e^{-rt}\, dt$$

where $c$ represents a continuous income function (in dollars per year) and the annual interest rate $r$ is compounded continuously. If the size of an annuity's payment is a constant number of dollars $P$, then $c(t)$ is equal to $P$ and the present value is

$$\text{Present value} = \int_0^{t_1} Pe^{-rt}\, dt = P \int_0^{t_1} e^{-rt}\, dt. \qquad \text{Present value of an annuity with payment } P$$

Suppose you wanted to start an annuity, such as a scholarship fund, that pays the same amount each year *forever*? Because the annuity continues indefinitely, the number of years $t_1$ approaches infinity. Such an annuity is called a **perpetual annuity** or a **perpetuity.** This situation can be represented by the following improper integral.

$$\text{Present value} = P \int_0^{\infty} e^{-rt}\, dt \qquad \text{Present value of a perpetuity with payment } P$$

This integral is simplified as follows.

$$P \int_0^{\infty} e^{-rt}\, dt = P \lim_{b \to \infty} \int_0^{b} e^{-rt}\, dt \qquad \text{Definition of improper integral}$$

$$= P \lim_{b \to \infty} \left[ -\frac{e^{-rt}}{r} \right]_0^b \qquad \text{Find antiderivative.}$$

$$= P \lim_{b \to \infty} \left( -\frac{e^{-rb}}{r} + \frac{1}{r} \right) \qquad \text{Apply Fundamental Theorem.}$$

$$= P \left( 0 + \frac{1}{r} \right) \qquad \text{Evaluate limit.}$$

$$= \frac{P}{r} \qquad \text{Simplify.}$$

So, the improper integral converges to $P/r$. As shown in Figure 6.27, this implies that the region lying between the graph of $y = Pe^{-rt}$ and the $t$-axis for $t \geq 0$ has an area equal to the annual payment $P$ divided by the annual interest rate $r$.
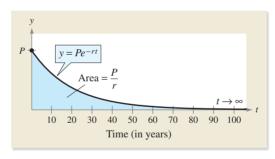


**FIGURE 6.27**

The present value of a perpetuity is defined as follows.

---

**Present Value of a Perpetuity**

If $P$ represents the size of each annual payment in dollars and the annual interest rate is $r$ (compounded continuously), then the present value of a perpetuity is

$$\text{Present value} = P \int_0^\infty e^{-rt}\, dt = \frac{P}{r}.$$

---

This definition is useful in determining the amount of money needed to start an endowment, such as a scholarship fund, as shown in Example 8.

*Example 8*
MAKE A DECISION   **Finding Present Value**   (R)

You want to start a scholarship fund at your alma mater. You plan to give one $9000 scholarship annually beginning one year from now, and you have at most $120,000 to start the fund. You also want the scholarship to be given out indefinitely. Assuming an annual interest rate of 8% (compounded continuously), do you have enough money for the scholarship fund?

**SOLUTION**   To answer this question, you must find the present value of the scholarship fund. Because the scholarship is to be given out each year indefinitely, the time period is infinite. The fund is a perpetuity with $P = 9000$ and $r = 0.08$. The present value is

$$\text{Present value} = \frac{P}{r}$$

$$= \frac{9000}{0.08}$$

$$= 112,500.$$

The amount you need to start the scholarship fund is $112,500. Yes, you have enough money to start the scholarship fund.

✓ **CHECKPOINT 8**

In Example 8, do you have enough money to start a scholarship fund that pays $10,000 annually? Explain why or why not. ■

---

**CONCEPT CHECK**

1. Integrals are improper integrals if they have either of what two characteristics?

2. Describe the different types of improper integrals.

3. Define the term *converges* when working with improper integrals.

4. Define the term *diverges* when working with improper integrals.

**Skills Review 6.5**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.5, 4.1, and 4.4.

In Exercises 1–6, find the limit.

**1.** $\lim\limits_{x \to 2} (2x + 5)$

**2.** $\lim\limits_{x \to 1} \left( \dfrac{1}{x} + 2x^2 \right)$

**3.** $\lim\limits_{x \to -4} \dfrac{x + 4}{x^2 - 16}$

**4.** $\lim\limits_{x \to 0} \dfrac{x^2 - 2x}{x^3 + 3x^2}$

**5.** $\lim\limits_{x \to 1} \dfrac{1}{\sqrt{x - 1}}$

**6.** $\lim\limits_{x \to -3} \dfrac{x^2 + 2x - 3}{x + 3}$

In Exercises 7–10, evaluate the expression (a) when $x = b$ and (b) when $x = 0$.

**7.** $\dfrac{4}{3}(2x - 1)^3$

**8.** $\dfrac{1}{x - 5} + \dfrac{3}{(x - 2)^2}$

**9.** $\ln(5 - 3x^2) - \ln(x + 1)$

**10.** $e^{3x^2} + e^{-3x^2}$

## Exercises 6.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, decide whether the integral is improper. Explain your reasoning.

**1.** $\displaystyle\int_0^1 \dfrac{dx}{3x - 2}$
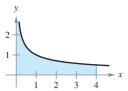
**2.** $\displaystyle\int_1^3 \dfrac{dx}{x^2}$

**3.** $\displaystyle\int_0^1 \dfrac{2x - 5}{x^2 - 5x + 6}\, dx$

**4.** $\displaystyle\int_1^\infty x^2\, dx$
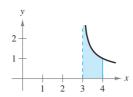
In Exercises 5–10, explain why the integral is improper and determine whether it diverges or converges. Evaluate the integral if it converges.

**5.** $\displaystyle\int_0^4 \dfrac{1}{\sqrt{x}}\, dx$
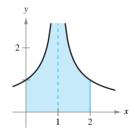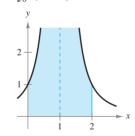
**6.** $\displaystyle\int_3^4 \dfrac{1}{\sqrt{x - 3}}\, dx$
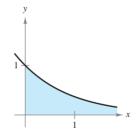
**7.** $\displaystyle\int_0^2 \dfrac{1}{(x - 1)^{2/3}}\, dx$
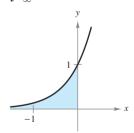
**8.** $\displaystyle\int_0^2 \dfrac{1}{(x - 1)^2}\, dx$

**9.** $\displaystyle\int_0^\infty e^{-x}\, dx$

**10.** $\displaystyle\int_{-\infty}^0 e^{2x}\, dx$

In Exercises 11–22, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

**11.** $\displaystyle\int_1^\infty \dfrac{1}{x^2}\, dx$

**12.** $\displaystyle\int_1^\infty \dfrac{1}{\sqrt[3]{x}}\, dx$

**13.** $\displaystyle\int_0^\infty e^{x/3}\, dx$

**14.** $\displaystyle\int_0^\infty \dfrac{5}{e^{2x}}\, dx$

**15.** $\displaystyle\int_5^\infty \dfrac{x}{\sqrt{x^2 - 16}}\, dx$

**16.** $\displaystyle\int_{1/2}^\infty \dfrac{1}{\sqrt{2x - 1}}\, dx$

**17.** $\displaystyle\int_{-\infty}^0 e^{-x}\, dx$

**18.** $\displaystyle\int_{-\infty}^{-1} \dfrac{1}{x^2}\, dx$

**19.** $\displaystyle\int_1^\infty \dfrac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

**20.** $\displaystyle\int_{-\infty}^0 \dfrac{x}{x^2 + 1}\, dx$

**21.** $\displaystyle\int_{-\infty}^\infty 2xe^{-3x^2}\, dx$

**22.** $\displaystyle\int_{-\infty}^\infty x^2 e^{-x^3}\, dx$

In Exercises 23–32, determine whether the improper integral diverges or converges. Evaluate the integral if it converges, and check your results with the results obtained by using the integration capabilities of a graphing utility.

**23.** $\displaystyle\int_0^1 \frac{1}{1-x}\,dx$

**24.** $\displaystyle\int_0^{27} \frac{5}{\sqrt[3]{x}}\,dx$

**25.** $\displaystyle\int_0^9 \frac{1}{\sqrt{9-x}}\,dx$

**26.** $\displaystyle\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx$

**27.** $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

**28.** $\displaystyle\int_0^1 \frac{1}{x}\,dx$

**29.** $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$

**30.** $\displaystyle\int_0^2 \frac{1}{(x-1)^{4/3}}\,dx$

**31.** $\displaystyle\int_3^4 \frac{1}{\sqrt{x^2-9}}\,dx$

**32.** $\displaystyle\int_3^5 \frac{1}{x^2\sqrt{x^2-9}}\,dx$

In Exercises 33 and 34, consider the region satisfying the inequalities. Find the area of the region.

**33.** $y \le \dfrac{1}{x^2}, y \ge 0, x \ge 1$

**34.** $y \le e^{-x}, y \ge 0, x \ge 0$

Ⓢ In Exercises 35–38, use a spreadsheet to complete the table for the specified values of $a$ and $n$ to demonstrate that

$$\lim_{x\to\infty} x^n e^{-ax} = 0, \quad a > 0, n > 0.$$

| $x$ | 1 | 10 | 25 | 50 |
|-----|---|----|----|----|
| $x^n e^{-ax}$ | | | | |

**35.** $a = 1, n = 1$

**36.** $a = 2, n = 4$

**37.** $a = \frac{1}{2}, n = 2$

**38.** $a = \frac{1}{2}, n = 5$

In Exercises 39–42, use the results of Exercises 35–38 to evaluate the improper integral.

**39.** $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$

**40.** $\displaystyle\int_0^\infty (x-1)e^{-x}\,dx$

**41.** $\displaystyle\int_0^\infty xe^{-2x}\,dx$

**42.** $\displaystyle\int_0^\infty xe^{-x}\,dx$

**43. Women's Height**   The mean height of American women between the ages of 30 and 39 is 64.5 inches, and the standard deviation is 2.7 inches. Find the probability that a 30- to 39-year-old woman chosen at random is

(a)  between 5 and 6 feet tall.

(b)  5 feet 8 inches or taller.

(c)  6 feet or taller.

*(Source: U.S. National Center for Health Statistics)*

**44. Quality Control**   A company manufactures wooden yardsticks. The lengths of the yardsticks are normally distributed with a mean of 36 inches and a standard deviation of 0.2 inch. Find the probability that a yardstick is

(a)  longer than 35.5 inches.    (b)  longer than 35.9 inches.

**Endowment**   In Exercises 45 and 46, determine the amount of money required to set up a charitable endowment that pays the amount $P$ each year indefinitely for the annual interest rate $r$ compounded continuously.

**45.** $P = \$5000, r = 7.5\%$

**46.** $P = \$12,000, r = 6\%$

**47. *MAKE A DECISION: SCHOLARSHIP FUND***   You want to start a scholarship fund at your alma mater. You plan to give one $18,000 scholarship annually beginning one year from now and you have at most $400,000 to start the fund. You also want the scholarship to be given out indefinitely. Assuming an annual interest rate of 5% compounded continuously, do you have enough money for the scholarship fund?

**48. *MAKE A DECISION: CHARITABLE FOUNDATION***   A charitable foundation wants to help schools buy computers. The foundation plans to donate $35,000 each year to one school beginning one year from now, and the foundation has at most $500,000 to start the fund. The foundation wants the donation to be given out indefinitely. Assuming an annual interest rate of 8% compounded continuously, does the foundation have enough money to fund the donation?

**49. Present Value**   A business is expected to yield a continuous flow of profit at the rate of $500,000 per year. If money will earn interest at the nominal rate of 9% per year compounded continuously, what is the present value of the business (a) for 20 years and (b) forever?

**50. Present Value**   Repeat Exercise 49 for a farm that is expected to produce a profit of $75,000 per year. Assume that money will earn interest at the nominal rate of 8% compounded continuously.

**Capitalized Cost**   In Exercises 51 and 52, find the capitalized cost $C$ of an asset (a) for $n = 5$ years, (b) for $n = 10$ years, and (c) forever. The capitalized cost is given by

$$C = C_0 + \int_0^n c(t)e^{-rt}\,dt$$

where $C_0$ is the original investment, $t$ is the time in years, $r$ is the annual interest rate compounded continuously, and $c(t)$ is the annual cost of maintenance (measured in dollars). [*Hint:* For part (c), see Exercises 35–38.]

**51.** $C_0 = \$650,000, c(t) = 25,000, r = 10\%$

**52.** $C_0 = \$650,000, c(t) = 25,000(1 + 0.08t), r = 12\%$

## Algebra Review

### Algebra and Integration Techniques

Integration techniques involve many different algebraic skills. Study the examples in this Algebra Review. Be sure that you understand the algebra used in each step.

#### *Example 1*   Algebra and Integration Techniques

Perform each operation and simplify.

**a.** $\dfrac{2}{x-3} - \dfrac{1}{x+2}$ 　　　　　　　　　　　　　　　　Example 1, page 430

$\quad = \dfrac{2(x+2)}{(x-3)(x+2)} - \dfrac{(x-3)}{(x-3)(x+2)}$ 　　　　Rewrite with common denominator.

$\quad = \dfrac{2(x+2)-(x-3)}{(x-3)(x+2)}$ 　　　　　　　　　Rewrite as single fraction.

$\quad = \dfrac{2x+4-x+3}{x^2-x-6}$ 　　　　　　　　　　　Multiply factors.

$\quad = \dfrac{x+7}{x^2-x-6}$ 　　　　　　　　　　　　　　Combine like terms.

**b.** $\dfrac{6}{x} - \dfrac{1}{x+1} + \dfrac{9}{(x+1)^2}$ 　　　　　　　　　　　Example 2, page 431

$\quad = \dfrac{6(x+1)^2}{x(x+1)^2} - \dfrac{x(x+1)}{x(x+1)^2} + \dfrac{9x}{x(x+1)^2}$ 　　Rewrite with common denominator.

$\quad = \dfrac{6(x+1)^2 - x(x+1) + 9x}{x(x+1)^2}$ 　　　　　Rewrite as single fraction.

$\quad = \dfrac{6x^2 + 12x + 6 - x^2 - x + 9x}{x^3 + 2x^2 + x}$ 　　　Multiply factors.

$\quad = \dfrac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}$ 　　　　　　　　　　Combine like terms.

**c.** $6\ln|x| - \ln|x+1| + 9\dfrac{(x+1)^{-1}}{-1}$ 　　　　　　Example 2, page 431

$\quad = \ln|x|^6 - \ln|x+1| + 9\dfrac{(x+1)^{-1}}{-1}$ 　　　　$m\ln n = \ln n^m$

$\quad = \ln|x^6| - \ln|x+1| + 9\dfrac{(x+1)^{-1}}{-1}$ 　　　　Property of absolute value

$\quad = \ln\dfrac{|x^6|}{|x+1|} + 9\dfrac{(x+1)^{-1}}{-1}$ 　　　　　　$\ln m - \ln n = \ln\dfrac{m}{n}$

$\quad = \ln\left|\dfrac{x^6}{x+1}\right| + 9\dfrac{(x+1)^{-1}}{-1}$ 　　　　　　$\dfrac{|a|}{|b|} = \left|\dfrac{a}{b}\right|$

$\quad = \ln\left|\dfrac{x^6}{x+1}\right| - 9(x+1)^{-1}$ 　　　　　　Rewrite sum as difference.

$\quad = \ln\left|\dfrac{x^6}{x+1}\right| - \dfrac{9}{x+1}$ 　　　　　　　Rewrite with positive exponent.

## *Example 2*  Algebra and Integration Techniques

Perform each operation and simplify.

**a.** $x + 1 + \dfrac{1}{x^3} + \dfrac{1}{x - 1}$

**b.** $x^2 e^x - 2(x - 1)e^x$

**c.** Solve for $y$:  $\ln |y| - \ln |L - y| = kt + C$

**SOLUTION**

**a.**  $x + 1 + \dfrac{1}{x^3} + \dfrac{1}{x - 1}$    Example 3, page 432

$= \dfrac{(x + 1)(x^3)(x - 1)}{x^3(x - 1)} + \dfrac{x - 1}{x^3(x - 1)} + \dfrac{x^3}{x^3(x - 1)}$

$= \dfrac{(x + 1)(x^3)(x - 1) + (x - 1) + x^3}{x^3(x - 1)}$    Rewrite as single fraction.

$= \dfrac{(x^2 - 1)(x^3) + x - 1 + x^3}{x^3(x - 1)}$    $(x + 1)(x - 1) = x^2 - 1$

$= \dfrac{x^5 - x^3 + x - 1 + x^3}{x^4 - x^3}$    Multiply factors.

$= \dfrac{x^5 + x - 1}{x^4 - x^3}$    Combine like terms.

**b.**  $x^2 e^x - 2(x - 1)e^x$    Example 5, page 445

$= x^2 e^x - 2(xe^x - e^x)$    Multiply factors.

$= x^2 e^x - 2xe^x + 2e^x$    Multiply factors.

$= e^x(x^2 - 2x + 2)$    Factor.

**c.**    $\ln |y| - \ln |L - y| = kt + C$    Example 4, page 433

$-\ln |y| + \ln |L - y| = -kt - C$    Multiply each side by $-1$.

$\ln \left| \dfrac{L - y}{y} \right| = -kt - C$    $\ln x - \ln y = \ln \dfrac{x}{y}$

$\left| \dfrac{L - y}{y} \right| = e^{-kt - C}$    Exponentiate each side.

$\left| \dfrac{L - y}{y} \right| = e^{-C} e^{-kt}$    $x^{n+m} = x^n x^m$

$\dfrac{L - y}{y} = \pm e^{-C} e^{-kt}$    Property of absolute value

$L - y = be^{-kt} y$    Let $\pm e^{-C} = b$ and multiply each side by $y$.

$L = y + be^{-kt} y$    Add $y$ to each side.

$L = y(1 + be^{-kt})$    Factor.

$\dfrac{L}{1 + be^{-kt}} = y$    Divide.

# Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 474.
Answers to odd-numbered Review Exercises are given in the back of the text.*

| Section 6.1 | Review Exercises |
|---|---|
| ■ Use integration by parts to find indefinite integrals.<br><br>$$\int u \, dv = uv - \int v \, du$$ | *1–4* |
| ■ Use integration by parts repeatedly to find indefinite integrals. | *5, 6* |
| ■ Find the present value of future income. | *7–14* |

| Section 6.2 | |
|---|---|
| ■ Use partial fractions to find indefinite integrals. | *15–20* |
| ■ Use logistic growth functions to model real-life situations.<br><br>$$y = \frac{L}{1 + be^{-kt}}$$ | *21, 22* |

| Section 6.3 | |
|---|---|
| ■ Use integration tables to find indefinite and definite integrals. | *23–30* |
| ■ Use reduction formulas to find indefinite integrals. | *31–34* |
| ■ Use integration tables to solve real-life problems. | *35, 36* |

| Section 6.4 | |
|---|---|
| ■ Use the Trapezoidal Rule to approximate definite integrals.<br><br>$$\int_a^b f(x) \, dx \approx \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$ | *37–40* |
| ■ Use Simpson's Rule to approximate definite integrals.<br><br>$$\int_a^b f(x) \, dx \approx \left(\frac{b-a}{3n}\right)[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)]$$ | *41–44* |
| ■ Analyze the sizes of the errors when approximating definite integrals with the Trapezoidal Rule.<br><br>$$|E| \le \frac{(b-a)^3}{12n^2}[\max|f''(x)|], \quad a \le x \le b$$ | *45, 46* |
| ■ Analyze the sizes of the errors when approximating definite integrals with Simpson's Rule.<br><br>$$|E| \le \frac{(b-a)^5}{180n^4}[\max|f^{(4)}(x)|], \quad a \le x \le b$$ | *47, 48* |

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* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

## Section 6.5

■ Evaluate improper integrals with infinite limits of integration.

$$\int_{a}^{\infty} f(x)\, dx = \lim_{b \to \infty} \int_{a}^{b} f(x)\, dx, \qquad \int_{-\infty}^{b} f(x)\, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\, dx,$$

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_{c}^{\infty} f(x)\, dx$$

■ Evaluate improper integrals with infinite integrands.

$$\int_{a}^{b} f(x)\, dx = \lim_{c \to b^{-}} \int_{a}^{c} f(x)\, dx, \qquad \int_{a}^{b} f(x)\, dx = \lim_{c \to a^{+}} \int_{c}^{b} f(x)\, dx,$$

$$\int_{a}^{b} f(x)\, dx = \int_{a}^{c} f(x)\, dx + \int_{c}^{b} f(x)\, dx$$

■ Use improper integrals to solve real-life problems.

## Study Strategies

■ **Use a Variety of Approaches**   To be efficient at finding antiderivatives, you need to use a variety of approaches.

1. Check to see whether the integral fits one of the basic integration formulas—you should have these formulas memorized.

2. Try an integration technique such as substitution, integration by parts, or partial fractions to rewrite the integral in a form that fits one of the basic integration formulas.

3. Use a table of integrals.

4. Use a symbolic integration utility.

■ **Use Numerical Integration**   When solving a definite integral, remember that you cannot apply the Fundamental Theorem of Calculus unless you can find an antiderivative of the integrand. This is not always possible—even with a symbolic integration utility. In such cases, you can use a numerical technique such as the Midpoint Rule, the Trapezoidal Rule, or Simpson's Rule to approximate the value of the integral.

■ **Improper Integrals**   When solving integration problems, remember that the symbols used to denote definite integrals are the same as those used to denote improper integrals. Evaluating an improper integral as a definite integral can lead to an incorrect value. For instance, if you evaluated the integral

$$\int_{-2}^{1} \frac{1}{x^2}\, dx$$

as though it were a definite integral, you would obtain a value of $-\frac{3}{2}$. This is not, however, correct. This integral is actually a divergent improper integral.

## Review Exercises

In Exercises 1–4, use integration by parts to find the indefinite integral.

**1.** $\displaystyle\int \frac{\ln x}{\sqrt{x}}\, dx$

**2.** $\displaystyle\int \sqrt{x}\,\ln x\, dx$

**3.** $\displaystyle\int (x + 1)e^x\, dx$

**4.** $\displaystyle\int \ln\!\left(\frac{x}{x + 1}\right) dx$

In Exercises 5 and 6, use integration by parts repeatedly to find the indefinite integral. Use a symbolic integration utility to verify your answer.

**5.** $\displaystyle\int 2x^2 e^{2x}\, dx$

**6.** $\displaystyle\int (\ln x)^3\, dx$

**Present Value**  In Exercises 7–10, find the present value of the income given by $c(t)$ (measured in dollars) over $t_1$ years at the given annual inflation rate $r$.

**7.** $c(t) = 20{,}000$, $r = 4\%$, $t_1 = 5$ years

**8.** $c(t) = 10{,}000 + 1500t$, $r = 6\%$, $t_1 = 10$ years

**9.** $c(t) = 24{,}000t$, $r = 5\%$, $t_1 = 10$ years

**10.** $c(t) = 20{,}000 + 100e^{t/2}$, $r = 5\%$, $t_1 = 5$ years

**B 11. Economics: Present Value**  Calculate the present value of each scenario.

(a) $2000 per year for 5 years at interest rates of 5%, 10%, and 15%

(b) A lottery ticket that pays $200,000 per year after taxes over 20 years, assuming an inflation rate of 8%

*(Source: Adapted from Boyes/Melvin, Economics, Third Edition)*

**B 12. Finance: Present Value**  You receive $2000 at the end of each year for the next 3 years to help with college expenses. Assuming an annual interest rate of 6%, what is the present value of that stream of payments?  *(Source: Adapted from Garman/Forgue, Personal Finance, Eighth Edition)*

**B 13. Finance: Present Value**  Determine the amount a person planning for retirement would need to deposit today to be able to withdraw $12,000 each year for the next 10 years from an account earning 6% interest.  *(Source: Adapted from Garman/Forgue, Personal Finance, Eighth Edition)*

**B 14. Finance: Present Value**  A person invests $100,000 earning 6% interest. If $10,000 is withdrawn each year, use present value to determine how many years it will take for the fund to run out.  *(Source: Adapted from Garman/ Forgue, Personal Finance, Eighth Edition)*

In Exercises 15–20, use partial fractions to find the indefinite integral.

**15.** $\displaystyle\int \frac{1}{x(x + 5)}\, dx$

**16.** $\displaystyle\int \frac{4x - 2}{3(x - 1)^2}\, dx$

**17.** $\displaystyle\int \frac{x - 28}{x^2 - x - 6}\, dx$

**18.** $\displaystyle\int \frac{4x^2 - x - 5}{x^2(x + 5)}\, dx$

**19.** $\displaystyle\int \frac{x^2}{x^2 + 2x - 15}\, dx$

**20.** $\displaystyle\int \frac{x^2 + 2x - 12}{x(x + 3)}\, dx$

**21. Sales**  A new product initially sells 1250 units per week. After 24 weeks, the number of sales increases to 6500. The sales can be modeled by logistic growth with a limit of 10,000 units per week.

(a) Find a logistic growth model for the number of units.

(b) Use the model to complete the table.

| Time, $t$ | 0 | 3 | 6 | 12 | 24 |
|---|---|---|---|---|---|
| Sales, $y$ | | | | | |

(c) Use the graph shown below to approximate the time $t$ when sales will be 7500.

**New Product Sales**



**22. Biology**  A conservation society has introduced a population of 300 ring-necked pheasants into a new area. After 5 years, the population has increased to 966. The population can be modeled by logistic growth with a limit of 2700 pheasants.

(a) Find a logistic growth model for the population of ring-necked pheasants.

(b) How many pheasants were present after 4 years?

(c) How long will it take to establish a population of 1750 pheasants?
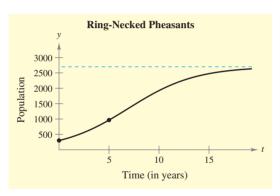
Figure for 22

In Exercises 23–30, use the table of integrals in Section 6.3 to find or evaluate the integral.

23. $\displaystyle\int \frac{x}{(2 + 3x)^2}\,dx$

24. $\displaystyle\int \frac{x}{\sqrt{2 + 3x}}\,dx$

25. $\displaystyle\int \frac{\sqrt{x^2 + 25}}{x}\,dx$

26. $\displaystyle\int \frac{1}{x(4 + 3x)}\,dx$

27. $\displaystyle\int \frac{1}{x^2 - 4}\,dx$

28. $\displaystyle\int (\ln 3x)^2\,dx$

29. $\displaystyle\int_0^3 \frac{x}{\sqrt{1 + x}}\,dx$

30. $\displaystyle\int_1^3 \frac{1}{x^2\sqrt{16 - x^2}}\,dx$

In Exercises 31–34, use a reduction formula from the table of integrals in Section 6.3 to find the indefinite integral.

31. $\displaystyle\int \frac{\sqrt{1 + x}}{x}\,dx$

32. $\displaystyle\int \frac{1}{(x^2 - 9)^2}\,dx$

33. $\displaystyle\int (x - 5)^3 e^{x-5}\,dx$

34. $\displaystyle\int (\ln x)^4\,dx$

35. **Probability**   The probability of recall in an experiment is found to be

$$P(a \le x \le b) = \int_a^b \frac{96}{11}\left(\frac{x}{\sqrt{9 + 16x}}\right) dx, \quad 0 \le a \le b \le 1$$

where $x$ represents the percent of recall (see figure).

(a) Find the probability that a randomly chosen individual will recall between 0% and 80% of the material.

(b) Find the probability that a randomly chosen individual will recall between 0% and 50% of the material.
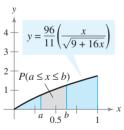


Figure for 35



Figure for 36

36. **Probability**   The probability of locating between $a$ and $b$ percent of oil and gas deposits in a region is
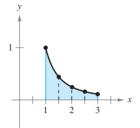
$$P(a \le x \le b) = \int_a^b 1.5x^2 e^{x^{1.5}}\,dx \text{ (see figure).}$$
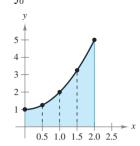
(a) Find the probability that between 40% and 60% of the deposits will be found.

(b) Find the probability that between 0% and 50% of the deposits will be found.

In Exercises 37–40, use the Trapezoidal Rule to approximate the definite integral.

37. $\displaystyle\int_1^3 \frac{1}{x^2}\,dx, \; n = 4$
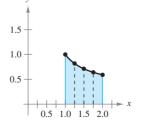
38. $\displaystyle\int_0^2 (x^2 + 1)\,dx, \; n = 4$





39. $\displaystyle\int_1^2 \frac{1}{1 + \ln x}\,dx, \; n = 4$

40. $\displaystyle\int_0^2 \frac{1}{\sqrt{1 + x^3}}\,dx, \; n = 8$
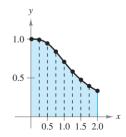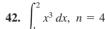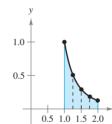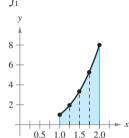




In Exercises 41–44, use Simpson's Rule to approximate the definite integral.

41. $\displaystyle\int_1^2 \frac{1}{x^3}\,dx, \; n = 4$
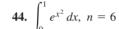
42. $\displaystyle\int_1^2 x^3\,dx, \; n = 4$

**43.** $\int_0^1 \frac{x^{3/2}}{2 - x^2} \, dx, \quad n = 4$



**44.** $\int_0^1 e^{x^2} \, dx, \quad n = 6$



**53.** $\int_0^4 \frac{1}{\sqrt{4x}} \, dx$



**54.** $\int_1^2 \frac{x}{16(x - 1)^2} \, dx$
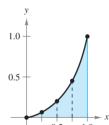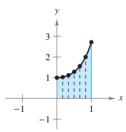


In Exercises 45 and 46, use the error formula to find bounds for the error in approximating the integral using the Trapezoidal Rule.

**45.** $\int_0^2 e^{2x} \, dx, \quad n = 4$

**46.** $\int_0^2 e^{2x} \, dx, \quad n = 8$

In Exercises 47 and 48, use the error formula to find bounds for the error in approximating the integral using Simpson's Rule.
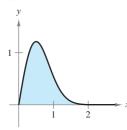
**47.** $\int_2^4 \frac{1}{x - 1} \, dx, \quad n = 4$
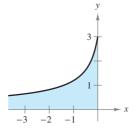
**48.** $\int_2^4 \frac{1}{x - 1} \, dx, \quad n = 8$

In Exercises 49–56, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

**49.** $\int_0^\infty 4xe^{-2x^2} \, dx$
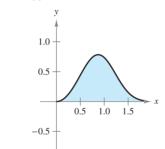


**50.** $\int_{-\infty}^0 \frac{3}{(1 - 3x)^{2/3}} \, dx$
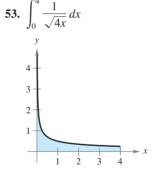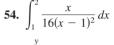


**51.** $\int_{-\infty}^0 \frac{1}{3x^2} \, dx$



**52.** $\int_0^\infty 2x^2 e^{-x^3} \, dx$



**55.** $\int_2^3 \frac{1}{\sqrt{x - 2}} \, dx$
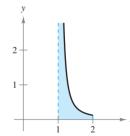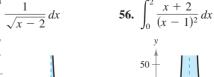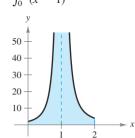


**56.** $\int_0^2 \frac{x + 2}{(x - 1)^2} \, dx$



**57. Present Value**   You are considering buying a franchise that yields a continuous income stream of \$100,000 per year. Find the present value of the franchise (a) for 15 years and (b) forever. Assume that money earns 6% interest per year, compounded continuously.

**58. Capitalized Cost**   A company invests \$1.5 million in a new manufacturing plant that will cost \$75,000 per year in maintenance. Find the capitalized cost for (a) 20 years and (b) forever. Assume that money earns 6% interest, compounded continuously.

**59. SAT Scores**   In 2006, the Scholastic Aptitude Test (SAT) math scores for college-bound seniors roughly followed a normal distribution

$$y = 0.0035e^{-(x - 518)^2/26,450}, \quad 200 \le x \le 800$$

where $x$ is the SAT score for mathematics. Find the probability that a senior chosen at random had an SAT score (a) between 500 and 650, (b) 650 or better, and (c) 750 or better.   *(Source: College Board)*

**60. ACT Scores**   In 2006, the ACT composite scores for college-bound seniors followed a normal distribution

$$y = 0.0831e^{-(x - 21.1)^2/46.08}, \quad 1 \le x \le 36$$

where $x$ is the composite ACT score. Find the probability that a senior chosen at random had an ACT score (a) between 16.3 and 25.9, (b) 25.9 or better, and (c) 30.7 or better.   *(Source: ACT, Inc.)*

# Chapter Test

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, use integration by parts to find the indefinite integral.

**1.** $\displaystyle\int xe^{x+1}\,dx$

**2.** $\displaystyle\int 9x^2 \ln x\,dx$

**3.** $\displaystyle\int x^2\,e^{-x/3}\,dx$

**4.** The earnings per share $E$ (in dollars) for Home Depot from 2000 through 2006 can be modeled by

$$E = -2.62 + 0.495\sqrt{t}\,\ln t, \quad 10 \le t \le 16$$

where $t$ is the year, with $t = 10$ corresponding to 2000. Find the average earnings per share for the years 2000 through 2006.  *(Source: The Home Depot, Inc.)*

In Exercises 5–7, use partial fractions to find the indefinite integral.

**5.** $\displaystyle\int \frac{18}{x^2 - 81}\,dx$

**6.** $\displaystyle\int \frac{3x}{(3x + 1)^2}\,dx$

**7.** $\displaystyle\int \frac{x + 4}{x^2 + 2x}\,dx$

In Exercises 8–10, use the table of integrals in Section 6.3 to find the indefinite integral.

**8.** $\displaystyle\int \frac{x}{(7 + 2x)^2}\,dx$

**9.** $\displaystyle\int \frac{3x^2}{1 + e^{x^3}}\,dx$

**10.** $\displaystyle\int \frac{2x^3}{\sqrt{1 + 5x^2}}\,dx$

In Exercises 11–13, evaluate the definite integral.

**11.** $\displaystyle\int_0^1 \ln(3 - 2x)\,dx$

**12.** $\displaystyle\int_5^{10} \frac{28}{x^2 - x - 12}\,dx$

**13.** $\displaystyle\int_{-3}^{-1} \frac{\sqrt{x^2 + 16}}{x}\,dx$

**14.** Use the Trapezoidal Rule with $n = 4$ to approximate $\displaystyle\int_1^2 \frac{1}{x^2\sqrt{x^2 + 4}}\,dx$. Compare your result with the exact value of the definite integral.

**15.** Use Simpson's Rule with $n = 4$ to approximate $\displaystyle\int_0^1 9xe^{3x}\,dx$. Compare your result with the exact value of the definite integral.

In Exercises 16–18, determine whether the improper integral converges or diverges. Evaluate the integral if it converges.

**16.** $\displaystyle\int_0^{\infty} e^{-3x}\,dx$

**17.** $\displaystyle\int_0^9 \frac{2}{\sqrt{x}}\,dx$

**18.** $\displaystyle\int_{-\infty}^0 \frac{1}{(4x - 1)^{2/3}}\,dx$

**19.** A magazine publisher offers two subscription plans. Plan A is a one-year subscription for $19.95. Plan B is a lifetime subscription (lasting indefinitely) for $149.

(a) A subscriber considers using plan A indefinitely. Assuming an annual inflation rate of 4%, find the present value of the money the subscriber will spend using plan A.

(b) Based on your answer to part (a), which plan should the subscriber use? Explain.

# 7

# Functions of Several Variables



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A spherical building can be represented by an equation involving three variables. (See Section 7.1, Exercise 61.)

## Applications

Functions of several variables have many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Modeling Data, Exercise 59, page 495
- Make a Decision: Monthly Payments, Exercise 51, page 504
- Milk Consumption, Exercise 65, page 515
- Shareholder's Equity, Exercise 66, page 515
- Make a Decision: Revenue, Exercise 33, page 544

## Section 7.1

# The Three-Dimensional Coordinate System

- Plot points in space.
- Find distances between points in space and find midpoints of line segments in space.
- Write the standard forms of the equations of spheres and find the centers and radii of spheres.
- Sketch the coordinate plane traces of surfaces.

## The Three-Dimensional Coordinate System

Recall from Section 1.1 that the Cartesian plane is determined by two perpendicular number lines called the *x*-axis and the *y*-axis. These axes together with their point of intersection (the origin) allow you to develop a two-dimensional coordinate system for identifying points in a plane. To identify a point in space, you must introduce a third dimension to the model. The geometry of this three-dimensional model is called **solid analytic geometry.**
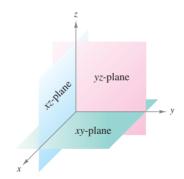
**FIGURE 7.1**

You can construct a **three-dimensional coordinate system** by passing a *z*-axis perpendicular to both the *x*- and *y*-axes at the origin. Figure 7.1 shows the positive portion of each coordinate axis. Taken as pairs, the axes determine three **coordinate planes:** the **xy-plane,** the **xz-plane,** and the **yz-plane.** These three coordinate planes separate the three-dimensional coordinate system into eight **octants.** The first octant is the one for which all three coordinates are positive. In this three-dimensional system, a point *P* in space is determined by an ordered triple $(x, y, z)$, where *x*, *y*, and *z* are as follows.

$x$ = directed distance from *yz*-plane to *P*

$y$ = directed distance from *xz*-plane to *P*

$z$ = directed distance from *xy*-plane to *P*

A three-dimensional coordinate system can have either a **left-handed** or a **right-handed** orientation. To determine the orientation of a system, imagine that you are standing at the origin, with your arms pointing in the direction of the positive *x*- and *y*-axes, and with the *z*-axis pointing up, as shown in Figure 7.2. The system is right-handed or left-handed depending on which hand points along the *x*-axis. In this text, you will work exclusively with the right-handed system.



Right-handed system    Left-handed system

**FIGURE 7.2**

*Example 1* **Plotting Points in Space**

Plot each point in space.

**a.** $(2, -3, 3)$

**b.** $(-2, 6, 2)$

**c.** $(1, 4, 0)$

**d.** $(2, 2, -3)$

**SOLUTION** To plot the point $(2, -3, 3)$, notice that $x = 2$, $y = -3$, and $z = 3$. To help visualize the point (see Figure 7.3), locate the point $(2, -3)$ in the $xy$-plane (denoted by a cross). The point $(2, -3, 3)$ lies three units above the cross. The other three points are also shown in the figure.

✓ **CHECKPOINT 1**

Plot each point on the three-dimensional coordinate system.

**a.** $(2, 5, 1)$

**b.** $(-2, -4, 3)$

**c.** $(4, 0, -5)$ ◼
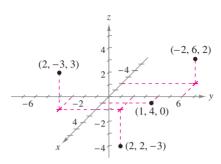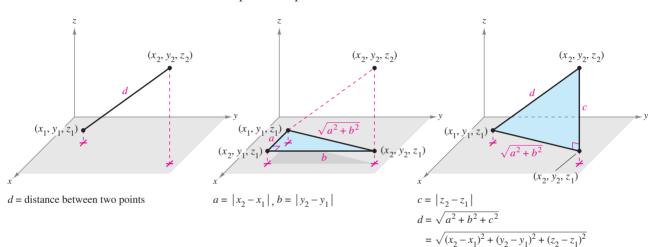


**FIGURE 7.3**

## The Distance and Midpoint Formulas

Many of the formulas established for the two-dimensional coordinate system can be extended to three dimensions. For example, to find the distance between two points in space, you can use the Pythagorean Theorem twice, as shown in Figure 7.4. By doing this, you will obtain the formula for the distance between two points in space.



$d$ = distance between two points

$a = |x_2 - x_1|$, $b = |y_2 - y_1|$

$c = |z_2 - z_1|$

$d = \sqrt{a^2 + b^2 + c^2}$

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

**FIGURE 7.4**

**Distance Formula in Space**

The distance between the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

---

**Example 2** **Finding the Distance Between Two Points**

Find the distance between $(1, 0, 2)$ and $(2, 4, -3)$.

**SOLUTION**

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} && \text{Write Distance Formula.}\\
&= \sqrt{(2 - 1)^2 + (4 - 0)^2 + (-3 - 2)^2} && \text{Substitute.}\\
&= \sqrt{1 + 16 + 25} && \text{Simplify.}\\
&= \sqrt{42} && \text{Simplify.}
\end{aligned}
$$

✓ **CHECKPOINT 2**

Find the distance between $(2, 3, -1)$ and $(0, 5, 3)$. ◼

Notice the similarity between the Distance Formula in the plane and the Distance Formula in space. The Midpoint Formulas in the plane and in space are also similar.

---

**Midpoint Formula in Space**

The midpoint of the line segment joining the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

---

**Example 3** **Using the Midpoint Formula**

Find the midpoint of the line segment joining $(5, -2, 3)$ and $(0, 4, 4)$.

**SOLUTION** Using the Midpoint Formula, the midpoint is

$$\left(\frac{5 + 0}{2}, \frac{-2 + 4}{2}, \frac{3 + 4}{2}\right) = \left(\frac{5}{2}, 1, \frac{7}{2}\right)$$

as shown in Figure 7.5.

✓ **CHECKPOINT 3**

Find the midpoint of the line segment joining $(3, -2, 0)$ and $(-8, 6, -4)$. ◼
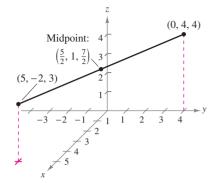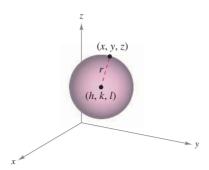


**FIGURE 7.5**

## The Equation of a Sphere

A **sphere** with center at $(h, k, l)$ and radius $r$ is defined to be the set of all points $(x, y, z)$ such that the distance between $(x, y, z)$ and $(h, k, l)$ is $r$, as shown in Figure 7.6. Using the Distance Formula, this condition can be written as

$$\sqrt{(x - h)^2 + (y - k)^2 + (z - l)^2} = r.$$

By squaring both sides of this equation, you obtain the standard equation of a sphere.



**FIGURE 7.6**    Sphere: Radius $r$, Center $(h, k, l)$

**Standard Equation of a Sphere**

The **standard equation of a sphere** whose center is $(h, k, l)$ and whose radius is $r$ is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

### *Example 4*   Finding the Equation of a Sphere

Find the standard equation of the sphere whose center is $(2, 4, 3)$ and whose radius is 3. Does this sphere intersect the $xy$-plane?

**SOLUTION**

| | |
|---|---|
| $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$ | Write standard equation. |
| $(x - 2)^2 + (y - 4)^2 + (z - 3)^2 = 3^2$ | Substitute. |
| $(x - 2)^2 + (y - 4)^2 + (z - 3)^2 = 9$ | Simplify. |

From the graph shown in Figure 7.7, you can see that the center of the sphere lies three units above the $xy$-plane. Because the sphere has a radius of 3, you can conclude that it does intersect the $xy$-plane—at the point $(2, 4, 0)$.
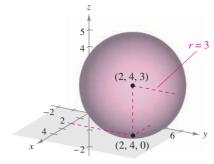


**FIGURE 7.7**

✓ **CHECKPOINT 4**

Find the standard equation of the sphere whose center is $(4, 3, 2)$ and whose radius is 5. ∎

*Example 5* **Finding the Equation of a Sphere**

Find the equation of the sphere that has the points $(3, -2, 6)$ and $(-1, 4, 2)$ as endpoints of a diameter.

**SOLUTION**    By the Midpoint Formula, the center of the sphere is

$$(h, k, l) = \left(\frac{3 + (-1)}{2}, \frac{-2 + 4}{2}, \frac{6 + 2}{2}\right)$$    Apply Midpoint Formula.

$$= (1, 1, 4).$$    Simplify.

By the Distance Formula, the radius is

$$r = \sqrt{(3 - 1)^2 + (-2 - 1)^2 + (6 - 4)^2}$$

$$= \sqrt{17}.$$    Simplify.

So, the standard equation of the sphere is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$    Write formula for a sphere.

$$(x - 1)^2 + (y - 1)^2 + (z - 4)^2 = 17.$$    Substitute.

**✓ CHECKPOINT 5**

Find the equation of the sphere that has the points $(-2, 5, 7)$ and $(4, 1, -3)$ as endpoints of a diameter. ■

*Example 6* **Finding the Center and Radius of a Sphere**

Find the center and radius of the sphere whose equation is

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 8 = 0.$$

**SOLUTION**    You can obtain the standard equation of the sphere by completing the square. To do this, begin by grouping terms with the same variable. Then add "the square of half the coefficient of each linear term" to each side of the equation. For instance, to complete the square of $(x^2 - 2x)$, add $\left[\frac{1}{2}(-2)\right]^2 = 1$ to each side.

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 8 = 0$$

$$(x^2 - 2x + \boxed{\phantom{0}}) + (y^2 + 4y + \boxed{\phantom{0}}) + (z^2 - 6z + \boxed{\phantom{0}}) = -8$$

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) + (z^2 - 6z + 9) = -8 + 1 + 4 + 9$$

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 6$$

So, the center of the sphere is $(1, -2, 3)$, and its radius is $\sqrt{6}$, as shown in Figure 7.8.



Sphere:
$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 6$

Center: $(1, -2, 3)$

$r = \sqrt{6}$
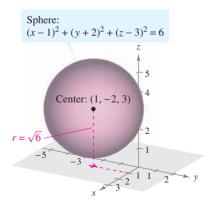
**FIGURE 7.8**

**✓ CHECKPOINT 6**

Find the center and radius of the sphere whose equation is

$$x^2 + y^2 + z^2 + 6x - 8y + 2z - 10 = 0. ■$$

Note in Example 6 that the points satisfying the equation of the sphere are "surface points," not "interior points." In general, the collection of points satisfying an equation involving $x$, $y$, and $z$ is called a **surface in space.**

## Traces of Surfaces

Finding the intersection of a surface with one of the three coordinate planes (or with a plane parallel to one of the three coordinate planes) helps visualize the surface. Such an intersection is called a **trace** of the surface. For example, the $xy$-trace of a surface consists of all points that are common to both the surface *and* the $xy$-plane. Similarly, the $xz$-trace of a surface consists of all points that are common to both the surface and the $xz$-plane.



$xy$-trace:
$(x-3)^2 + (y-2)^2 = 3^2$

Sphere:
$(x-3)^2 + (y-2)^2 + (z+4)^2 = 5^2$
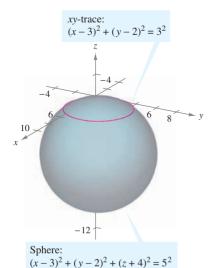
**FIGURE 7.9**

### *Example 7*    Finding a Trace of a Surface

Sketch the $xy$-trace of the sphere whose equation is

$$(x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 5^2.$$

**SOLUTION**    To find the $xy$-trace of this surface, use the fact that every point in the $xy$-plane has a $z$-coordinate of zero. This means that if you substitute $z = 0$ into the original equation, the resulting equation will represent the intersection of the surface with the $xy$-plane.

$(x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 5^2$    Write original equation.

$(x - 3)^2 + (y - 2)^2 + (0 + 4)^2 = 25$    Let $z = 0$ to find $xy$-trace.

$(x - 3)^2 + (y - 2)^2 + 16 = 25$

$(x - 3)^2 + (y - 2)^2 = 9$

$(x - 3)^2 + (y - 2)^2 = 3^2$    Equation of circle

From this equation, you can see that the $xy$-trace is a circle of radius 3, as shown in Figure 7.9.

### ✓ CHECKPOINT 7

Find the equation of the $xy$-trace of the sphere whose equation is

$$(x + 1)^2 + (y - 2)^2 + (z + 3)^2 = 5^2. \blacksquare$$

---

### CONCEPT CHECK

1. Name the three coordinate planes of a three-dimensional coordinate system formed by passing a $z$-axis perpendicular to both the $x$- and $y$-axes at the origin.

2. A point in the three-dimensional coordinate system has coordinates $(x_1, y_1, z_1)$. Describe what each coordinate measures.

3. Give the formula for the distance between the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$.

4. Give the standard equation of a sphere of radius $r$ centered at $(h, k, l)$.

| Skills Review 7.1 | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.1 and 1.2. |

In Exercises 1–4, find the distance between the points.

**1.** $(5, 1), (3, 5)$    **2.** $(2, 3), (-1, -1)$    **3.** $(-5, 4), (-5, -4)$    **4.** $(-3, 6), (-3, -2)$

In Exercises 5–8, find the midpoint of the line segment connecting the points.

**5.** $(2, 5), (6, 9)$    **6.** $(-1, -2), (3, 2)$    **7.** $(-6, 0), (6, 6)$    **8.** $(-4, 3), (2, -1)$

In Exercises 9 and 10, write the standard form of the equation of the circle.

**9.** Center: $(2, 3)$; radius: $2$    **10.** Endpoints of a diameter: $(4, 0), (-2, 8)$

## Exercises 7.1

In Exercises 1–4, plot the points on the same three-dimensional coordinate system.

**1.** (a) $(2, 1, 3)$
   (b) $(-1, 2, 1)$

**2.** (a) $(3, -2, 5)$
   (b) $\left(\frac{3}{2}, 4, -2\right)$

**3.** (a) $(5, -2, 2)$
   (b) $(5, -2, -2)$

**4.** (a) $(0, 4, -5)$
   (b) $(4, 0, 5)$
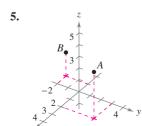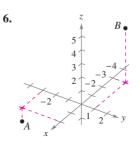
In Exercises 5 and 6, approximate the coordinates of the points.

**5.**



**6.**



In Exercises 7–10, find the coordinates of the point.

**7.** The point is located three units behind the $yz$-plane, four units to the right of the $xz$-plane, and five units above the $xy$-plane.

**8.** The point is located seven units in front of the $yz$-plane, two units to the left of the $xz$-plane, and one unit below the $xy$-plane.

**9.** The point is located on the $x$-axis, 10 units in front of the $yz$-plane.

**10.** The point is located in the $yz$-plane, three units to the right of the $xz$-plane, and two units above the $xy$-plane.

**11. Think About It**    What is the $z$-coordinate of any point in the $xy$-plane?

**12. Think About It**    What is the $x$-coordinate of any point in the $yz$-plane?

In Exercises 13–16, find the distance between the two points.

**13.** $(4, 1, 5), (8, 2, 6)$    **14.** $(-4, -1, 1), (2, -1, 5)$

**15.** $(-1, -5, 7), (-3, 4, -4)$    **16.** $(8, -2, 2), (8, -2, 4)$

In Exercises 17–20, find the coordinates of the midpoint of the line segment joining the two points.

**17.** $(6, -9, 1), (-2, -1, 5)$    **18.** $(4, 0, -6), (8, 8, 20)$
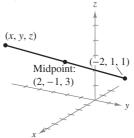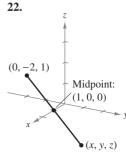
**19.** $(-5, -2, 5), (6, 3, -7)$    **20.** $(0, -2, 5), (4, 2, 7)$
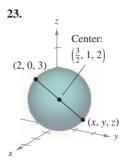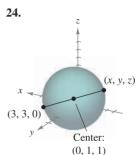
In Exercises 21–24, find $(x, y, z)$.

**21.**



**22.**



**23.**



**24.**
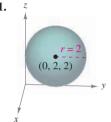
In Exercises 25–28, find the lengths of the sides of the triangle with the given vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither of these.
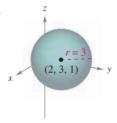
**25.** $(0, 0, 0)$, $(2, 2, 1)$, $(2, -4, 4)$

**26.** $(5, 3, 4)$, $(7, 1, 3)$, $(3, 5, 3)$

**27.** $(-2, 2, 4)$, $(-2, 2, 6)$, $(-2, 4, 8)$

**28.** $(5, 0, 0)$, $(0, 2, 0)$, $(0, 0, -3)$

**29. Think About It**    The triangle in Exercise 25 is translated five units upward along the $z$-axis. Determine the coordinates of the translated triangle.

**30. Think About It**    The triangle in Exercise 26 is translated three units to the right along the $y$-axis. Determine the coordinates of the translated triangle.

In Exercises 31–40, find the standard equation of the sphere.
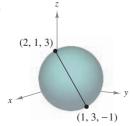
**31.**
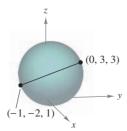


**32.**



**33.**



**34.**



**35.** Center: $(1, 1, 5)$; radius: $3$

**36.** Center: $(4, -1, 1)$; radius: $5$

**37.** Endpoints of a diameter: $(2, 0, 0)$, $(0, 6, 0)$

**38.** Endpoints of a diameter: $(1, 0, 0)$, $(0, 5, 0)$

**39.** Center: $(-2, 1, 1)$; tangent to the $xy$-plane

**40.** Center: $(1, 2, 0)$; tangent to the $yz$-plane

In Exercises 41–46, find the sphere's center and radius.

**41.** $x^2 + y^2 + z^2 - 5x = 0$

**42.** $x^2 + y^2 + z^2 - 8y = 0$

**43.** $x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$

**44.** $x^2 + y^2 + z^2 - 4y + 6z + 4 = 0$

**45.** $2x^2 + 2y^2 + 2z^2 - 4x - 12y - 8z + 3 = 0$

**46.** $4x^2 + 4y^2 + 4z^2 - 8x + 16y + 11 = 0$

In Exercises 47–50, sketch the $xy$-trace of the sphere.

**47.** $(x - 1)^2 + (y - 3)^2 + (z - 2)^2 = 25$

**48.** $(x + 1)^2 + (y + 2)^2 + (z - 2)^2 = 16$

**49.** $x^2 + y^2 + z^2 - 6x - 10y + 6z + 30 = 0$

**50.** $x^2 + y^2 + z^2 - 4y + 2z - 60 = 0$

In Exercises 51–54, sketch the $yz$-trace of the sphere.

**51.** $x^2 + (y + 3)^2 + z^2 = 25$

**52.** $(x + 2)^2 + (y - 3)^2 + z^2 = 9$

**53.** $x^2 + y^2 + z^2 - 4x - 4y - 6z - 12 = 0$

**54.** $x^2 + y^2 + z^2 - 6x - 10y + 6z + 30 = 0$

In Exercises 55–58, sketch the trace of the intersection of each plane with the given sphere.

**55.** $x^2 + y^2 + z^2 = 25$

(a) $z = 3$    (b) $x = 4$

**56.** $x^2 + y^2 + z^2 = 169$

(a) $x = 5$    (b) $y = 12$
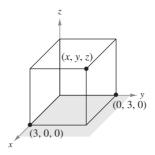
**57.** $x^2 + y^2 + z^2 - 4x - 6y + 9 = 0$

(a) $x = 2$    (b) $y = 3$

**58.** $x^2 + y^2 + z^2 - 8x - 6z + 16 = 0$

(a) $x = 4$    (b) $z = 3$

**59. Geology**    Crystals are classified according to their symmetry. Crystals shaped like cubes are classified as isometric. The vertices of an isometric crystal mapped onto a three-dimensional coordinate system are shown in the figure. Determine $(x, y, z)$.
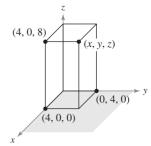


Figure for 59



Figure for 60

**60. Crystals**    Crystals shaped like rectangular prisms are classified as tetragonal. The vertices of a tetragonal crystal mapped onto a three-dimensional coordinate system are shown in the figure. Determine $(x, y, z)$.

**61. Architecture**    A spherical building has a diameter of 165 feet. The center of the building is placed at the origin of a three-dimensional coordinate system. What is the equation of the sphere?

# Section 7.2

# Surfaces in Space

- ■ Sketch planes in space.
- ■ Draw planes in space with different numbers of intercepts.
- ■ Classify quadric surfaces in space.

## Equations of Planes in Space

In Section 7.1, you studied one type of surface in space—a sphere. In this section, you will study a second type—a plane in space. The **general equation of a plane** in space is
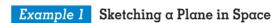
$$ax + by + cz = d.$$   General equation of a plane

Note the similarity of this equation to the general equation of a line in the plane. In fact, if you intersect the plane represented by this equation with each of the three coordinate planes, you will obtain traces that are lines, as shown in Figure 7.10.

In Figure 7.10, the points where the plane intersects the three coordinate axes are the $x$-, $y$-, and $z$-intercepts of the plane. By connecting these three points, you can form a triangular region, which helps you visualize the plane in space.

xz-trace: $ax + cz = d$

Plane: $ax + by + cz = d$

yz-trace: $by + cz = d$

xy-trace: $ax + by = d$

**FIGURE 7.10**

### Example 1   Sketching a Plane in Space

Find the $x$-, $y$-, and $z$-intercepts of the plane given by

$$3x + 2y + 4z = 12.$$

Then sketch the plane.

**SOLUTION**   To find the $x$-intercept, let both $y$ and $z$ be zero.

$$3x + 2(0) + 4(0) = 12$$   Substitute 0 for $y$ and $z$.

$$3x = 12$$   Simplify.

$$x = 4$$   Solve for $x$.

So, the $x$-intercept is $(4, 0, 0)$. To find the $y$-intercept, let $x$ and $z$ be zero and conclude that $y = 6$. So, the $y$-intercept is $(0, 6, 0)$. Similarly, by letting $x$ and $y$ be zero, you can determine that $z = 3$ and that the $z$-intercept is $(0, 0, 3)$. Figure 7.11 shows the triangular portion of the plane formed by connecting the three intercepts.
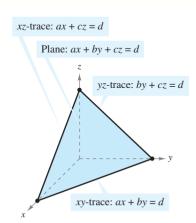
Plane: $3x + 2y + 4z = 12$

$(0, 0, 3)$

$(0, 6, 0)$

$(4, 0, 0)$

**FIGURE 7.11**   Sketch Made by Connecting Intercepts: (4, 0, 0), (0, 6, 0), (0, 0, 3)

### ✓ CHECKPOINT 1

Find the $x$-, $y$-, and $z$-intercepts of the plane given by
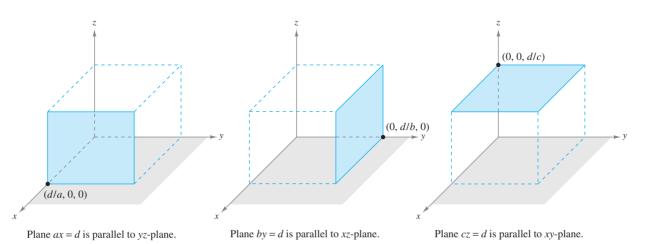
$$2x + 4y + z = 8.$$

Then sketch the plane. ■

## Drawing Planes in Space

The planes shown in Figures 7.10 and 7.11 have three intercepts. When this occurs, we suggest that you draw the plane by sketching the triangular region formed by connecting the three intercepts.

It is possible for a plane in space to have fewer than three intercepts. This occurs when one or more of the coefficients in the equation $ax + by + cz = d$ is zero. Figure 7.12 shows some planes in space that have only one intercept, and Figure 7.13 shows some that have only two intercepts. In each figure, note the use of dashed lines and shading to give the illusion of three dimensions.



Plane $ax = d$ is parallel to $yz$-plane.    Plane $by = d$ is parallel to $xz$-plane.    Plane $cz = d$ is parallel to $xy$-plane.

**FIGURE 7.12**    Planes Parallel to Coordinate Planes



Plane $ax + by = d$ is parallel to $z$-axis.    Plane $ax + cz = d$ is parallel to $y$-axis.    Plane $by + cz = d$ is parallel to $x$-axis.
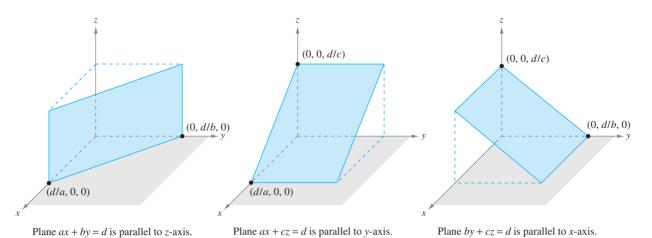
**FIGURE 7.13**    Planes Parallel to Coordinate Axes

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### DISCOVERY

What is the equation of each plane?

**a.** $xy$-plane    **b.** $xz$-plane    **c.** $yz$-plane

## Quadric Surfaces

A third common type of surface in space is a **quadric surface.** Every quadric surface has an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + G = 0. \qquad \text{Second-degree equation}$$

There are six basic types of quadric surfaces.

1. Elliptic cone

2. Elliptic paraboloid

3. Hyperbolic paraboloid

4. Ellipsoid

5. Hyperboloid of one sheet

6. Hyperboloid of two sheets

The six types are summarized on pages 490 and 491. Notice that each surface is pictured with two types of three-dimensional sketches. The computer-generated sketches use traces with hidden lines to give the illusion of three dimensions. The artist-rendered sketches use shading to create the same illusion.

All of the quadric surfaces on pages 490 and 491 are centered at the origin and have axes along the coordinate axes. Moreover, only one of several possible orientations of each surface is shown. If the surface has a different center or is oriented along a different axis, then its standard equation will change accordingly. For instance, the ellipsoid

$$\frac{x^2}{1^2} + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$$

has $(0, 0, 0)$ as its center, but the ellipsoid

$$\frac{(x-2)^2}{1^2} + \frac{(y+1)^2}{3^2} + \frac{(z-4)^2}{2^2} = 1$$

has $(2, -1, 4)$ as its center. A computer-generated graph of the first ellipsoid is shown in Figure 7.14.

### DISCOVERY

One way to help visualize a quadric surface is to determine the intercepts of the surface with the coordinate axes. What are the intercepts of the ellipsoid in Figure 7.14?



$$\frac{x^2}{1^2} + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$$
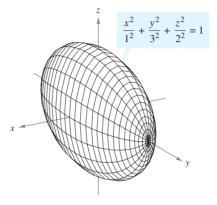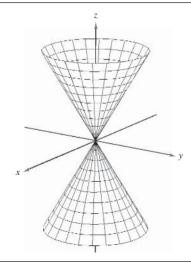
**FIGURE 7.14**
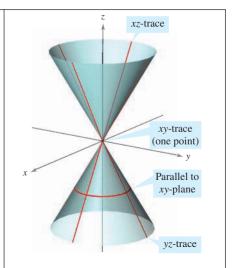
| | | |
|---|---|---|
|  | **Elliptic Cone**<br><br>$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$<br><br>*Trace*      *Plane*<br><br>Ellipse      Parallel to *xy*-plane<br>Hyperbola      Parallel to *xz*-plane<br>Hyperbola      Parallel to *yz*-plane<br><br>The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines. |  |
|  | **Elliptic Paraboloid**<br><br>$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$<br><br>*Trace*      *Plane*<br><br>Ellipse      Parallel to *xy*-plane<br>Parabola      Parallel to *xz*-plane<br>Parabola      Parallel to *yz*-plane<br><br>The axis of the paraboloid corresponds to the variable raised to the first power. |  |
|  | **Hyperbolic Paraboloid**<br><br>$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$<br><br>*Trace*      *Plane*<br><br>Hyperbola      Parallel to *xy*-plane<br>Parabola      Parallel to *xz*-plane<br>Parabola      Parallel to *yz*-plane<br><br>The axis of the paraboloid corresponds to the variable raised to the first power. |  |
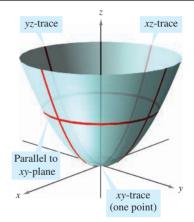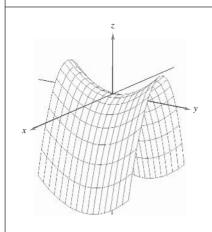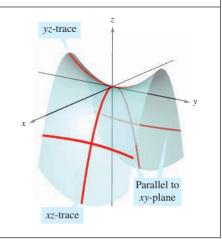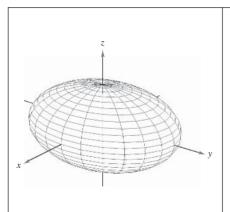
**Ellipsoid**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

*Trace*          *Plane*
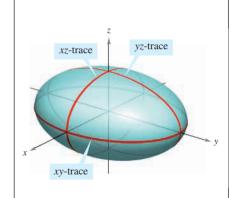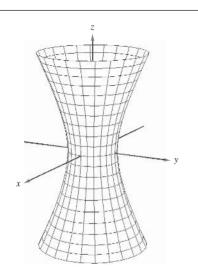
Ellipse          Parallel to $xy$-plane

Ellipse          Parallel to $xz$-plane

Ellipse          Parallel to $yz$-plane

The surface is a sphere if the coefficients $a$, $b$, and $c$ are equal and nonzero.
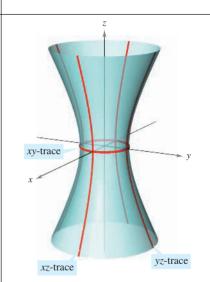


**Hyperboloid of One Sheet**
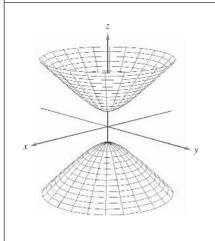
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

*Trace*          *Plane*

Ellipse          Parallel to $xy$-plane

Hyperbola          Parallel to $xz$-plane

Hyperbola          Parallel to $yz$-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.
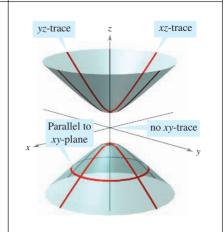


**Hyperboloid of Two Sheets**

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

*Trace*          *Plane*

Ellipse          Parallel to $xy$-plane

Hyperbola          Parallel to $xz$-plane

Hyperbola          Parallel to $yz$-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.

When classifying quadric surfaces, note that the two types of paraboloids have one variable raised to the first power. The other four types of quadric surfaces have equations that are of second degree in *all* three variables.



Surface:
$x - y^2 - z^2 = 0$

**FIGURE 7.15**   Elliptic Paraboloid

### *Example 2*   Classifying a Quadric Surface

Classify the surface given by $x - y^2 - z^2 = 0$. Describe the traces of the surface in the *xy*-plane, the *xz*-plane, and the plane given by $x = 1$.

**SOLUTION**   Because $x$ is raised only to the first power, the surface is a paraboloid whose axis is the *x*-axis, as shown in Figure 7.15. In standard form, the equation is

$$x = y^2 + z^2.$$

The traces in the *xy*-plane, the *xz*-plane, and the plane given by $x = 1$ are as shown.

| | | |
|---|---|---|
| Trace in *xy*-plane ($z = 0$): | $x = y^2$ | Parabola |
| Trace in *xz*-plane ($y = 0$): | $x = z^2$ | Parabola |
| Trace in plane $x = 1$: | $y^2 + z^2 = 1$ | Circle |

These three traces are shown in Figure 7.16. From the traces, you can see that the surface is an elliptic (or circular) paraboloid. If you have access to a three-dimensional graphing utility, try using it to graph this surface. If you do this, you will discover that sketching surfaces in space is not a simple task—even with a graphing utility.



*xy*-trace:
$x = y^2$

Parabola

*xz*-trace:
$x = z^2$

Parabola

Trace in the plane $x = 1$:
$y^2 + z^2 = 1$

Circle

**FIGURE 7.16**

### ✓ CHECKPOINT 2

Classify the surface given by $x^2 + y^2 - z^2 = 1$. Describe the traces of the surface in the *xy*-plane, the *yz*-plane, the *xz*-plane, and the plane given by $z = 3$. ∎

*Example 3*  **Classifying Quadric Surfaces**

Classify the surface given by each equation.

**a.** $x^2 - 4y^2 - 4z^2 - 4 = 0$

**b.** $x^2 + 4y^2 + z^2 - 4 = 0$

**SOLUTION**

**a.** The equation $x^2 - 4y^2 - 4z^2 - 4 = 0$ can be written in standard form as
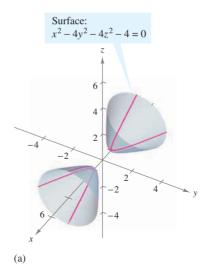
$$\frac{x^2}{4} - y^2 - z^2 = 1. \qquad \text{Standard form}$$
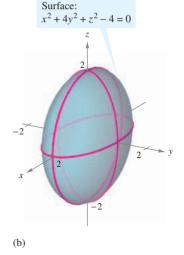
From the standard form, you can see that the graph is a hyperboloid of two sheets, with the $x$-axis as its axis, as shown in Figure 7.17(a).

**b.** The equation $x^2 + 4y^2 + z^2 - 4 = 0$ can be written in standard form as

$$\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1. \qquad \text{Standard form}$$

From the standard form, you can see that the graph is an ellipsoid, as shown in Figure 7.17(b).

✓ **CHECKPOINT 3**

Write each quadric surface in standard form and classify each equation.

**a.** $4x^2 + 9y^2 - 36z = 0$

**b.** $36x^2 + 16y^2 - 144z^2 = 0$ ■



Surface:
$x^2 - 4y^2 - 4z^2 - 4 = 0$

(a)



Surface:
$x^2 + 4y^2 + z^2 - 4 = 0$

(b)

**FIGURE 7.17**

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**CONCEPT CHECK**

1. Give the general equation of a plane in space.

2. List the six basic types of quadric surfaces.

3. Which types of quadric surfaces have equations that are of second degree in *all* three variables? Which types of quadric surfaces have equations that have one variable raised to the first power?

4. Is it possible for a plane in space to have fewer than three intercepts? If so, when does this occur?

**Skills Review 7.2**    The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.2 and 7.1.

In Exercises 1–4, find the $x$- and $y$-intercepts of the function.

**1.** $3x + 4y = 12$      **2.** $6x + y = -8$      **3.** $-2x + y = -2$      **4.** $-x - y = 5$

In Exercises 5–8, rewrite the expression by completing the square.

**5.** $x^2 + y^2 + z^2 - 2x - 4y - 6z + 15 = 0$      **6.** $x^2 + y^2 - z^2 - 8x + 4y - 6z + 11 = 0$

**7.** $z - 2 = x^2 + y^2 + 2x - 2y$      **8.** $x^2 + y^2 + z^2 - 6x + 10y + 26z = -202$

In Exercises 9 and 10, write the equation of the sphere in standard form.

**9.** $16x^2 + 16y^2 + 16z^2 = 4$      **10.** $9x^2 + 9y^2 + 9z^2 = 36$

## Exercises 7.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, find the intercepts and sketch the graph of the plane.

**1.** $4x + 2y + 6z = 12$      **2.** $3x + 6y + 2z = 6$

**3.** $3x + 3y + 5z = 15$      **4.** $x + y + z = 3$

**5.** $2x - y + 3z = 4$      **6.** $2x - y + z = 4$

**7.** $z = 8$      **8.** $x = 5$

**9.** $y + z = 5$      **10.** $x + 2y = 4$
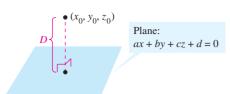
**11.** $x + y - z = 0$      **12.** $x - 3z = 3$

In Exercises 13–20, find the distance between the point and the plane (see figure). The distance $D$ between a point $(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Plane:
$ax + by + cz + d = 0$

**13.** $(0, 0, 0),\ 2x + 3y + z = 12$

**14.** $(0, 0, 0),\ 8x - 4y + z = 8$

**15.** $(1, 5, -4),\ 3x - y + 2z = 6$

**16.** $(3, 2, 1),\ x - y + 2z = 4$

**17.** $(1, 0, -1),\ 2x - 4y + 3z = 12$

**18.** $(2, -1, 0),\ 3x + 3y + 2z = 6$

**19.** $(3, 2, -1),\ 2x - 3y + 4z = 24$

**20.** $(-2, 1, 0),\ 2x + 5y - z = 20$

In Exercises 21–30, determine whether the planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ are parallel, perpendicular, or neither. The planes are parallel if there exists a nonzero constant $k$ such that $a_1 = ka_2$, $b_1 = kb_2$, and $c_1 = kc_2$, and are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

**21.** $5x - 3y + z = 4,\ x + 4y + 7z = 1$

**22.** $3x + y - 4z = 3,\ -9x - 3y + 12z = 4$

**23.** $x - 5y - z = 1,\ 5x - 25y - 5z = -3$

**24.** $x + 3y + 2z = 6,\ 4x - 12y + 8z = 24$

**25.** $x + 2y = 3,\ 4x + 8y = 5$

**26.** $x + 3y + z = 7,\ x - 5z = 0$
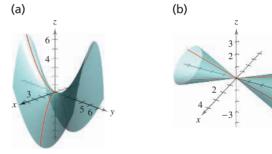
**27.** $2x + y = 3,\ 3x - 5z = 0$

**28.** $2x - z = 1,\ 4x + y + 8z = 10$

**29.** $x = 6,\ y = -1$
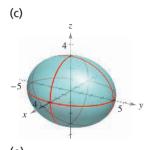
**30.** $x = -2,\ y = 4$
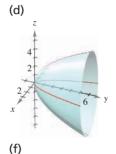
In Exercises 31–36, match the equation with its graph. [The graphs are labeled (a)–(f).]

(a)



(b)

(c)



(d)



(e)



(f)
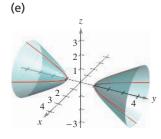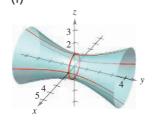


**31.** $\dfrac{x^2}{9} + \dfrac{y^2}{16} + \dfrac{z^2}{9} = 1$

**32.** $15x^2 - 4y^2 + 15z^2 = -4$

**33.** $4x^2 - y^2 + 4z^2 = 4$

**34.** $y^2 = 4x^2 + 9z^2$

**35.** $4x^2 - 4y + z^2 = 0$

**36.** $4x^2 - y^2 + 4z = 0$

In Exercises 37–40, describe the traces of the surface in the given planes.

| Surface | Planes |
|---|---|
| **37.** $x^2 - y - z^2 = 0$ | $xy$-plane, $y = 1$, $yz$-plane |
| **38.** $y = x^2 + z^2$ | $xy$-plane, $y = 1$, $yz$-plane |
| **39.** $\dfrac{x^2}{4} + y^2 + z^2 = 1$ | $xy$-plane, $xz$-plane, $yz$-plane |
| **40.** $y^2 + z^2 - x^2 = 1$ | $xy$-plane, $xz$-plane, $yz$-plane |

In Exercises 41–54, identify the quadric surface.

**41.** $x^2 + \dfrac{y^2}{4} + z^2 = 1$

**42.** $\dfrac{x^2}{9} + \dfrac{y^2}{16} + \dfrac{z^2}{16} = 1$

**43.** $25x^2 + 25y^2 - z^2 = 5$

**44.** $9x^2 + 4y^2 - 8z^2 = 72$

**45.** $x^2 - y + z^2 = 0$

**46.** $z = 4x^2 + y^2$

**47.** $x^2 - y^2 + z = 0$

**48.** $z^2 - x^2 - \dfrac{y^2}{4} = 1$

**49.** $2x^2 - y^2 + 2z^2 = -4$

**50.** $z^2 = x^2 + \dfrac{y^2}{4}$

**51.** $z^2 = 9x^2 + y^2$
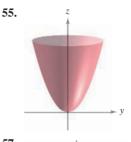
**52.** $4y = x^2 + z^2$

**53.** $3z = -y^2 + x^2$
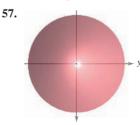
**54.** $z^2 = 2x^2 + 2y^2$
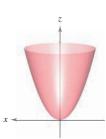
**Think About It**  In Exercises 55–58, each figure is a graph of the quadric surface $z = x^2 + y^2$. Match each of the four graphs with the point in space from which the paraboloid is viewed. The four points are (0, 0, 20), (0, 20, 0), (20, 0, 0), and (10, 10, 20).

**55.**



**56.**



**57.**



**58.**
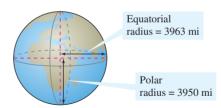


**59. Modeling Data**  Per capita consumptions (in gallons) of different types of plain milk in the United States from 1999 through 2004 are shown in the table. Consumption of reduced-fat (1%) and skim milks, reduced-fat milk (2%), and whole milk are represented by the variables $x$, $y$, and $z$, respectively.  *(Source: U.S. Department of Agriculture)*

| Year | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 |
|---|---|---|---|---|---|---|
| $x$ | 6.2 | 6.1 | 5.9 | 5.8 | 5.6 | 5.5 |
| $y$ | 7.3 | 7.1 | 7.0 | 7.0 | 6.9 | 6.9 |
| $z$ | 7.8 | 7.7 | 7.4 | 7.3 | 7.2 | 6.9 |

A model for the data in the table is given by $-1.25x + 0.125y + z = 0.95$.

(a) Complete a fourth row of the table using the model to approximate $z$ for the given values of $x$ and $y$. Compare the approximations with the actual values of $z$.

(b) According to this model, increases in consumption of milk types $y$ and $z$ would correspond to what kind of change in consumption of milk type $x$?

**60. Physical Science**  Because of the forces caused by its rotation, Earth is actually an oblate ellipsoid rather than a sphere. The equatorial radius is 3963 miles and the polar radius is 3950 miles. Find an equation of the ellipsoid. Assume that the center of Earth is at the origin and the $xy$-trace ($z = 0$) corresponds to the equator.



Equatorial
radius = 3963 mi

Polar
radius = 3950 mi

# Functions of Several Variables

- ■ Evaluate functions of several variables.
- ■ Find the domains and ranges of functions of several variables.
- ■ Read contour maps and sketch level curves of functions of two variables.
- ■ Use functions of several variables to answer questions about real-life situations.

## Functions of Several Variables

So far in this text, you have studied functions of a single independent variable. Many quantities in science, business, and technology, however, are functions not of one, but of two or more variables. For instance, the demand function for a product is often dependent on the price *and* the advertising, rather than on the price alone. The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples.

$$z = f(x, y) = x^2 + xy \qquad \text{Function of two variables}$$

$$\underbrace{\phantom{x, y}}_{\text{2 variables}}$$

and

$$w = f(x, y, z) = x + 2y - 3z \qquad \text{Function of three variables}$$

$$\underbrace{\phantom{x, y, z}}_{\text{3 variables}}$$

---

**Definition of a Function of Two Variables**

Let $D$ be a set of ordered pairs of real numbers. If to each ordered pair $(x, y)$ in $D$ there corresponds a unique real number $f(x, y)$, then $f$ is called a **function of $x$ and $y$.** The set $D$ is the **domain** of $f$, and the corresponding set of $z$-values is the **range** of $f$. Functions of three, four, or more variables are defined similarly.

---

### Example 1 Evaluating Functions of Several Variables

**a.** For $f(x, y) = 2x^2 - y^2$, you can evaluate $f(2, 3)$ as shown.

$$f(2, 3) = 2(2)^2 - (3)^2$$
$$= 8 - 9$$
$$= -1$$

**b.** For $f(x, y, z) = e^x(y + z)$, you can evaluate $f(0, -1, 4)$ as shown.

$$f(0, -1, 4) = e^0(-1 + 4)$$
$$= (1)(3)$$
$$= 3$$

✓ **CHECKPOINT 1**
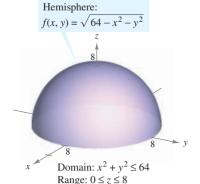
Find the function values of $f(x, y)$.

**a.** For $f(x, y) = x^2 + 2xy$, find $f(2, -1)$.

**b.** For $f(x, y, z) = \dfrac{2x^2z}{y^3}$, find $f(-3, 2, 1)$. ■

## The Graph of a Function of Two Variables

A function of two variables can be represented graphically as a surface in space by letting $z = f(x, y)$. When sketching the graph of a function of $x$ and $y$, remember that even though the graph is three-dimensional, the domain of the function is two-dimensional—it consists of the points in the $xy$-plane for which the function is defined. As with functions of a single variable, unless specifically restricted, the domain of a function of two variables is assumed to be the set of all points $(x, y)$ for which the defining equation has meaning. In other words, to each point $(x, y)$ in the domain of $f$ there corresponds a point $(x, y, z)$ on the surface, and conversely, to each point $(x, y, z)$ on the surface there corresponds a point $(x, y)$ in the domain of $f$.

Hemisphere:
$f(x, y) = \sqrt{64 - x^2 - y^2}$



Domain: $x^2 + y^2 \leq 64$
Range: $0 \leq z \leq 8$

**FIGURE 7.18**

### Example 2   Finding the Domain and Range of a Function

Find the domain and range of the function

$$f(x, y) = \sqrt{64 - x^2 - y^2}.$$

**SOLUTION**   Because no restrictions are given, the domain is assumed to be the set of all points for which the defining equation makes sense.

$$64 - x^2 - y^2 \geq 0 \qquad \text{Quantity inside radical must be nonnegative.}$$
$$x^2 + y^2 \leq 64 \qquad \text{Domain of the function}$$
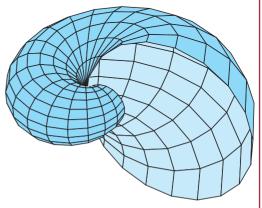
So, the domain is the set of all points that lie on or inside the circle given by $x^2 + y^2 = 8^2$. The range of $f$ is the set

$$0 \leq z \leq 8. \qquad \text{Range of the function}$$

As shown in Figure 7.18, the graph of the function is a hemisphere.

✓ **CHECKPOINT 2**

Find the domain and range of the function

$$f(x, y) = \sqrt{9 - x^2 - y^2}. \quad \blacksquare$$

### TECHNOLOGY

(T) Some three-dimensional graphing utilities can graph equations in $x$, $y$, and $z$. Others are programmed to graph only functions of $x$ and $y$. A surface in space represents the graph of a function of $x$ and $y$ only if each vertical line intersects the surface at most once. For instance, the surface shown in Figure 7.18 passes this vertical line test, but the surface at the right (drawn by *Mathematica*) does not represent the graph of a function of $x$ and $y$.
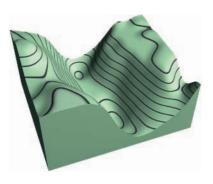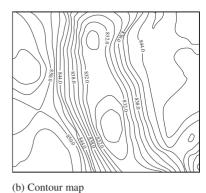


Some vertical lines intersect this surface more than once. So, the surface does not pass the Vertical Line Test and is not a function of $x$ and $y$.

## Contour Maps and Level Curves

A **contour map** of a surface is created by *projecting* traces, taken in evenly spaced planes that are parallel to the *xy*-plane, onto the *xy*-plane. Each projection is a **level curve** of the surface.

Contour maps are used to create weather maps, topographical maps, and population density maps. For instance, Figure 7.19(a) shows a graph of a "mountain and valley" surface given by $z = f(x, y)$. Each of the level curves in Figure 7.19(b) represents the intersection of the surface $z = f(x, y)$ with a plane $z = c$, where $c = 828, 830, \ldots, 854$.



(a) Surface

(b) Contour map

**FIGURE 7.19**

*Example 3*    Reading a Contour Map    Ⓡ

The "contour map" in Figure 7.20 was computer generated using data collected by satellite instrumentation. Color is used to show the "ozone hole" in Earth's atmosphere. The purple and blue areas represent the lowest levels of ozone and the green areas represent the highest level. Describe the areas that have the lowest levels of ozone.    *(Source: National Aeronautics and Space Administration)*

**SOLUTION**    The lowest levels of ozone are over Antarctica and the Antarctic Ocean. The ozone layer acts to protect life on Earth by blocking harmful ultraviolet rays from the sun. The "ozone hole" in the polar region of the Southern Hemisphere is an area in which there is a severe depletion of the ozone levels in the atmosphere. It is primarily caused by compounds that release chlorine and bromine gases into the atmosphere.



**FIGURE 7.20**

✓**CHECKPOINT 3**

When the level curves of a contour map are close together, is the surface represented by the contour map steep or nearly level? When the level curves of a contour map are far apart, is the surface represented by the contour map steep or nearly level? ■
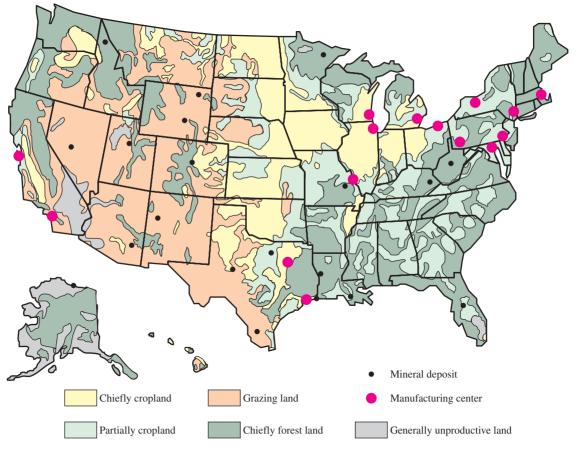
*Example 4*  **Reading a Contour Map**  Ⓡ

The contour map shown in Figure 7.21 represents the economy of the United States. Discuss the use of color to represent the level curves.  *(Source: U.S. Census Bureau)*

**SOLUTION**  You can see from the key that the light yellow regions are mainly used in crop production. The gray areas represent regions that are unproductive. Manufacturing centers are denoted by large red dots and mineral deposits are denoted by small black dots.

One advantage of such a map is that it allows you to "see" the components of the country's economy at a glance. From the map it is clear that the Midwest is responsible for most of the crop production in the United States.



● Mineral deposit
● Manufacturing center

| | | |
|---|---|---|
| Chiefly cropland | Grazing land | |
| Partially cropland | Chiefly forest land | Generally unproductive land |

**FIGURE 7.21**

✓**CHECKPOINT 4**

Use Figure 7.21 to describe how Alaska contributes to the U.S. economy. Does Alaska contain any manufacturing centers? Does Alaska contain any mineral deposits? ■

## Applications

The **Cobb-Douglas production function** is used in economics to represent the numbers of units produced by varying amounts of labor and capital. Let $x$ represent the number of units of labor and let $y$ represent the number of units of capital. Then, the number of units produced is modeled by

$$f(x, y) = Cx^a y^{1-a}$$

where $C$ is a constant and $0 < a < 1$.

### Example 5 Using a Production Function ®

A manufacturer estimates that its production (measured in units of a product) can be modeled by $f(x, y) = 100x^{0.6}y^{0.4}$, where the labor $x$ is measured in person-hours and the capital $y$ is measured in thousands of dollars.

**a.** What is the production level when $x = 1000$ and $y = 500$?

**b.** What is the production level when $x = 2000$ and $y = 1000$?

**c.** How does doubling the amounts of labor and capital from part (a) to part (b) affect the production?

**SOLUTION**

**a.** When $x = 1000$ and $y = 500$, the production level is
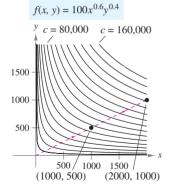
$$f(1000, 500) = 100(1000)^{0.6}(500)^{0.4}$$
$$\approx 75{,}786 \text{ units.}$$

**b.** When $x = 2000$ and $y = 1000$, the production level is

$$f(2000, 1000) = 100(2000)^{0.6}(1000)^{0.4}$$
$$\approx 151{,}572 \text{ units.}$$

**c.** When the amounts of labor and capital are doubled, the production level also doubles. In Exercise 42, you are asked to show that this is characteristic of the Cobb-Douglas production function.

A contour graph of this function is shown in Figure 7.22.



$f(x, y) = 100x^{0.6}y^{0.4}$

$c = 80,000$ $c = 160,000$

(1000, 500)
(2000, 1000)

**FIGURE 7.22** Level Curves (at Increments of 10,000)

### ✓ CHECKPOINT 5

Use the Cobb-Douglas production function in Example 5 to find the production levels when $x = 1500$ and $y = 1000$ and when $x = 1000$ and $y = 1500$. Use your results to determine which variable has a greater influence on production.

---

**STUDY TIP**

In Figure 7.22, note that the level curves of the function

$$f(x, y) = 100x^{0.6}y^{0.4}$$

occur at increments of 10,000.

*Example 6*  **Finding Monthly Payments**  Ⓡ

The monthly payment $M$ for an installment loan of $P$ dollars taken out over $t$ years at an annual interest rate of $r$ is given by

$$M = f(P, r, t) = \frac{\dfrac{Pr}{12}}{1 - \left[\dfrac{1}{1 + (r/12)}\right]^{12t}}.$$

**a.** Find the monthly payment for a home mortgage of $100,000 taken out for 30 years at an annual interest rate of 7%.

**b.** Find the monthly payment for a car loan of $22,000 taken out for 5 years at an annual interest rate of 8%.

**SOLUTION**

**a.** If $P = \$100,000$, $r = 0.07$, and $t = 30$, then the monthly payment is

$$M = f(100{,}000, 0.07, 30)$$

$$= \frac{\dfrac{(100{,}000)(0.07)}{12}}{1 - \left[\dfrac{1}{1 + (0.07/12)}\right]^{12(30)}}$$

$$\approx \$665.30.$$

**b.** If $P = \$22,000$, $r = 0.08$, and $t = 5$, then the monthly payment is

$$M = f(22{,}000, 0.08, 5)$$

$$= \frac{\dfrac{(22{,}000)(0.08)}{12}}{1 - \left[\dfrac{1}{1 + (0.08/12)}\right]^{12(5)}}$$

$$\approx \$446.08.$$

For many Americans, buying a house is the largest single purchase they will ever make. During the 1970s, 1980s, and 1990s, the annual interest rate on home mortgages varied drastically. It was as high as 18% and as low as 5%. Such variations can change monthly payments by hundreds of dollars.

Kayte M. Deioma/PhotoEdit

✓ **CHECKPOINT 6**

**a.** Find the monthly payment $M$ for a home mortgage of $100,000 taken out for 30 years at an annual interest rate of 8%.

**b.** Find the total amount of money you will pay for the mortgage.  ■

**CONCEPT CHECK**

1. The function $f(x, y) = x + y$ is a function of how many variables?

2. What is a graph of a function of two variables?

3. Give a description of the domain of a function of two variables.

4. How is a contour map created? What is a level curve?

In Exercises 1–4, evaluate the function when $x = -3$.

**1.** $f(x) = 5 - 2x$

**2.** $f(x) = -x^2 + 4x + 5$

**3.** $y = \sqrt{4x^2 - 3x + 4}$

**4.** $y = \sqrt[3]{34 - 4x + 2x^2}$

In Exercises 5–8, find the domain of the function.

**5.** $f(x) = 5x^2 + 3x - 2$

**6.** $g(x) = \dfrac{1}{2x} - \dfrac{2}{x + 3}$

**7.** $h(y) = \sqrt{y - 5}$

**8.** $f(y) = \sqrt{y^2 - 5}$

In Exercises 9 and 10, evaluate the expression.

**9.** $(476)^{0.65}$

**10.** $(251)^{0.35}$

## Exercises 7.3

In Exercises 1–14, find the function values.

**1.** $f(x, y) = \dfrac{x}{y}$

  (a) $f(3, 2)$      (b) $f(-1, 4)$      (c) $f(30, 5)$

  (d) $f(5, y)$      (e) $f(x, 2)$      (f) $f(5, t)$

**2.** $f(x, y) = 4 - x^2 - 4y^2$

  (a) $f(0, 0)$      (b) $f(0, 1)$      (c) $f(2, 3)$

  (d) $f(1, y)$      (e) $f(x, 0)$      (f) $f(t, 1)$

**3.** $f(x, y) = xe^y$

  (a) $f(5, 0)$      (b) $f(3, 2)$      (c) $f(2, -1)$

  (d) $f(5, y)$      (e) $f(x, 2)$      (f) $f(t, t)$

**4.** $g(x, y) = \ln|x + y|$

  (a) $g(2, 3)$      (b) $g(5, 6)$      (c) $g(e, 0)$

  (d) $g(0, 1)$      (e) $g(2, -3)$      (f) $g(e, e)$

**5.** $h(x, y, z) = \dfrac{xy}{z}$

  (a) $h(2, 3, 9)$           (b) $h(1, 0, 1)$

**6.** $f(x, y, z) = \sqrt{x + y + z}$

  (a) $f(0, 5, 4)$           (b) $f(6, 8, -3)$

**7.** $V(r, h) = \pi r^2 h$

  (a) $V(3, 10)$           (b) $V(5, 2)$

**8.** $F(r, N) = 500\left(1 + \dfrac{r}{12}\right)^N$

  (a) $F(0.09, 60)$           (b) $F(0.14, 240)$

**9.** $A(P, r, t) = P\left[\left(1 + \dfrac{r}{12}\right)^{12t} - 1\right]\left(1 + \dfrac{12}{r}\right)$

  (a) $A(100, 0.10, 10)$      (b) $A(275, 0.0925, 40)$

**10.** $A(P, r, t) = Pe^{rt}$

  (a) $A(500, 0.10, 5)$      (b) $A(1500, 0.12, 20)$

**11.** $f(x, y) = \displaystyle\int_x^y (2t - 3)\, dt$

  (a) $f(1, 2)$           (b) $f(1, 4)$

**12.** $g(x, y) = \displaystyle\int_x^y \dfrac{1}{t}\, dt$

  (a) $g(4, 1)$           (b) $g(6, 3)$

**13.** $f(x, y) = x^2 - 2y$

  (a) $f(x + \Delta x, y)$      (b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

**14.** $f(x, y) = 3xy + y^2$

  (a) $f(x + \Delta x, y)$      (b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

In Exercises 15–18, describe the region $R$ in the $xy$-plane that corresponds to the domain of the function, and find the range of the function.

**15.** $f(x, y) = \sqrt{16 - x^2 - y^2}$

**16.** $f(x, y) = x^2 + y^2 - 1$

**17.** $f(x, y) = e^{x/y}$

**18.** $f(x, y) = \ln(x + y)$

In Exercises 19–28, describe the region $R$ in the $xy$-plane that corresponds to the domain of the function.

**19.** $z = \sqrt{4 - x^2 - y^2}$

**20.** $z = \sqrt{4 - x^2 - 4y^2}$

**21.** $f(x, y) = x^2 + y^2$

**22.** $f(x, y) = \dfrac{x}{y}$

**23.** $f(x, y) = \dfrac{1}{xy}$

**24.** $g(x, y) = \dfrac{1}{x - y}$

**25.** $h(x, y) = x\sqrt{y}$
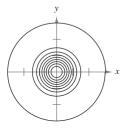
**26.** $f(x, y) = \sqrt{xy}$

**27.** $g(x, y) = \ln(4 - x - y)$

**28.** $f(x, y) = ye^{1/x}$
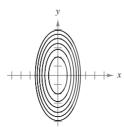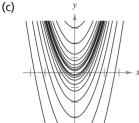
In Exercises 29–32, match the graph of the surface with one of the contour maps. [The contour maps are labeled (a)–(d).]

(a)

(b)

(c)

(d)

**29.** $f(x, y) = x^2 + \dfrac{y^2}{4}$

**30.** $f(x, y) = e^{1 - x^2 + y^2}$
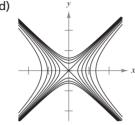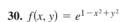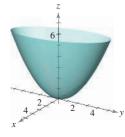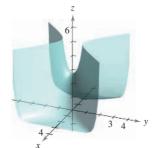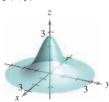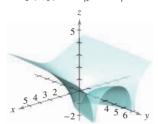




**31.** $f(x, y) = e^{1 - x^2 - y^2}$

**32.** $f(x, y) = \ln|y - x^2|$





In Exercises 33–40, describe the level curves of the function. Sketch the level curves for the given c-values.

| *Function* | *c-Values* |
|---|---|
| **33.** $z = x + y$ | $c = -1, 0, 2, 4$ |
| **34.** $z = 6 - 2x - 3y$ | $c = 0, 2, 4, 6, 8, 10$ |
| **35.** $z = \sqrt{25 - x^2 - y^2}$ | $c = 0, 1, 2, 3, 4, 5$ |
| **36.** $f(x, y) = x^2 + y^2$ | $c = 0, 2, 4, 6, 8$ |
| **37.** $f(x, y) = xy$ | $c = \pm 1, \pm 2, \ldots, \pm 6$ |
| **38.** $z = e^{xy}$ | $c = 1, 2, 3, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ |
| **39.** $f(x, y) = \dfrac{x}{x^2 + y^2}$ | $c = \pm\dfrac{1}{2}, \pm 1, \pm\dfrac{3}{2}, \pm 2$ |
| **40.** $f(x, y) = \ln(x - y)$ | $c = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$ |

**41. Cobb-Douglas Production Function**   A manufacturer estimates the Cobb-Douglas production function to be given by

$$f(x, y) = 100x^{0.75}y^{0.25}.$$

Estimate the production levels when $x = 1500$ and $y = 1000$.

**42. Cobb-Douglas Production Function**   Use the Cobb-Douglas production function (Example 5) to show that if both the number of units of labor and the number of units of capital are doubled, the production level is also doubled.

**43. Profit**   A sporting goods manufacturer produces regulation soccer balls at two plants. The costs of producing $x_1$ units at location 1 and $x_2$ units at location 2 are given by

$$C_1(x_1) = 0.02x_1^2 + 4x_1 + 500$$

and

$$C_2(x_2) = 0.05x_2^2 + 4x_2 + 275$$

respectively. If the product sells for $50 per unit, then the profit function for the product is given by

$$P(x_1, x_2) = 50(x_1 + x_2) - C_1(x_1) - C_2(x_2).$$

Find (a) $P(250, 150)$ and (b) $P(300, 200)$.

**44. Queuing Model**   The average amount of time that a customer waits in line for service is given by

$$W(x, y) = \dfrac{1}{x - y}, \quad y < x$$

where $y$ is the average arrival rate and $x$ is the average service rate ($x$ and $y$ are measured in the number of customers per hour). Evaluate $W$ at each point.

(a) $(15, 10)$   (b) $(12, 9)$   (c) $(12, 6)$   (d) $(4, 2)$

**45. Investment** In 2008, an investment of $1000 was made in a bond earning 10% compounded annually. The investor pays tax at rate $R$, and the annual rate of inflation is $I$. In the year 2018, the value $V$ of the bond in constant 2008 dollars is given by

$$V(I, R) = 1000 \left[ \frac{1 + 0.10(1 - R)}{1 + I} \right]^{10}.$$
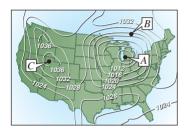
Use this function of two variables and a spreadsheet to complete the table.

|          | Inflation Rate | | |
|----------|------|------|------|
| Tax Rate | 0 | 0.03 | 0.05 |
| 0        |   |   |   |
| 0.28     |   |   |   |
| 0.35     |   |   |   |

**46. Investment** A principal of $1000 is deposited in a savings account that earns an interest rate of $r$ (written as a decimal), compounded continuously. The amount $A(r, t)$ after $t$ years is $A(r, t) = 1000 \, e^{rt}$. Use this function of two variables and a spreadsheet to complete the table.
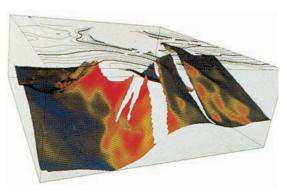
|      | Number of Years | | | |
|------|---|----|----|----|
| Rate | 5 | 10 | 15 | 20 |
| 0.02 |   |    |    |    |
| 0.04 |   |    |    |    |
| 0.06 |   |    |    |    |
| 0.08 |   |    |    |    |

**47. Meteorology** Meteorologists measure the atmospheric pressure in millibars. From these observations they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). On the map, the closer the isobars the higher the wind speed. Match points $A$, $B$, and $C$ with (a) highest pressure, (b) lowest pressure, and (c) highest wind velocity.



**48. Geology** The contour map below represents color-coded seismic amplitudes of a fault horizon and a projected contour map, which is used in earthquake studies. *(Source: Adapted from Shipman/Wilson/Todd, An Introduction to Physical Science, Tenth Edition)*



Shipman, *An Introduction to Physical Science* 10/e, 2003, Houghton Mifflin Company

(a) Discuss the use of color to represent the level curves.

(b) Do the level curves correspond to equally spaced amplitudes? Explain your reasoning.

**49. Earnings per Share** The earnings per share $z$ (in dollars) for Starbucks Corporation from 1998 through 2006 can be modeled by $z = 0.106x - 0.036y - 0.005$, where $x$ is sales (in billions of dollars) and $y$ is the shareholder's equity (in billions of dollars). *(Source: Starbucks Corporation)*

(a) Find the earnings per share when $x = 8$ and $y = 5$.

(b) Which of the two variables in this model has the greater influence on the earnings per share? Explain.

**50. Shareholder's Equity** The shareholder's equity $z$ (in billions of dollars) for Wal-Mart Corporation from 2000 to 2006 can be modeled by $z = 0.205x - 0.073y - 0.728$, where $x$ is net sales (in billions of dollars) and $y$ is the total assets (in billions of dollars). *(Source: Wal-Mart Corporation)*

(a) Find the shareholder's equity when $x = 300$ and $y = 130$.

(b) Which of the two variables in this model has the greater influence on shareholder's equity? Explain.

**51. MAKE A DECISION: MONTHLY PAYMENTS** You are taking out a home mortgage for $120,000, and you are given the options below. Which option would you choose? Explain your reasoning.

(a) A fixed annual rate of 8%, over a term of 20 years.

(b) A fixed annual rate of 7%, over a term of 30 years.

(c) An adjustable annual rate of 7%, over a term of 20 years. The annual rate can fluctuate—each year it is set at 1% above the prime rate.

(d) A fixed annual rate of 7%, over a term of 15 years.

## Section 7.4

# Partial Derivatives

- Find the first partial derivatives of functions of two variables.
- Find the slopes of surfaces in the $x$- and $y$-directions and use partial derivatives to answer questions about real-life situations.
- Find the partial derivatives of functions of several variables.
- Find higher-order partial derivatives.

## Functions of Two Variables

Real-life applications of functions of several variables are often concerned with how changes in one of the variables will affect the values of the functions. For instance, an economist who wants to determine the effect of a tax increase on the economy might make calculations using different tax rates while holding all other variables, such as unemployment, constant.

You can follow a similar procedure to find the rate of change of a function $f$ with respect to one of its independent variables. That is, you find the derivative of $f$ with respect to one independent variable, while holding the other variable(s) constant. This process is called **partial differentiation,** and each derivative is called a **partial derivative.** A function of several variables has as many partial derivatives as it has independent variables.

**STUDY TIP**

Note that this definition indicates that partial derivatives of a function of two variables are determined by temporarily considering one variable to be fixed. For instance, if $z = f(x, y)$, then to find $\partial z/\partial x$, you consider $y$ to be constant and differentiate with respect to $x$. Similarly, to find $\partial z/\partial y$, you consider $x$ to be constant and differentiate with respect to $y$.

**Partial Derivatives of a Function of Two Variables**

If $z = f(x, y)$, then the **first partial derivatives of $f$ with respect to $x$ and $y$** are the functions $\partial z/\partial x$ and $\partial z/\partial y$, defined as shown.

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \qquad \textcolor{red}{y \text{ is held constant.}}$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \qquad \textcolor{red}{x \text{ is held constant.}}$$

### *Example 1*  Finding Partial Derivatives

Find $\partial z/\partial x$ and $\partial z/\partial y$ for the function $z = 3x - x^2y^2 + 2x^3y$.

**SOLUTION**

$$\frac{\partial z}{\partial x} = 3 - 2xy^2 + 6x^2y \qquad \textcolor{red}{\text{Hold } y \text{ constant and differentiate with respect to } x.}$$

$$\frac{\partial z}{\partial y} = -2x^2y + 2x^3 \qquad \textcolor{red}{\text{Hold } x \text{ constant and differentiate with respect to } y.}$$

**✓CHECKPOINT 1**

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = 2x^2 - 4x^2y^3 + y^4$. ■

**Notation for First Partial Derivatives**

The first partial derivatives of $z = f(x, y)$ are denoted by

$$\frac{\partial z}{\partial x} = f_x(x, y) = z_x = \frac{\partial}{\partial x}[f(x, y)]$$

and

$$\frac{\partial z}{\partial y} = f_y(x, y) = z_y = \frac{\partial}{\partial y}[f(x, y)].$$

The values of the first partial derivatives at the point $(a, b)$ are denoted by

$$\frac{\partial z}{\partial x}\bigg|_{(a, b)} = f_x(a, b) \quad \text{and} \quad \frac{\partial z}{\partial y}\bigg|_{(a, b)} = f_y(a, b).$$

**TECHNOLOGY**

(T) Symbolic differentiation utilities can be used to find partial derivatives of a function of two variables. Try using a symbolic differentiation utility to find the first partial derivatives of the function in Example 2.

*Example 2* **Finding and Evaluating Partial Derivatives**

Find the first partial derivatives of $f(x, y) = xe^{x^2 y}$ and evaluate each at the point $(1, \ln 2)$.

**SOLUTION**   To find the first partial derivative with respect to $x$, hold $y$ constant and differentiate using the Product Rule.

$$f_x(x, y) = x\frac{\partial}{\partial x}[e^{x^2 y}] + e^{x^2 y}\frac{\partial}{\partial x}[x] \qquad \text{Apply Product Rule.}$$

$$= x(2xy)e^{x^2 y} + e^{x^2 y} \qquad \text{$y$ is held constant.}$$

$$= e^{x^2 y}(2x^2 y + 1) \qquad \text{Simplify.}$$

At the point $(1, \ln 2)$, the value of this derivative is

$$f_x(1, \ln 2) = e^{(1)^2(\ln 2)}[2(1)^2(\ln 2) + 1] \qquad \text{Substitute for $x$ and $y$.}$$

$$= 2(2 \ln 2 + 1) \qquad \text{Simplify.}$$

$$\approx 4.773. \qquad \text{Use a calculator.}$$

To find the first partial derivative with respect to $y$, hold $x$ constant and differentiate to obtain

$$f_y(x, y) = x(x^2)e^{x^2 y} \qquad \text{Apply Constant Multiple Rule.}$$

$$= x^3 e^{x^2 y}. \qquad \text{Simplify.}$$

At the point $(1, \ln 2)$, the value of this derivative is

$$f_y(1, \ln 2) = (1)^3 e^{(1)^2(\ln 2)} \qquad \text{Substitute for $x$ and $y$.}$$

$$= 2. \qquad \text{Simplify.}$$

✓**CHECKPOINT 2**

Find the first partial derivatives of $f(x, y) = x^2 y^3$ and evaluate each at the point $(1, 2)$. ■

## Graphical Interpretation of Partial Derivatives

At the beginning of this course, you studied graphical interpretations of the derivative of a function of a single variable. There, you found that $f'(x_0)$ represents the slope of the tangent line to the graph of $y = f(x)$ at the point $(x_0, y_0)$. The partial derivatives of a function of two variables also have useful graphical interpretations. Consider the function

$z = f(x, y).$          Function of two variables

As shown in Figure 7.23(a), the graph of this function is a surface in space. If the variable $y$ is fixed, say at $y = y_0$, then

$z = f(x, y_0)$          Function of one variable

is a function of one variable. The graph of this function is the curve that is the intersection of the plane $y = y_0$ and the surface $z = f(x, y)$. On this curve, the partial derivative

$f_x(x, y_0)$          Slope in $x$-direction
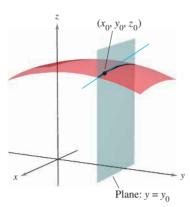
represents the slope in the plane $y = y_0$, as shown in Figure 7.23(a). In a similar way, if the variable $x$ is fixed, say at $x = x_0$, then

$z = f(x_0, y)$          Function of one variable

is a function of one variable. Its graph is the intersection of the plane $x = x_0$ and the surface $z = f(x, y)$. On this curve, the partial derivative

$f_y(x_0, y)$          Slope in $y$-direction

represents the slope in the plane $x = x_0$, as shown in Figure 7.23(b).



(a) $f_x(x, y_0)$ = slope in $x$-direction          (b) $f_y(x_0, y)$ = slope in $y$-direction

**FIGURE 7.23**

### DISCOVERY

How can partial derivatives be used to find *relative extrema* of graphs of functions of two variables?

Find the slopes of the surface given by

$$f(x, y) = 4x^2 + 9y^2 + 36$$

at the point $(1, -1, 49)$ in the $x$-direction and the $y$-direction. ■

*Example 3*   **Finding Slopes in the $x$- and $y$-Directions**

Find the slopes of the surface given by

$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$ in (a) the $x$-direction and (b) the $y$-direction.

**SOLUTION**

a. To find the slope in the $x$-direction, hold $y$ constant and differentiate with respect to $x$ to obtain

$$f_x(x, y) = -x. \qquad \text{Partial derivative with respect to } x$$

At the point $\left(\frac{1}{2}, 1, 2\right)$, the slope in the $x$-direction is

$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2} \qquad \text{Slope in } x\text{-direction}$$

as shown in Figure 7.24(a).

b. To find the slope in the $y$-direction, hold $x$ constant and differentiate with respect to $y$ to obtain

$$f_y(x, y) = -2y. \qquad \text{Partial derivative with respect to } y$$

At the point $\left(\frac{1}{2}, 1, 2\right)$, the slope in the $y$-direction is

$$f_y\left(\frac{1}{2}, 1\right) = -2 \qquad \text{Slope in } y\text{-direction}$$

as shown in Figure 7.24(b).
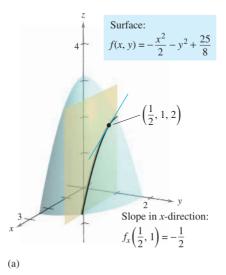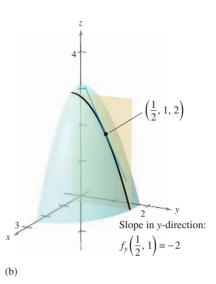
**DISCOVERY**

Find the partial derivatives $f_x$ and $f_y$ at $(0, 0)$ for the function in Example 3. What are the slopes of $f$ in the $x$- and $y$-directions at $(0, 0)$? Describe the shape of the graph of $f$ at this point.



Surface:
$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

$\left(\frac{1}{2}, 1, 2\right)$

(a)   Slope in $x$-direction:
$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2}$$

(b)   Slope in $y$-direction:
$$f_y\left(\frac{1}{2}, 1\right) = -2$$

**FIGURE 7.24**

Consumer products in the same market or in related markets can be classified as **complementary** or **substitute products.** If two products have a complementary relationship, an increase in the sale of one product will be accompanied by an increase in the sale of the other product. For instance, DVD players and DVDs have a complementary relationship.

If two products have a substitute relationship, an increase in the sale of one product will be accompanied by a decrease in the sale of the other product. For instance, videocassette recorders and DVD players both compete in the same home entertainment market and you would expect a drop in the price of one to be a deterrent to the sale of the other.

**Example 4**  **Examining Demand Functions**  Ⓡ

The demand functions for two products are represented by

$$x_1 = f(p_1, p_2) \quad \text{and} \quad x_2 = g(p_1, p_2)$$

where $p_1$ and $p_2$ are the prices per unit for the two products, and $x_1$ and $x_2$ are the numbers of units sold. The graphs of two different demand functions for $x_1$ are shown below. Use them to classify the products as complementary or substitute products.



(a)                                            (b)

**FIGURE 7.25**

**SOLUTION**

a. Notice that Figure 7.25(a) represents the demand for the *first product*. From the graph of this function, you can see that for a fixed price $p_1$, an increase in $p_2$ results in an increase in the demand for the first product. Remember that an increase in $p_2$ will also result in a decrease in the demand for the second product. So, if $\partial f/\partial p_2 > 0$, the two products have a *substitute* relationship.

b. Notice that Figure 7.25(b) represents a different demand for the *first product*. From the graph of this function, you can see that for a fixed price $p_1$, an increase in $p_2$ results in a decrease in the demand for the first product. Remember that an increase in $p_2$ will also result in a decrease in the demand for the second product. So, if $\partial f/\partial p_2 < 0$, the two products have a *complementary* relationship.

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✓**CHECKPOINT 4**

Determine if the demand functions below describe a complementary or a substitute product relationship.
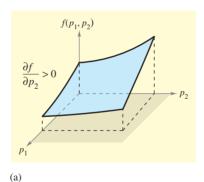
$$x_1 = 100 - 2p_1 + 1.5p_2$$
$$x_2 = 145 + \tfrac{1}{2}p_1 - \tfrac{3}{4}p_2 \ ■$$



AP/World Wide Photos

In 2007, Subway was chosen as the number one franchise by Entrepreneur Magazine. By the end of the year 2006, Subway had a total of 26,197 franchises worldwide. What type of product would be complementary to a Subway sandwich? What type of product would be a substitute?

## Functions of Three Variables

The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, the function $w = f(x, y, z)$ has three partial derivatives, each of which is formed by considering two of the variables to be constant. That is, to define the partial derivative of $w$ with respect to $x$, consider $y$ *and* $z$ to be constant and write

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}.$$

To define the partial derivative of $w$ with respect to $y$, consider $x$ *and* $z$ to be constant and write

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}.$$

To define the partial derivative of $w$ with respect to $z$, consider $x$ *and* $y$ to be constant and write

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}.$$

### Example 5   Finding Partial Derivatives of a Function

Find the three partial derivatives of the function

$$w = xe^{xy + 2z}.$$

**SOLUTION**   Holding $y$ and $z$ constant, you obtain

$$\frac{\partial w}{\partial x} = x\frac{\partial}{\partial x}[e^{xy+2z}] + e^{xy+2z}\frac{\partial}{\partial x}[x] \qquad \text{Apply Product Rule.}$$

$$= x(ye^{xy+2z}) + e^{xy+2z}(1) \qquad \text{Hold } y \text{ and } z \text{ constant.}$$

$$= (xy + 1)e^{xy+2z}. \qquad \text{Simplify.}$$

Holding $x$ and $z$ constant, you obtain

$$\frac{\partial w}{\partial y} = x(x)e^{xy+2z} \qquad \text{Hold } x \text{ and } z \text{ constant.}$$

$$= x^2e^{xy+2z}. \qquad \text{Simplify.}$$

Holding $x$ and $y$ constant, you obtain

$$\frac{\partial w}{\partial z} = x(2)e^{xy+2z} \qquad \text{Hold } x \text{ and } y \text{ constant.}$$

$$= 2xe^{xy+2z}. \qquad \text{Simplify.}$$

### ✓CHECKPOINT 5

Find the three partial derivatives of the function

$$w = x^2y \ln(xz). \blacksquare$$

---

**TECHNOLOGY**

(T) A symbolic differentiation utility can be used to find the partial derivatives of a function of three or more variables. Try using a symbolic differentiation utility to find the partial derivative $f_y(x, y, z)$ for the function in Example 5.

---

**STUDY TIP**

Note that in Example 5 the Product Rule was used only when finding the partial derivative with respect to $x$. Can you see why?

## Higher-Order Partial Derivatives

As with ordinary derivatives, it is possible to take second, third, and higher partial derivatives of a function of several variables, provided such derivatives exist. Higher-order derivatives are denoted by the order in which the differentiation occurs. For instance, there are four different ways to find a second partial derivative of $z = f(x, y)$.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \text{Differentiate twice with respect to } x.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} \qquad \text{Differentiate twice with respect to } y.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \qquad \text{Differentiate first with respect to } x \text{ and then with respect to } y.$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \qquad \text{Differentiate first with respect to } y \text{ and then with respect to } x.$$

The third and fourth cases are **mixed partial derivatives.** Notice that with the two types of notation for mixed partials, different conventions are used for indicating the order of differentiation. For instance, the partial derivative

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \qquad \text{Right-to-left order}$$

indicates differentiation with respect to $x$ first, but the partial derivative

$$(f_y)_x = f_{yx} \qquad \text{Left-to-right order}$$

indicates differentiation with respect to $y$ first. To remember this, note that in each case you differentiate first with respect to the variable "nearest" $f$.

### Example 6 Finding Second Partial Derivatives

Find the second partial derivatives of

$$f(x, y) = 3xy^2 - 2y + 5x^2y^2$$

and determine the value of $f_{xy}(-1, 2)$.

**SOLUTION** Begin by finding the first partial derivatives.

$$f_x(x, y) = 3y^2 + 10xy^2 \qquad f_y(x, y) = 6xy - 2 + 10x^2y$$

Then, differentiating with respect to $x$ and $y$ produces

$$f_{xx}(x, y) = 10y^2, \qquad f_{yy}(x, y) = 6x + 10x^2$$
$$f_{xy}(x, y) = 6y + 20xy, \qquad f_{yx}(x, y) = 6y + 20xy.$$

Finally, the value of $f_{xy}(x, y)$ at the point $(-1, 2)$ is

$$f_{xy}(-1, 2) = 6(2) + 20(-1)(2) = 12 - 40 = -28.$$

✓**CHECKPOINT 6**

Find the second partial derivatives of

$$f(x, y) = 4x^2y^2 + 2x + 4y^2. \ ∎$$

A function of two variables has two first partial derivatives and four second partial derivatives. For a function of three variables, there are three first partials

$$f_x, f_y, \quad \text{and} \quad f_z$$

and nine second partials

$$f_{xx}, f_{xy}, f_{xz}, f_{yx}, f_{yy}, f_{yz}, f_{zx}, f_{zy}, \quad \text{and} \quad f_{zz}$$

of which six are mixed partials. To find partial derivatives of order three and higher, follow the same pattern used to find second partial derivatives. For instance, if $z = f(x, y)$, then

$$z_{xxx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3} \quad \text{and} \quad z_{xxy} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial y \partial x^2}.$$

### *Example 7*  Finding Second Partial Derivatives

Find the second partial derivatives of

$$f(x, y, z) = ye^x + x \ln z.$$

**SOLUTION**    Begin by finding the first partial derivatives.

$$f_x(x, y, z) = ye^x + \ln z, \qquad f_y(x, y, z) = e^x, \qquad f_z(x, y, z) = \frac{x}{z}$$

Then, differentiate with respect to $x$, $y$, and $z$ to find the nine second partial derivatives.

$$f_{xx}(x, y, z) = ye^x, \qquad f_{xy}(x, y, z) = e^x, \qquad f_{xz}(x, y, z) = \frac{1}{z}$$

$$f_{yx}(x, y, z) = e^x, \qquad f_{yy}(x, y, z) = 0, \qquad f_{yz}(x, y, z) = 0$$

$$f_{zx}(x, y, z) = \frac{1}{z}, \qquad f_{zy}(x, y, z) = 0, \qquad f_{zz}(x, y, z) = -\frac{x}{z^2}$$

**✓ CHECKPOINT 7**

Find the second partial derivatives of $f(x, y, z) = xe^y + 2xz + y^2$. ■

---

**CONCEPT CHECK**

1. Write the notation that denotes the first partial derivative of $z = f(x, y)$ with respect to $x$.

2. Write the notation that denotes the first partial derivative of $z = f(x, y)$ with respect to $y$.

3. Let $f$ be a function of two variables $x$ and $y$. Describe the procedure for finding the first partial derivatives.

4. Define the first partial derivatives of a function $f$ of two variables $x$ and $y$.

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.2, 2.4, 2.5, 4.3, and 4.5.

In Exercises 1–8, find the derivative of the function.

**1.** $f(x) = \sqrt{x^2 + 3}$

**2.** $g(x) = (3 - x^2)^3$

**3.** $g(t) = te^{2t+1}$

**4.** $f(x) = e^{2x}\sqrt{1 - e^{2x}}$

**5.** $f(x) = \ln(3 - 2x)$

**6.** $u(t) = \ln\sqrt{t^3 - 6t}$

**7.** $g(x) = \dfrac{5x^2}{(4x - 1)^2}$

**8.** $f(x) = \dfrac{(x + 2)^3}{(x^2 - 9)^2}$

In Exercises 9 and 10, evaluate the derivative at the point (2, 4).

**9.** $f(x) = x^2 e^{x-2}$

**10.** $g(x) = x\sqrt{x^2 - x + 2}$

## Exercises 7.4

In Exercises 1–14, find the first partial derivatives with respect to $x$ and with respect to $y$.

**1.** $z = 3x + 5y - 1$

**2.** $z = x^2 - 2y$

**3.** $f(x, y) = 3x - 6y^2$

**4.** $f(x, y) = x + 4y^{3/2}$

**5.** $f(x, y) = \dfrac{x}{y}$

**6.** $z = x\sqrt{y}$

**7.** $f(x, y) = \sqrt{x^2 + y^2}$

**8.** $f(x, y) = \dfrac{xy}{x^2 + y^2}$

**9.** $z = x^2 e^{2y}$

**10.** $z = xe^{x+y}$

**11.** $h(x, y) = e^{-(x^2+y^2)}$

**12.** $g(x, y) = e^{x/y}$

**13.** $z = \ln\dfrac{x + y}{x - y}$

**14.** $g(x, y) = \ln(x^2 + y^2)$

In Exercises 15–20, let $f(x, y) = 3x^2 y e^{x-y}$ and $g(x, y) = 3xy^2 e^{y-x}$. Find each of the following.

**15.** $f_x(x, y)$

**16.** $f_y(x, y)$

**17.** $g_x(x, y)$

**18.** $g_y(x, y)$

**19.** $f_x(1, 1)$

**20.** $g_x(-2, -2)$

In Exercises 21–28, evaluate $f_x$ and $f_y$ at the point.

| Function | Point |
|---|---|
| **21.** $f(x, y) = 3x^2 + xy - y^2$ | $(2, 1)$ |
| **22.** $f(x, y) = x^2 - 3xy + y^2$ | $(1, -1)$ |
| **23.** $f(x, y) = e^{3xy}$ | $(0, 4)$ |
| **24.** $f(x, y) = e^x y^2$ | $(0, 2)$ |

| Function | Point |
|---|---|
| **25.** $f(x, y) = \dfrac{xy}{x - y}$ | $(2, -2)$ |
| **26.** $f(x, y) = \dfrac{4xy}{\sqrt{x^2 + y^2}}$ | $(1, 0)$ |
| **27.** $f(x, y) = \ln(x^2 + y^2)$ | $(1, 0)$ |
| **28.** $f(x, y) = \ln\sqrt{xy}$ | $(-1, -1)$ |

In Exercises 29–32, find the first partial derivatives with respect to $x$, $y$, and $z$.

**29.** $w = xyz$

**30.** $w = x^2 - 3xy + 4yz + z^3$

**31.** $w = \dfrac{2z}{x + y}$

**32.** $w = \sqrt{x^2 + y^2 + z^2}$

In Exercises 33–38, evaluate $w_x$, $w_y$, and $w_z$ at the point.

| Function | Point |
|---|---|
| **33.** $w = \sqrt{x^2 + y^2 + z^2}$ | $(2, -1, 2)$ |
| **34.** $w = \dfrac{xy}{x + y + z}$ | $(1, 2, 0)$ |
| **35.** $w = \ln\sqrt{x^2 + y^2 + z^2}$ | $(3, 0, 4)$ |
| **36.** $w = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$ | $(0, 0, 0)$ |
| **37.** $w = 2xz^2 + 3xyz - 6y^2 z$ | $(1, -1, 2)$ |
| **38.** $w = xye^{z^2}$ | $(2, 1, 0)$ |

In Exercises 39–42, find values of $x$ and $y$ such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.

**39.** $f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$

**40.** $f(x, y) = 3x^3 - 12xy + y^3$
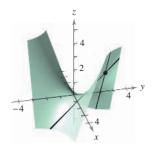
**41.** $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y} + xy$
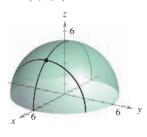
**42.** $f(x, y) = \ln(x^2 + y^2 + 1)$

In Exercises 43–46, find the slope of the surface at the given point in (a) the $x$-direction and (b) the $y$-direction.

**43.** $z = xy$

$(1, 2, 2)$

**44.** $z = \sqrt{25 - x^2 - y^2}$

$(3, 0, 4)$



**45.** $z = 4 - x^2 - y^2$
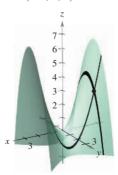
$(1, 1, 2)$

**46.** $z = x^2 - y^2$

$(-2, 1, 3)$



In Exercises 47–54, find the four second partial derivatives. Observe that the second mixed partials are equal.

**47.** $z = x^2 - 2xy + 3y^2$

**48.** $z = y^3 - 4xy^2 - 1$

**49.** $z = \dfrac{e^{2xy}}{4x}$

**50.** $z = \dfrac{x^2 - y^2}{2xy}$

**51.** $z = x^3 - 4y^2$

**52.** $z = \sqrt{9 - x^2 - y^2}$

**53.** $z = \dfrac{1}{x - y}$

**54.** $z = \dfrac{x}{x + y}$

In Exercises 55–58, evaluate the second partial derivatives $f_{xx}$, $f_{xy}$, $f_{yy}$, and $f_{yx}$ at the point.

| Function | Point |
|---|---|
| **55.** $f(x, y) = x^4 - 3x^2y^2 + y^2$ | $(1, 0)$ |
| **56.** $f(x, y) = \sqrt{x^2 + y^2}$ | $(0, 2)$ |
| **57.** $f(x, y) = \ln(x - y)$ | $(2, 1)$ |
| **58.** $f(x, y) = x^2 e^y$ | $(-1, 0)$ |

**59. Marginal Cost** A company manufactures two models of bicycles: a mountain bike and a racing bike. The cost function for producing $x$ mountain bikes and $y$ racing bikes is given by

$$C = 10\sqrt{xy} + 149x + 189y + 675.$$

(a) Find the marginal costs ($\partial C/\partial x$ and $\partial C/\partial y$) when $x = 120$ and $y = 160$.

(b) When additional production is required, which model of bicycle results in the cost increasing at a higher rate? How can this be determined from the cost model?

**60. Marginal Revenue** A pharmaceutical corporation has two plants that produce the same over-the-counter medicine. If $x_1$ and $x_2$ are the numbers of units produced at plant 1 and plant 2, respectively, then the total revenue for the product is given by

$$R = 200x_1 + 200x_2 - 4x_1^2 - 8x_1x_2 - 4x_2^2.$$

When $x_1 = 4$ and $x_2 = 12$, find

(a) the marginal revenue for plant 1, $\partial R/\partial x_1$.

(b) the marginal revenue for plant 2, $\partial R/\partial x_2$.

**61. Marginal Productivity** Consider the Cobb-Douglas production function $f(x, y) = 200x^{0.7}y^{0.3}$. When $x = 1000$ and $y = 500$, find

(a) the marginal productivity of labor, $\partial f/\partial x$.

(b) the marginal productivity of capital, $\partial f/\partial y$.

**62. Marginal Productivity** Repeat Exercise 61 for the production function given by $f(x, y) = 100x^{0.75}y^{0.25}$.

**Complementary and Substitute Products** In Exercises 63 and 64, determine whether the demand functions describe complementary or substitute product relationships. Using the notation of Example 4, let $x_1$ and $x_2$ be the demands for products $p_1$ and $p_2$, respectively.

**63.** $x_1 = 150 - 2p_1 - \frac{5}{2}p_2$, $\quad x_2 = 350 - \frac{3}{2}p_1 - 3p_2$

**64.** $x_1 = 150 - 2p_1 + 1.8p_2$, $\quad x_2 = 350 + \frac{3}{4}p_1 - 1.9p_2$

**65. Milk Consumption**   A model for the per capita consumptions (in gallons) of different types of plain milk in the United States from 1999 through 2004 is

$$z = 1.25x - 0.125y + 0.95.$$

Consumption of reduced-fat (1%) and skim milks, reduced-fat milk (2%), and whole milk are represented by variables $x$, $y$, and $z$, respectively.   *(Source: U.S. Department of Agriculture)*

(a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(b) Interpret the partial derivatives in the context of the problem.

**66. Shareholder's Equity**   The shareholder's equity $z$ (in billions of dollars) for Wal-Mart Corporation from 2000 through 2006 can be modeled by

$$z = 0.205x - 0.073y - 0.728$$

where $x$ is net sales (in billions of dollars) and $y$ is the total assets (in billions of dollars).   *(Source: Wal-Mart Corporation)*

(a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(b) Interpret the partial derivatives in the context of the problem.

**B 67. Psychology**   Early in the twentieth century, an intelligence test called the *Stanford-Binet Test* (more commonly known as the *IQ test*) was developed. In this test, an individual's mental age $M$ is divided by the individual's chronological age $C$ and the quotient is multiplied by 100. The result is the individual's *IQ*.

$$IQ(M, C) = \frac{M}{C} \times 100$$

Find the partial derivatives of *IQ* with respect to $M$ and with respect to $C$. Evaluate the partial derivatives at the point $(12, 10)$ and interpret the result.   *(Source: Adapted from Bernstein/Clark-Stewart/Roy/Wickens, Psychology, Fourth Edition)*

**68. Investment**   The value of an investment of $1000 earning 10% compounded annually is

$$V(I, R) = 1000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^{10}$$

where $I$ is the annual rate of inflation and $R$ is the tax rate for the person making the investment. Calculate $V_I(0.03, 0.28)$ and $V_R(0.03, 0.28)$. Determine whether the tax rate or the rate of inflation is the greater "negative" factor on the growth of the investment.

**69. Think About It**   Let $N$ be the number of applicants to a university, $p$ the charge for food and housing at the university, and $t$ the tuition. Suppose that $N$ is a function of $p$ and $t$ such that $\partial N/\partial p < 0$ and $\partial N/\partial t < 0$. How would you interpret the fact that both partials are negative?

**70. Marginal Utility**   The utility function $U = f(x, y)$ is a measure of the utility (or satisfaction) derived by a person from the consumption of two products $x$ and $y$. Suppose the utility function is given by $U = -5x^2 + xy - 3y^2$.

(a) Determine the marginal utility of product $x$.

(b) Determine the marginal utility of product $y$.

(c) When $x = 2$ and $y = 3$, should a person consume one more unit of product $x$ or one more unit of product $y$? Explain your reasoning.

(T) (d) Use a three-dimensional graphing utility to graph the function. Interpret the marginal utilities of products $x$ and $y$ graphically.

**Business Capsule**



Photo courtesy of Izzy and Coco Tihanyi

In 1996, twin sisters Izzy and Coco Tihanyi started Surf Diva, a surf school and apparel company for women and girls, in La Jolla, California. To advertise their business, they would donate surf lessons and give the surf report on local radio stations in exchange for air time. Today, they have schools in Japan and Costa Rica, and their clothing line can be found in surf and specialty shops, sporting goods stores, and airport gift shops. Sales from their surf schools have increased nearly 13% per year, and product sales are expected to double each year.

**71. Research Project**   Use your school's library, the Internet, or some other reference source to research a company that increased the demand for its product by creative advertising. Write a paper about the company. Use graphs to show how a change in demand is related to a change in the marginal utility of a product or service.

# Extrema of Functions of Two Variables

- ■ Understand the relative extrema of functions of two variables.
- ■ Use the First-Partials Test to find the relative extrema of functions of two variables.
- ■ Use the Second-Partials Test to find the relative extrema of functions of two variables.
- ■ Use relative extrema to answer questions about real-life situations.

## Relative Extrema

Earlier in the text, you learned how to use derivatives to find the relative minimum and relative maximum values of a function of a single variable. In this section, you will learn how to use partial derivatives to find the relative minimum and relative maximum values of a function of two variables.

---

**Relative Extrema of a Function of Two Variables**

Let $f$ be a function defined on a region containing $(x_0, y_0)$. The function $f$ has a **relative maximum** at $(x_0, y_0)$ if there is a circular region $R$ centered at $(x_0, y_0)$ such that

$$f(x, y) \leq f(x_0, y_0) \qquad \text{\textit{f} has a relative maximum at } (x_0, y_0).$$

for all $(x, y)$ in $R$. The function $f$ has a **relative minimum** at $(x_0, y_0)$ if there is a circular region $R$ centered at $(x_0, y_0)$ such that

$$f(x, y) \geq f(x_0, y_0) \qquad \text{\textit{f} has a relative minimum at } (x_0, y_0).$$

for all $(x, y)$ in $R$.

---

To say that $f$ has a relative maximum at $(x_0, y_0)$ means that the point $(x_0, y_0, z_0)$ is at least as high as all nearby points on the graph of $z = f(x, y)$. Similarly, $f$ has a relative minimum at $(x_0, y_0)$ if $(x_0, y_0, z_0)$ is at least as low as all nearby points on the graph. (See Figure 7.26.)

Surface:
$f(x, y) = -(x^2 + y^2)$



**FIGURE 7.27**    $f$ has an absolute maximum at $(0, 0, 0)$.
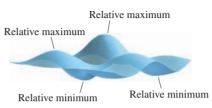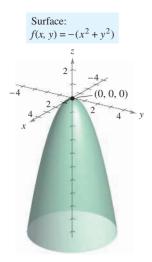


**FIGURE 7.26**    Relative Extrema

As in single-variable calculus, you need to distinguish between relative extrema and absolute extrema of a function of two variables. The number $f(x_0, y_0)$ is an absolute maximum of $f$ in the region $R$ if it is greater than or equal to all other function values in the region. For instance, the function $f(x, y) = -(x^2 + y^2)$ graphs as a paraboloid, opening downward, with vertex at $(0, 0, 0)$. (See Figure 7.27.) The number $f(0, 0) = 0$ is an absolute maximum of the function over the entire $xy$-plane. An absolute minimum of $f$ in a region is defined similarly.

## The First-Partials Test for Relative Extrema

To locate the relative extrema of a function of two variables, you can use a procedure that is similar to the First-Derivative Test used for functions of a single variable.

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**First-Partials Test for Relative Extrema**
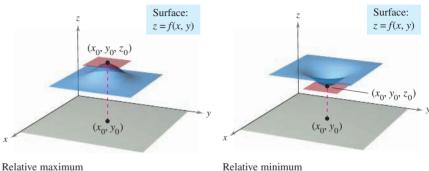
If $f$ has a relative extremum at $(x_0, y_0)$ on an open region $R$ in the $xy$-plane, and the first partial derivatives of $f$ exist in $R$, then

$$f_x(x_0, y_0) = 0$$

and

$$f_y(x_0, y_0) = 0$$

as shown in Figure 7.28.

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Relative maximum            Relative minimum
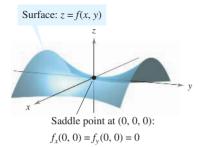
**FIGURE 7.28**

An *open* region in the $xy$-plane is similar to an open interval on the real number line. For instance, the region $R$ consisting of the interior of the circle $x^2 + y^2 = 1$ is an open region. If the region $R$ consists of the interior of the circle *and* the points on the circle, then it is a *closed* region.

A point $(x_0, y_0)$ is a **critical point** of $f$ if $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ is undefined or if

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0. \qquad \text{Critical point}$$

The First-Partials Test states that if the first partial derivatives exist, then you need only examine values of $f(x, y)$ at critical points to find the relative extrema. As is true for a function of a single variable, however, the critical points of a function of two variables do not always yield relative extrema. For instance, the point $(0, 0)$ is a critical point of the surface shown in Figure 7.29, but $f(0, 0)$ is not a relative extremum of the function. Such points are called **saddle points** of the function.



Surface: $z = f(x, y)$

Saddle point at $(0, 0, 0)$:
$f_x(0, 0) = f_y(0, 0) = 0$

**FIGURE 7.29**

Surface:
$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20$



**FIGURE 7.30**

Surface:
$f(x, y) = 1 - (x^2 + y^2)^{1/3}$



**FIGURE 7.31**    $f_x(x, y)$ and $f_y(x, y)$ are undefined at $(0, 0)$.

✓**CHECKPOINT 2**

Find the relative extrema of

$$f(x, y) = \sqrt{1 - \frac{x^2}{16} - \frac{y^2}{4}}.$$ ∎

*Example 1*    **Finding Relative Extrema**

Find the relative extrema of

$$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20.$$

**SOLUTION**    Begin by finding the first partial derivatives of $f$.

$$f_x(x, y) = 4x + 8 \quad \text{and} \quad f_y(x, y) = 2y - 6$$
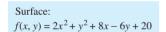
Because these partial derivatives are defined for all points in the $xy$-plane, the only critical points are those for which both first partial derivatives are zero. To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0, and solve the resulting system of equations.
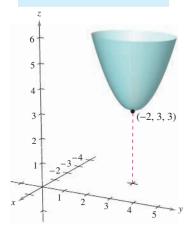
$$4x + 8 = 0 \qquad \text{Set } f_x(x, y) \text{ equal to 0.}$$
$$2y - 6 = 0 \qquad \text{Set } f_y(x, y) \text{ equal to 0.}$$

The solution of this system is $x = -2$ and $y = 3$. So, the point $(-2, 3)$ is the only critical number of $f$. From the graph of the function, shown in Figure 7.30, you can see that this critical point yields a relative minimum of the function. So, the function has only one relative extremum, which is

$$f(-2, 3) = 3. \qquad \text{Relative minimum}$$
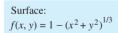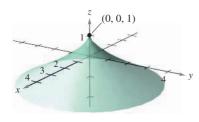
✓**CHECKPOINT 1**

Find the relative extrema of $f(x, y) = x^2 + 2y^2 + 16x - 8y + 8$. ∎

Example 1 shows a relative minimum occurring at one type of critical point—the type for which both $f_x(x, y)$ and $f_y(x, y)$ are zero. The next example shows a relative maximum that occurs at the other type of critical point—the type for which either $f_x(x, y)$ or $f_y(x, y)$ is undefined.

*Example 2*    **Finding Relative Extrema**

Find the relative extrema of

$$f(x, y) = 1 - (x^2 + y^2)^{1/3}.$$

**SOLUTION**    Begin by finding the first partial derivatives of $f$.

$$f_x(x, y) = -\frac{2x}{3(x^2 + y^2)^{2/3}} \quad \text{and} \quad f_y(x, y) = -\frac{2y}{3(x^2 + y^2)^{2/3}}$$

These partial derivatives are defined for all points in the $xy$-plane *except* the point $(0, 0)$. So, $(0, 0)$ is a critical point of $f$. Moreover, this is the only critical point, because there are no other values of $x$ and $y$ for which either partial is undefined or for which both partials are zero. From the graph of the function, shown in Figure 7.31, you can see that this critical point yields a relative maximum of the function. So, the function has only one relative extremum, which is

$$f(0, 0) = 1. \qquad \text{Relative maximum}$$

**Algebra Review**

For help in solving the system of equations

$$y - x^3 = 0$$

$$x - y^3 = 0$$

in Example 3, see Example 1(a) in the *Chapter 7 Algebra Review*, on page 561.
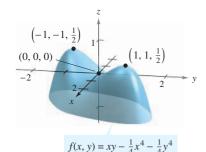
## The Second-Partials Test for Relative Extrema

For functions such as those in Examples 1 and 2, you can determine the *types* of extrema at the critical points by sketching the graph of the function. For more complicated functions, a graphical approach is not so easy to use. The **Second-Partials Test** is an analytical test that can be used to determine whether a critical number yields a relative minimum, a relative maximum, or neither.

**Second-Partials Test for Relative Extrema**

Let $f$ have continuous second partial derivatives on an open region containing $(a, b)$ for which $f_x(a, b) = 0$ and $f_y(a, b) = 0$. To test for relative extrema of $f$, consider the quantity

$$d = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a **relative minimum** at $(a, b)$.
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a **relative maximum** at $(a, b)$.
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point.**
4. The test gives no information if $d = 0$.

*Example 3* **Applying the Second-Partials Test**

Find the relative extrema and saddle points of $f(x, y) = xy - \frac{1}{4}x^4 - \frac{1}{4}y^4$.

**SOLUTION** Begin by finding the critical points of $f$. Because $f_x(x, y) = y - x^3$ and $f_y(x, y) = x - y^3$ are defined for all points in the $xy$-plane, the only critical points are those for which both first partial derivatives are zero. By solving the equations $y - x^3 = 0$ and $x - y^3 = 0$ simultaneously, you can determine that the critical points are $(1, 1)$, $(-1, -1)$, and $(0, 0)$. Furthermore, because

$$f_{xx}(x, y) = -3x^2, \quad f_{yy}(x, y) = -3y^2, \quad \text{and} \quad f_{xy}(x, y) = 1$$

you can use the quantity $d = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$ to classify the critical points as shown.

| Critical Point | $d$ | $f_{xx}(x, y)$ | Conclusion |
| --- | --- | --- | --- |
| $(1, 1)$ | $(-3)(-3) - 1 = 8$ | $-3$ | Relative maximum |
| $(-1, -1)$ | $(-3)(-3) - 1 = 8$ | $-3$ | Relative maximum |
| $(0, 0)$ | $(0)(0) - 1 = -1$ | $0$ | Saddle point |

The graph of $f$ is shown in Figure 7.32.



$f(x, y) = xy - \frac{1}{4}x^4 - \frac{1}{4}y^4$

**FIGURE 7.32**

✓**CHECKPOINT 3**

Find the relative extrema and saddle points of $f(x, y) = \dfrac{y^2}{16} - \dfrac{x^2}{4}$. ■

**FIGURE 7.33**
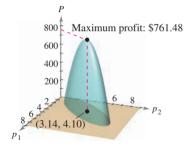
## Application of Extrema

**Example 4** Finding a Maximum Profit ®

A company makes two substitute products whose demand functions are given by

$$x_1 = 200(p_2 - p_1)$$  Demand for product 1

$$x_2 = 500 + 100p_1 - 180p_2$$  Demand for product 2

where $p_1$ and $p_2$ are the prices per unit (in dollars) and $x_1$ and $x_2$ are the numbers of units sold. The costs of producing the two products are $0.50 and $0.75 per unit, respectively. Find the prices that will yield a maximum profit.

**SOLUTION**  The cost and revenue functions are as shown.

$$C = 0.5x_1 + 0.75x_2$$  Write cost function.

$$= 0.5(200)(p_2 - p_1) + 0.75(500 + 100p_1 - 180p_2)$$  Substitute.

$$= 375 - 25p_1 - 35p_2$$  Simplify.

$$R = p_1x_1 + p_2x_2$$  Write revenue function.

$$= p_1(200)(p_2 - p_1) + p_2(500 + 100p_1 - 180p_2)$$  Substitute.

$$= -200p_1^2 - 180p_2^2 + 300p_1p_2 + 500p_2$$  Simplify.

This implies that the profit function is

$$P = R - C$$  Write profit function.

$$= -200p_1^2 - 180p_2^2 + 300p_1p_2 + 500p_2 - (375 - 25p_1 - 35p_2)$$

$$= -200p_1^2 - 180p_2^2 + 300p_1p_2 + 25p_1 + 535p_2 - 375.$$

The maximum profit occurs when the two first partial derivatives are zero.

$$\frac{\partial P}{\partial p_1} = -400p_1 + 300p_2 + 25 = 0$$

$$\frac{\partial P}{\partial p_2} = 300p_1 - 360p_2 + 535 = 0$$

By solving this system simultaneously, you can conclude that the solution is $p_1 = \$3.14$ and $p_2 = \$4.10$. From the graph of the function shown in Figure 7.33, you can see that this critical number yields a maximum. So, the maximum profit is

$$P(3.14, 4.10) = \$761.48.$$

✓ **CHECKPOINT 4**

Find the prices that will yield a maximum profit for the products in Example 4 if the costs of producing the two products are $0.75 and $0.50 per unit, respectively. ■
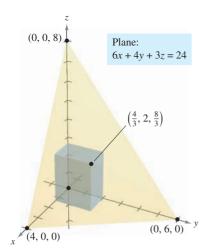
**FIGURE 7.34**

*Example 5*  **Finding a Maximum Volume**

Consider all possible rectangular boxes that are resting on the *xy*-plane with one vertex at the origin and the opposite vertex in the plane $6x + 4y + 3z = 24$, as shown in Figure 7.34. Of all such boxes, which has the greatest volume?

**SOLUTION**  Because one vertex of the box lies in the plane given by $6x + 4y + 3z = 24$ or $z = \frac{1}{3}(24 - 6x - 4y)$, you can write the volume of the box as

$$V = xyz \qquad\qquad \text{Volume} = (\text{width})(\text{length})(\text{height})$$

$$= xy\left(\tfrac{1}{3}\right)(24 - 6x - 4y) \qquad \text{Substitute for } z.$$

$$= \tfrac{1}{3}(24xy - 6x^2y - 4xy^2). \qquad \text{Simplify.}$$

To find the critical numbers, set the first partial derivatives equal to zero.

$$V_x = \tfrac{1}{3}(24y - 12xy - 4y^2) \qquad \text{Partial with respect to } x$$

$$= \tfrac{1}{3}y(24 - 12x - 4y) = 0 \qquad \text{Factor and set equal to 0.}$$

$$V_y = \tfrac{1}{3}(24x - 6x^2 - 8xy) \qquad \text{Partial with respect to } y$$

$$= \tfrac{1}{3}x(24 - 6x - 8y) = 0 \qquad \text{Factor and set equal to 0.}$$

The four solutions of this system are $(0, 0)$, $(0, 6)$, $(4, 0)$, and $\left(\tfrac{4}{3}, 2\right)$. Using the Second-Partials Test, you can determine that the maximum volume occurs when the width is $x = \tfrac{4}{3}$ and the length is $y = 2$. For these values, the height of the box is

$$z = \tfrac{1}{3}\left[24 - 6\left(\tfrac{4}{3}\right) - 4(2)\right] = \tfrac{8}{3}.$$

So, the maximum volume is

$$V = xyz = \left(\tfrac{4}{3}\right)(2)\left(\tfrac{8}{3}\right) = \tfrac{64}{9} \text{ cubic units.}$$

✓**CHECKPOINT 5**

Find the maximum volume of a box that is resting on the *xy*-plane with one vertex at the origin and the opposite vertex in the plane $2x + 4y + z = 8$. ■

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**CONCEPT CHECK**

1. Given a function of two variables $f$, state how you can determine whether $(x_0, y_0)$ is a critical point of $f$.

2. The point $(a, b, f(a, b))$ is a saddle point if what is true?

3. If $d > 0$ and $f_{xx}(a, b) > 0$, then what does $f$ have at $(a, b)$: a relative minimum or a relative maximum?

4. If $d > 0$ and $f_{xx}(a, b) < 0$, then what does $f$ have at $(a, b)$: a relative minimum or a relative maximum?

In Exercises 1–8, solve the system of equations.

**1.** $\begin{cases} 5x = 15 \\ 3x - 2y = 5 \end{cases}$
**2.** $\begin{cases} \frac{1}{2}y = 3 \\ -x + 5y = 19 \end{cases}$
**3.** $\begin{cases} x + y = 5 \\ x - y = -3 \end{cases}$
**4.** $\begin{cases} x + y = 8 \\ 2x - y = 4 \end{cases}$

**5.** $\begin{cases} 2x - y = 8 \\ 3x - 4y = 7 \end{cases}$
**6.** $\begin{cases} 2x - 4y = 14 \\ 3x + y = 7 \end{cases}$
**7.** $\begin{cases} x^2 + x = 0 \\ 2yx + y = 0 \end{cases}$
**8.** $\begin{cases} 3y^2 + 6y = 0 \\ xy + x + 2 = 0 \end{cases}$

In Exercises 9–14, find all first and second partial derivatives of the function.

**9.** $z = 4x^3 - 3y^2$
**10.** $z = 2x^5 - y^3$
**11.** $z = x^4 - \sqrt{xy} + 2y$

**12.** $z = 2x^2 - 3xy + y^2$
**13.** $z = ye^{xy^2}$
**14.** $z = xe^{xy}$

## Exercises 7.5

In Exercises 1–4, find any critical points and relative extrema of the function.

**1.** $f(x, y) = x^2 - y^2 + 4x - 8y - 11$

**2.** $f(x, y) = x^2 + y^2 + 2x - 6y + 6$

**3.** $f(x, y) = \sqrt{x^2 + y^2 + 1}$

**4.** $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2}$

In Exercises 5–20, examine the function for relative extrema and saddle points.

**5.** $f(x, y) = (x - 1)^2 + (y - 3)^2$

**6.** $f(x, y) = 9 - (x - 3)^2 - (y + 2)^2$

**7.** $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

**8.** $f(x, y) = -x^2 - 5y^2 + 8x - 10y - 13$

**9.** $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

**10.** $f(x, y) = x^2 + 6xy + 10y^2 - 4y + 4$
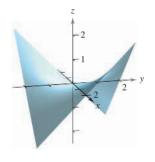
**11.** $f(x, y) = 3x^2 + 2y^2 - 12x - 4y + 7$

**12.** $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$
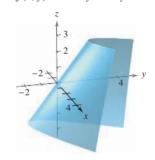
**13.** $f(x, y) = x^2 - y^2 + 4x - 4y - 8$

**14.** $f(x, y) = x^2 - 3xy - y^2$
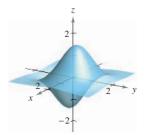
**15.** $f(x, y) = \frac{1}{2}xy$
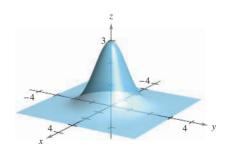
**16.** $f(x, y) = x + y + 2xy - x^2 - y^2$
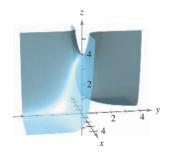


**17.** $f(x, y) = (x + y)e^{1-x^2-y^2}$
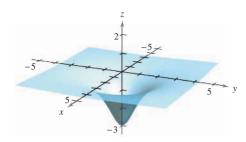


**18.** $f(x, y) = 3e^{-(x^2+y^2)}$

**19.** $f(x, y) = 4e^{xy}$



**20.** $f(x, y) = -\dfrac{3}{x^2 + y^2 + 1}$



**Think About It** In Exercises 21–24, determine whether there is a relative maximum, a relative minimum, a saddle point, or insufficient information to determine the nature of the function $f(x, y)$ at the critical point $(x_0, y_0)$.

**21.** $f_{xx}(x_0, y_0) = 9$, $f_{yy}(x_0, y_0) = 4$, $f_{xy}(x_0, y_0) = 6$

**22.** $f_{xx}(x_0, y_0) = -3$, $f_{yy}(x_0, y_0) = -8$, $f_{xy}(x_0, y_0) = 2$

**23.** $f_{xx}(x_0, y_0) = -9$, $f_{yy}(x_0, y_0) = 6$, $f_{xy}(x_0, y_0) = 10$

**24.** $f_{xx}(x_0, y_0) = 25$, $f_{vv}(x_0, y_0) = 8$, $f_{xv}(x_0, y_0) = 10$

In Exercises 25–30, find the critical points and test for relative extrema. List the critical points for which the Second-Partials Test fails.

**25.** $f(x, y) = (xy)^2$

**26.** $f(x, y) = \sqrt{x^2 + y^2}$

**27.** $f(x, y) = x^3 + y^3$

**28.** $f(x, y) = x^3 + y^3 - 3x^2 + 6y^2 + 3x + 12y + 7$

**29.** $f(x, y) = x^{2/3} + y^{2/3}$

**30.** $f(x, y) = (x^2 + y^2)^{2/3}$

In Exercises 31 and 32, find the critical points of the function and, from the form of the function, determine whether a relative maximum or a relative minimum occurs at each point.

**31.** $f(x, y, z) = (x - 1)^2 + (y + 3)^2 + z^2$

**32.** $f(x, y, z) = 6 - [x(y + 2)(z - 1)]^2$

In Exercises 33–36, find three positive numbers $x$, $y$, and $z$ that satisfy the given conditions.

**33.** The sum is 30 and the product is a maximum.

**34.** The sum is 32 and $P = xy^2z$ is a maximum.

**35.** The sum is 30 and the sum of the squares is a minimum.

**36.** The sum is 1 and the sum of the squares is a minimum.

**37. Revenue** A company manufactures two types of sneakers: running shoes and basketball shoes. The total revenue from $x_1$ units of running shoes and $x_2$ units of basketball shoes is

$$R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$$

where $x_1$ and $x_2$ are in thousands of units. Find $x_1$ and $x_2$ so as to maximize the revenue.

**38. Revenue** A retail outlet sells two types of riding lawn mowers, the prices of which are $p_1$ and $p_2$. Find $p_1$ and $p_2$ so as to maximize total revenue, where $R = 515p_1 + 805p_2 + 1.5p_1p_2 - 1.5p_1^2 - p_2^2$.

**Revenue** In Exercises 39 and 40, find $p_1$ and $p_2$ so as to maximize the total revenue $R = x_1p_1 + x_2p_2$ for a retail outlet that sells two competitive products with the given demand functions.

**39.** $x_1 = 1000 - 2p_1 + p_2$, $x_2 = 1500 + 2p_1 - 1.5p_2$

**40.** $x_1 = 1000 - 4p_1 + 2p_2$, $x_2 = 900 + 4p_1 - 3p_2$

**41. Profit** A corporation manufactures a high-performance automobile engine product at two locations. The cost of producing $x_1$ units at location 1 is

$$C_1 = 0.05x_1^2 + 15x_1 + 5400$$

and the cost of producing $x_2$ units at location 2 is

$$C_2 = 0.03x_2^2 + 15x_2 + 6100.$$

The demand function for the product is

$$p = 225 - 0.4(x_1 + x_2)$$

and the total revenue function is

$$R = [225 - 0.4(x_1 + x_2)](x_1 + x_2).$$

Find the production levels at the two locations that will maximize the profit

$$P = R - C_1 - C_2.$$

**42. Profit** A corporation manufactures candles at two locations. The cost of producing $x_1$ units at location 1 is

$$C_1 = 0.02x_1^2 + 4x_1 + 500$$

and the cost of producing $x_2$ units at location 2 is

$$C_2 = 0.05x_2^2 + 4x_2 + 275.$$

The candles sell for \$15 per unit. Find the quantity that should be produced at each location to maximize the profit

$$P = 15(x_1 + x_2) - C_1 - C_2.$$

**43. Volume** Find the dimensions of a rectangular package of maximum volume that may be sent by a shipping company assuming that the sum of the length and the girth (perimeter of a cross section) cannot exceed 96 inches.

**44. Volume** Repeat Exercise 43 assuming that the sum of the length and the girth cannot exceed 144 inches.

**45. Cost** A manufacturer makes a wooden storage crate that has an open top. The volume of each crate is 6 cubic feet. Material costs are $0.15 per square foot for the base of the crate and $0.10 per square foot for the sides. Find the dimensions that minimize the cost of each crate. What is the minimum cost?

**46. Cost** A home improvement contractor is painting the walls and ceiling of a rectangular room. The volume of the room is 668.25 cubic feet. The cost of wall paint is $0.06 per square foot and the cost of ceiling paint is $0.11 per square foot. Find the room dimensions that result in a minimum cost for the paint. What is the minimum cost for the paint?

**47. Hardy-Weinberg Law** Common blood types are determined genetically by the three alleles A, B, and O. (An allele is any of a group of possible mutational forms of a gene.) A person whose blood type is AA, BB, or OO is homozygous. A person whose blood type is AB, AO, or BO is heterozygous. The Hardy-Weinberg Law states that the proportion $P$ of heterozygous individuals in any given population is modeled by

$$P(p, q, r) = 2pq + 2pr + 2qr$$

where $p$ represents the percent of allele A in the population, $q$ represents the percent of allele B in the population, and $r$ represents the percent of allele O in the population. Use the fact that $p + q + r = 1$ (the sum of the three must equal 100%) to show that the maximum proportion of heterozygous individuals in any population is $\frac{2}{3}$.

**48. Biology** A lake is to be stocked with smallmouth and largemouth bass. Let $x$ represent the number of smallmouth bass and let $y$ represent the number of largemouth bass in the lake. The weight of each fish is dependent on the population densities. After a six-month period, the weight of a single smallmouth bass is given by

$$W_1 = 3 - 0.002x - 0.001y$$

and the weight of a single largemouth bass is given by

$$W_2 = 4.5 - 0.004x - 0.005y.$$

Assuming that no fish die during the six-month period, how many smallmouth and largemouth bass should be stocked in the lake so that the *total* weight $T$ of bass in the lake is a maximum?

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Bass help to keep a pond healthy. A suitable quantity of bass keeps other fish populations in check, and helps balance the food chain.

**49. Cost** An automobile manufacturer has determined that its annual labor and equipment cost (in millions of dollars) can be modeled by

$$C(x, y) = 2x^2 + 3y^2 - 15x - 20y + 4xy + 39$$

where $x$ is the amount spent per year on labor and $y$ is the amount spent per year on equipment (both in millions of dollars). Find the values of $x$ and $y$ that minimize the annual labor and equipment cost. What is this cost?

**50. Medicine** In order to treat a certain bacterial infection, a combination of two drugs is being tested. Studies have shown that the duration of the infection in laboratory tests can be modeled by

$$D(x, y) = x^2 + 2y^2 - 18x - 24y + 2xy + 120$$

where $x$ is the dosage in hundreds of milligrams of the first drug and $y$ is the dosage in hundreds of milligrams of the second drug. Determine the partial derivatives of $D$ with respect to $x$ and with respect to $y$. Find the amount of each drug necessary to minimize the duration of the infection.

**True or False?** In Exercises 51 and 52, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**51.** A saddle point always occurs at a critical point.

**52.** If $f(x, y)$ has a relative maximum at $(x_0, y_0, z_0)$, then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

## Mid-Chapter Quiz

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, (a) plot the points on a three-dimensional coordinate system, (b) find the distance between the points, and (c) find the coordinates of the midpoint of the line segment joining the points.

**1.** $(1, 3, 2), (-1, 2, 0)$     **2.** $(-1, 4, 3), (5, 1, -6)$     **3.** $(0, -3, 3), (3, 0, -3)$

In Exercises 4 and 5, find the standard equation of the sphere.

**4.** Center: $(2, -1, 3)$; radius: 4

**5.** Endpoints of a diameter: $(0, 3, 1), (2, 5, -5)$

**6.** Find the center and radius of the sphere whose equation is

$$x^2 + y^2 + z^2 - 8x - 2y - 6z - 23 = 0.$$

In Exercises 7–9, find the intercepts and sketch the graph of the plane.

**7.** $2x + 3y + z = 6$     **8.** $x - 2z = 4$     **9.** $z = -5$

In Exercises 10–12, identify the quadric surface.

**10.** $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$     **11.** $z^2 - x^2 - y^2 = 25$     **12.** $81z - 9x^2 - y^2 = 0$

In Exercises 13–15, find $f(1, 0)$ and $f(4, -1)$.

**13.** $f(x, y) = x - 9y^2$     **14.** $f(x, y) = \sqrt{4x^2 + y}$     **15.** $f(x, y) = \ln(x + 3y)$

**16.** The contour map shows level curves of equal temperature (isotherms), measured in degrees Fahrenheit, across North America on a spring day. Use the map to find the approximate range of temperatures in (a) the Great Lakes region, (b) the United States, and (c) Mexico.

Figure for 16

In Exercises 17 and 18, find $f_x$ and $f_y$ and evaluate each at the point $(-2, 3)$.

**17.** $f(x, y) = x^2 + 2y^2 - 3x - y + 1$     **18.** $f(x, y) = \dfrac{3x - y^2}{x + y}$

In Exercises 19 and 20, find any critical points, relative extrema, and saddle points of the function.

**19.** $f(x, y) = 3x^2 + y^2 - 2xy - 6x + 2y$     **20.** $f(x, y) = -x^3 + 4xy - 2y^2 + 1$

**21.** A company manufactures two types of wood burning stoves: a freestanding model and a fireplace-insert model. The total cost (in thousands of dollars) for producing $x$ freestanding stoves and $y$ fireplace-insert stoves can be modeled by

$$C(x, y) = \tfrac{1}{16}x^2 + y^2 - 10x - 40y + 820.$$

Find the values of $x$ and $y$ that minimize the total cost. What is this cost?

**22.** **Physical Science** Assume that Earth is a sphere with a radius of 3963 miles. If the center of Earth is placed at the origin of a three-dimensional coordinate system, what is the equation of the sphere? Lines of longitude that run north-south could be represented by what trace(s)? What shape would each of these traces form? Why? Lines of latitude that run east-west could be represented by what trace(s)? Why? What shape would each of these traces form? Why?

# Lagrange Multipliers

- Use Lagrange multipliers with one constraint to find extrema of functions of several variables and to answer questions about real-life situations.
- Use Lagrange multipliers with two constraints to find extrema of functions of several variables.



Plane:
$6x + 4y + 3z = 24$

$(0, 0, 8)$

$\left(\frac{4}{3}, 2, \frac{8}{3}\right)$

$(0, 6, 0)$

$(4, 0, 0)$

**FIGURE 7.35**

## Lagrange Multipliers with One Constraint

In Example 5 in Section 7.5, you were asked to find the dimensions of the rectangular box of maximum volume that would fit in the first octant beneath the plane

$$6x + 4y + 3z = 24$$

as shown again in Figure 7.35. Another way of stating this problem is to say that you are asked to find the maximum of

$$V = xyz \qquad\qquad \text{Objective function}$$

subject to the constraint

$$6x + 4y + 3z - 24 = 0. \qquad\qquad \text{Constraint}$$

This type of problem is called a **constrained optimization** problem. In Section 7.5, you answered this question by solving for $z$ in the constraint equation and then rewriting $V$ as a function of two variables.

In this section, you will study a different (and often better) way to solve constrained optimization problems. This method involves the use of variables called **Lagrange multipliers,** named after the French mathematician Joseph Louis Lagrange (1736–1813).

When using the Method of Lagrange Multipliers for functions of three variables, $F$ has the form

$$F(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z).$$

The system of equations used in Step 1 are as follows.

$$F_x(x, y, z, \lambda) = 0$$
$$F_y(x, y, z, \lambda) = 0$$
$$F_z(x, y, z, \lambda) = 0$$
$$F_\lambda(x, y, z, \lambda) = 0$$

### Method of Lagrange Multipliers

If $f(x, y)$ has a maximum or minimum subject to the constraint $g(x, y) = 0$, then it will occur at one of the critical numbers of the function $F$ defined by

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y).$$

The variable $\lambda$ (the lowercase Greek letter lambda) is called a **Lagrange multiplier.** To find the minimum or maximum of $f$, use the following steps.

1. Solve the following system of equations.

$$F_x(x, y, \lambda) = 0 \qquad F_y(x, y, \lambda) = 0 \qquad F_\lambda(x, y, \lambda) = 0$$

2. Evaluate $f$ at each solution point obtained in the first step. The greatest value yields the maximum of $f$ subject to the constraint $g(x, y) = 0$, and the least value yields the minimum of $f$ subject to the constraint $g(x, y) = 0$.

The Method of Lagrange Multipliers gives you a way of finding critical points but does not tell you whether these points yield minima, maxima, or neither. To make this distinction, you must rely on the context of the problem.

*Example 1* **Using Lagrange Multipliers: One Constraint**

Find the maximum of

$$V = xyz \qquad \text{Objective function}$$

subject to the constraint

$$6x + 4y + 3z - 24 = 0. \qquad \text{Constraint}$$

**SOLUTION** First, let $f(x, y, z) = xyz$ and $g(x, y, z) = 6x + 4y + 3z - 24$. Then, define a new function $F$ as

$$F(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$$
$$= xyz - \lambda(6x + 4y + 3z - 24).$$

To find the critical numbers of $F$, set the partial derivatives of $F$ with respect to $x$, $y$, $z$, and $\lambda$ equal to zero and obtain

$$F_x(x, y, z, \lambda) = yz - 6\lambda = 0$$
$$F_y(x, y, z, \lambda) = xz - 4\lambda = 0$$
$$F_z(x, y, z, \lambda) = xy - 3\lambda = 0$$
$$F_\lambda(x, y, z, \lambda) = -6x - 4y - 3z + 24 = 0.$$

Solving for $\lambda$ in the first equation and substituting into the second and third equations produces the following.

$$xz - 4\left(\frac{yz}{6}\right) = 0 \quad \Longrightarrow \quad y = \frac{3}{2}x$$

$$xy - 3\left(\frac{yz}{6}\right) = 0 \quad \Longrightarrow \quad z = 2x$$

Next, substitute for $y$ and $z$ in the equation $F_\lambda(x, y, z, \lambda) = 0$ and solve for $x$.

$$F_\lambda(x, y, z, \lambda) = 0$$
$$-6x - 4\left(\tfrac{3}{2}x\right) - 3(2x) + 24 = 0$$
$$-18x = -24$$
$$x = \tfrac{4}{3}$$

Using this $x$-value, you can conclude that the critical values are $x = \frac{4}{3}$, $y = 2$, and $z = \frac{8}{3}$, which implies that the maximum is

$$V = xyz \qquad \text{Write objective function.}$$
$$= \left(\frac{4}{3}\right)(2)\left(\frac{8}{3}\right) \qquad \text{Substitute values of } x, y, \text{ and } z.$$
$$= \frac{64}{9} \text{ cubic units.} \qquad \text{Maximum volume}$$

---

**Algebra Review**

The most difficult aspect of many Lagrange multiplier problems is the complicated algebra needed to solve the system of equations arising from $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$. There is no general way to proceed in every case, so you should study the examples carefully, and refer to the *Chapter 7 Algebra Review* on pages 561 and 562.
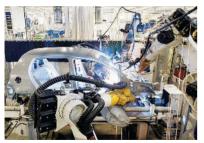
✓ **CHECKPOINT 1**

Find the maximum volume of $V = xyz$ subject to the constraint $2x + 4y + z - 8 = 0$. ■

For many industrial applications, a simple robot can cost more than a year's wages and benefits for one employee. So, manufacturers must carefully balance the amount of money spent on labor and capital.

*Example 2*
**MAKE A DECISION** **Finding a Maximum Production Level** Ⓡ

A manufacturer's production is modeled by the Cobb-Douglas function

$$f(x, y) = 100x^{3/4}y^{1/4} \qquad \text{Objective function}$$

where $x$ represents the units of labor and $y$ represents the units of capital. Each labor unit costs \$150 and each capital unit costs \$250. The total expenses for labor and capital cannot exceed \$50,000. Will the maximum production level exceed 16,000 units?

**SOLUTION** Because total labor and capital expenses cannot exceed \$50,000, the constraint is

$$150x + 250y = 50{,}000 \qquad \text{Constraint}$$
$$150x + 250y - 50{,}000 = 0. \qquad \text{Write in standard form.}$$

To find the maximum production level, begin by writing the function

$$F(x, y, \lambda) = 100x^{3/4}y^{1/4} - \lambda(150x + 250y - 50{,}000).$$

Next, set the partial derivatives of this function equal to zero.

$$F_x(x, y, \lambda) = 75x^{-1/4}y^{1/4} - 150\lambda = 0 \qquad \text{Equation 1}$$
$$F_y(x, y, \lambda) = 25x^{3/4}y^{-3/4} - 250\lambda = 0 \qquad \text{Equation 2}$$
$$F_\lambda(x, y, \lambda) = -150x - 250y + 50{,}000 = 0 \qquad \text{Equation 3}$$

The strategy for solving such a system must be customized to the particular system. In this case, you can solve for $\lambda$ in the first equation, substitute into the second equation, solve for $x$, substitute into the third equation, and solve for $y$.

$$75x^{-1/4}y^{1/4} - 150\lambda = 0 \qquad \text{Equation 1}$$
$$\lambda = \tfrac{1}{2}x^{-1/4}y^{1/4} \qquad \text{Solve for } \lambda.$$
$$25x^{3/4}y^{-3/4} - 250\left(\tfrac{1}{2}\right)x^{-1/4}y^{1/4} = 0 \qquad \text{Substitute in Equation 2.}$$
$$25x - 125y = 0 \qquad \text{Multiply by } x^{1/4}y^{3/4}.$$
$$x = 5y \qquad \text{Solve for } x.$$
$$-150(5y) - 250y + 50{,}000 = 0 \qquad \text{Substitute in Equation 3.}$$
$$-1000y = -50{,}000 \qquad \text{Simplify.}$$
$$y = 50 \qquad \text{Solve for } y.$$

Using this value for $y$, it follows that $x = 5(50) = 250$. So, the maximum production level of

$$f(250, 50) = 100(250)^{3/4}(50)^{1/4} \qquad \text{Substitute for } x \text{ and } y.$$
$$\approx 16{,}719 \text{ units} \qquad \text{Maximum production}$$

occurs when $x = 250$ units of labor and $y = 50$ units of capital. Yes, the maximum production level will exceed 16,000 units.

### TECHNOLOGY

Ⓣ You can use a spreadsheet to solve constrained optimization problems. Spreadsheet software programs have a built-in algorithm that finds absolute extrema of functions. Be sure you enter each constraint and the objective function into the spreadsheet. You should also enter initial values of the variables you are working with. Try using a spreadsheet to solve the problem in Example 2. What is your result? (Consult the user's manual of a spreadsheet software program for specific instructions on how to solve a constrained optimization problem.)

✓**CHECKPOINT 2**

In Example 2, suppose that each labor unit costs \$200 and each capital unit costs \$250. Find the maximum production level if labor and capital cannot exceed \$50,000. ∎

Economists call the Lagrange multiplier obtained in a production function the **marginal productivity of money.** For instance, in Example 2, the marginal productivity of money when $x = 250$ and $y = 50$ is

$$\lambda = \tfrac{1}{2}x^{-1/4}y^{1/4} = \tfrac{1}{2}(250)^{-1/4}(50)^{1/4} \approx 0.334.$$

This means that if one additional dollar is spent on production, approximately 0.334 additional unit of the product can be produced.

### Example 3    Finding a Maximum Production Level    ®

In Example 2, suppose that $70,000 is available for labor and capital. What is the maximum number of units that can be produced?

**SOLUTION**    You could rework the entire problem, as demonstrated in Example 2. However, because the only change in the problem is the availability of additional money to spend on labor and capital, you can use the fact that the marginal productivity of money is

$$\lambda \approx 0.334.$$

Because an additional $20,000 is available and the maximum production in Example 2 was 16,719 units, you can conclude that the maximum production is now

$$16,719 + (0.334)(20,000) \approx 23,400 \text{ units.}$$
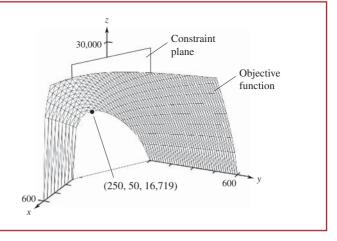
Try using the procedure demonstrated in Example 2 to confirm this result.

### ✓ CHECKPOINT 3

In Example 3, suppose that $80,000 is available for labor and capital. What is the maximum number of units that can be produced? ■

## TECHNOLOGY

Ⓣ    You can use a three-dimensional graphing utility to confirm graphically the results of Examples 2 and 3. Begin by graphing the surface $f(x, y) = 100x^{3/4}y^{1/4}$. Then graph the vertical plane given by $150x + 250y = 50,000$. As shown at the right, the maximum production level corresponds to the highest point on the intersection of the surface and the plane.

In Example 4 in Section 7.5, you found the maximum profit for two substitute products whose demand functions are given by

$$x_1 = 200(p_2 - p_1)$$  Demand for product 1
$$x_2 = 500 + 100p_1 - 180p_2.$$  Demand for product 2

With this model, the total demand, $x_1 + x_2$, is completely determined by the prices $p_1$ and $p_2$. In many real-life situations, this assumption is too simplistic; regardless of the prices of the substitute brands, the annual total demands for some products, such as toothpaste, are relatively constant. In such situations, the total demand is **limited,** and variations in price do not affect the total demand as much as they affect the market share of the substitute brands.

### Example 4 Finding a Maximum Profit ®

A company makes two substitute products whose demand functions are given by

$$x_1 = 200(p_2 - p_1)$$  Demand for product 1
$$x_2 = 500 + 100p_1 - 180p_2$$  Demand for product 2

where $p_1$ and $p_2$ are the prices per unit (in dollars) and $x_1$ and $x_2$ are the numbers of units sold. The costs of producing the two products are \$0.50 and \$0.75 per unit, respectively. The total demand is limited to 200 units per year. Find the prices that will yield a maximum profit.

**SOLUTION** From Example 4 in Section 7.5, the profit function is modeled by

$$P = -200p_1^2 - 180p_2^2 + 300p_1p_2 + 25p_1 + 535p_2 - 375.$$

The total demand for the two products is

$$x_1 + x_2 = 200(p_2 - p_1) + 500 + 100p_1 - 180p_2$$
$$= -100p_1 + 20p_2 + 500.$$
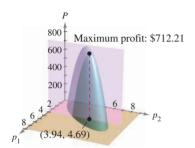
Because the total demand is limited to 200 units,

$$-100p_1 + 20p_2 + 500 = 200.$$  Constraint

Using Lagrange multipliers, you can determine that the maximum profit occurs when $p_1 = \$3.94$ and $p_2 = \$4.69$. This corresponds to an annual profit of \$712.21.

✓ **CHECKPOINT 4**

In Example 4, suppose the total demand is limited to 250 units per year. Find the prices that will yield a maximum profit. ∎

**FIGURE 7.36**

**STUDY TIP**

The constrained optimization problem in Example 4 is represented graphically in Figure 7.36. The graph of the objective function is a paraboloid and the graph of the constraint is a vertical plane. In the "unconstrained" optimization problem on page 520, the maximum profit occurred at the vertex of the paraboloid. In this "constrained" problem, however, the maximum profit corresponds to the highest point on the curve that is the intersection of the paraboloid and the vertical "constraint" plane.

## Lagrange Multipliers with Two Constraints

In Examples 1 through 4, each of the optimization problems contained only one constraint. When an optimization problem has two constraints, you need to introduce a second Lagrange multiplier. The customary symbol for this second multiplier is $\mu$, the Greek letter mu.

### Example 5    Using Lagrange Multipliers: Two Constraints

Find the minimum value of

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \text{Objective function}$$

subject to the constraints

$$x + y - 3 = 0 \qquad \text{Constraint 1}$$
$$x + z - 5 = 0. \qquad \text{Constraint 2}$$

**SOLUTION**    Begin by forming the function

$$F(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 - \lambda(x + y - 3) - \mu(x + z - 5).$$

Next, set the five partial derivatives equal to zero, and solve the resulting system of equations for $x$, $y$, and $z$.

$$F_x(x, y, z, \lambda, \mu) = 2x - \lambda - \mu = 0 \qquad \text{Equation 1}$$
$$F_y(x, y, z, \lambda, \mu) = 2y - \lambda = 0 \qquad \text{Equation 2}$$
$$F_z(x, y, z, \lambda, \mu) = 2z - \mu = 0 \qquad \text{Equation 3}$$
$$F_\lambda(x, y, z, \lambda, \mu) = -x - y + 3 = 0 \qquad \text{Equation 4}$$
$$F_\mu(x, y, z, \lambda, \mu) = -x - z + 5 = 0 \qquad \text{Equation 5}$$

✓ **CHECKPOINT 5**

Find the minimum value of
$f(x, y, z) = x^2 + y^2 + z^2$
subject to the constraints

$$x + y - 2 = 0$$
$$x + z - 4 = 0. \ \blacksquare$$

Solving this system of equations produces $x = \frac{8}{3}$, $y = \frac{1}{3}$, and $z = \frac{7}{3}$. So, the minimum value of $f(x, y, z)$ is

$$f\left(\frac{8}{3}, \frac{1}{3}, \frac{7}{3}\right) = \left(\frac{8}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{7}{3}\right)^2$$
$$= \frac{38}{3}.$$

---

**CONCEPT CHECK**

1. Lagrange multipliers are named after what French mathematician?
2. What do economists call the Lagrange multiplier obtained in a production function?
3. Explain what is meant by constrained optimization problems.
4. Explain the Method of Lagrange Multipliers for solving constrained optimization problems.

In Exercises 1–6, solve the system of linear equations.

**1.** $\begin{cases} 4x - 6y = 3 \\ 2x + 3y = 2 \end{cases}$

**2.** $\begin{cases} 6x - 6y = 5 \\ -3x - y = 1 \end{cases}$

**3.** $\begin{cases} 5x - y = 25 \\ x - 5y = 15 \end{cases}$

**4.** $\begin{cases} 4x - 9y = 5 \\ -x + 8y = -2 \end{cases}$

**5.** $\begin{cases} 2x - y + z = 3 \\ 2x + 2y + z = 4 \\ -x + 2y + 3z = -1 \end{cases}$

**6.** $\begin{cases} -x - 4y + 6z = -2 \\ x - 3y - 3z = 4 \\ 3x + y + 3z = 0 \end{cases}$

In Exercises 7–10, find all first partial derivatives.

**7.** $f(x, y) = x^2 y + xy^2$

**8.** $f(x, y) = 25(xy + y^2)^2$

**9.** $f(x, y, z) = x(x^2 - 2xy + yz)$

**10.** $f(x, y, z) = z(xy + xz + yz)$

## Exercises 7.6

In Exercises 1–12, use Lagrange multipliers to find the given extremum. In each case, assume that $x$ and $y$ are positive.

| *Objective Function* | *Constraint* |
|---|---|
| **1.** Maximize $f(x, y) = xy$ | $x + y = 10$ |
| **2.** Maximize $f(x, y) = xy$ | $2x + y = 4$ |
| **3.** Minimize $f(x, y) = x^2 + y^2$ | $x + y - 4 = 0$ |
| **4.** Minimize $f(x, y) = x^2 + y^2$ | $-2x - 4y + 5 = 0$ |
| **5.** Maximize $f(x, y) = x^2 - y^2$ | $2y - x^2 = 0$ |
| **6.** Minimize $f(x, y) = x^2 - y^2$ | $x - 2y + 6 = 0$ |
| **7.** Maximize $f(x, y) = 2x + 2xy + y$ | $2x + y = 100$ |
| **8.** Minimize $f(x, y) = 3x + y + 10$ | $x^2 y = 6$ |
| **9.** Maximize $f(x, y) = \sqrt{6 - x^2 - y^2}$ | $x + y - 2 = 0$ |
| **10.** Minimize $f(x, y) = \sqrt{x^2 + y^2}$ | $2x + 4y - 15 = 0$ |
| **11.** Maximize $f(x, y) = e^{xy}$ | $x^2 + y^2 - 8 = 0$ |
| **12.** Minimize $f(x, y) = 2x + y$ | $xy = 32$ |

In Exercises 13–18, use Lagrange multipliers to find the given extremum. In each case, assume that $x$, $y$, and $z$ are positive.

**13.** Minimize $f(x, y, z) = 2x^2 + 3y^2 + 2z^2$

Constraint: $x + y + z - 24 = 0$

**14.** Maximize $f(x, y, z) = xyz$

Constraint: $x + y + z - 6 = 0$

**15.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraint: $x + y + z = 1$

**16.** Minimize $f(x, y) = x^2 - 8x + y^2 - 12y + 48$

Constraint: $x + y = 8$

**17.** Maximize $f(x, y, z) = x + y + z$

Constraint: $x^2 + y^2 + z^2 = 1$

**18.** Maximize $f(x, y, z) = x^2 y^2 z^2$

Constraint: $x^2 + y^2 + z^2 = 1$

In Exercises 19–22, use Lagrange multipliers to find the given extremum of $f$ subject to two constraints. In each case, assume that $x$, $y$, and $z$ are nonnegative.

**19.** Maximize $f(x, y, z) = xyz$

Constraints: $x + y + z = 32$, $x - y + z = 0$

**20.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraints: $x + 2z = 6$, $x + y = 12$

**21.** Maximize $f(x, y, z) = xyz$

Constraints: $x^2 + z^2 = 5$, $x - 2y = 0$

**22.** Maximize $f(x, y, z) = xy + yz$

Constraints: $x + 2y = 6$, $x - 3z = 0$

Ⓢ In Exercises 23 and 24, use a spreadsheet to find the given extremum. In each case, assume that $x$, $y$, and $z$ are nonnegative.

**23.** Maximize $f(x, y, z) = xyz$

Constraints: $x + 3y = 6$, $x - 2z = 0$

**24.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraints: $x + 2y = 8$, $x + z = 4$

In Exercises 25–28, find three positive numbers $x$, $y$, and $z$ that satisfy the given conditions.

**25.** The sum is 120 and the product is maximum.

**26.** The sum is 120 and the sum of the squares is minimum.

**27.** The sum is $S$ and the product is maximum.

**28.** The sum is $S$ and the sum of the squares is minimum.

In Exercises 29–32, find the minimum distance from the curve or surface to the given point. (*Hint:* Start by minimizing the square of the distance.)

**29.** Line: $x + y = 6$, $(0, 0)$

Minimize $d^2 = x^2 + y^2$

**30.** Circle: $(x - 4)^2 + y^2 = 4$, $(0, 10)$

Minimize $d^2 = x^2 + (y - 10)^2$
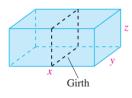
**31.** Plane: $x + y + z = 1$, $(2, 1, 1)$

Minimize $d^2 = (x - 2)^2 + (y - 1)^2 + (z - 1)^2$

**32.** Cone: $z = \sqrt{x^2 + y^2}$, $(4, 0, 0)$

Minimize $d^2 = (x - 4)^2 + y^2 + z^2$

**33. Volume** Find the dimensions of the rectangular package of largest volume subject to the constraint that the sum of the length and the girth cannot exceed 108 inches (see figure). (*Hint:* Maximize $V = xyz$ subject to the constraint $x + 2y + 2z = 108$.)



Figure for 33    Figure for 34

**34. Cost** In redecorating an office, the cost for new carpeting is $3 per square foot and the cost of wallpapering a wall is $1 per square foot. Find the dimensions of the largest office that can be redecorated for $1296 (see figure). (*Hint:* Maximize $V = xyz$ subject to $3xy + 2xz + 2yz = 1296$.)

**35. Cost** A cargo container (in the shape of a rectangular solid) must have a volume of 480 cubic feet. Use Lagrange multipliers to find the dimensions of the container of this size that has a minimum cost, if the bottom will cost $5 per square foot to construct and the sides and top will cost $3 per square foot to construct.

**36. Cost** A manufacturer has an order for 1000 units of fine paper that can be produced at two locations. Let $x_1$ and $x_2$ be the numbers of units produced at the two plants. Find the number of units that should be produced at each plant to minimize the cost if the cost function is given by

$C = 0.25x_1^2 + 25x_1 + 0.05x_2^2 + 12x_2$.

**37. Cost** A manufacturer has an order for 2000 units of all-terrain vehicle tires that can be produced at two locations. Let $x_1$ and $x_2$ be the numbers of units produced at the two plants. The cost function is modeled by

$C = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2$.

Find the number of units that should be produced at each plant to minimize the cost.

**38. Hardy-Weinberg Law** Repeat Exercise 47 in Section 7.5 using Lagrange multipliers—that is, maximize

$P(p, q, r) = 2pq + 2pr + 2qr$

subject to the constraint

$p + q + r = 1$.

**39. Least-Cost Rule** The production function for a company is given by

$f(x, y) = 100x^{0.25}y^{0.75}$

where $x$ is the number of units of labor and $y$ is the number of units of capital. Suppose that labor costs $48 per unit, capital costs $36 per unit, and management sets a production goal of 20,000 units.

(a) Find the numbers of units of labor and capital needed to meet the production goal while minimizing the cost.

(b) Show that the conditions of part (a) are met when

$$\frac{\text{Marginal productivity of labor}}{\text{Marginal productivity of capital}} = \frac{\text{unit price of labor}}{\text{unit price of capital}}.$$

This proportion is called the *Least-Cost Rule* (or Equimarginal Rule).

**40. Least-Cost Rule** Repeat Exercise 39 for the production function given by

$f(x, y) = 100x^{0.6}y^{0.4}$.

**41. Production** The production function for a company is given by

$f(x, y) = 100x^{0.25}y^{0.75}$

where $x$ is the number of units of labor and $y$ is the number of units of capital. Suppose that labor costs $48 per unit and capital costs $36 per unit. The total cost of labor and capital is limited to $100,000.

(a) Find the maximum production level for this manufacturer.

(b) Find the marginal productivity of money.

(c) Use the marginal productivity of money to find the maximum number of units that can be produced if $125,000 is available for labor and capital.

**42. Production** Repeat Exercise 41 for the production function given by

$f(x, y) = 100x^{0.6}y^{0.4}$.

**43. Biology**   A microbiologist must prepare a culture medium in which to grow a certain type of bacteria. The percent of salt contained in this medium is given by

$$S = 12xyz$$

where $x$, $y$, and $z$ are the nutrient solutions to be mixed in the medium. For the bacteria to grow, the medium must be 13% salt. Nutrient solutions $x$, $y$, and $z$ cost \$1, \$2, and \$3 per liter, respectively. How much of each nutrient solution should be used to minimize the cost of the culture medium?

**44. Biology**   Repeat Exercise 43 for a salt-content model of

$$S = 0.01x^2y^2z^2.$$

**45. Animal Shelter**   An animal shelter buys two different brands of dog food. The number of dogs that can be fed from $x$ pounds of the first brand and $y$ pounds of the second brand is given by the model
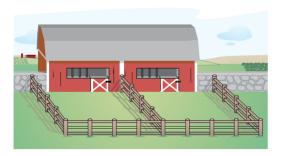
$$D(x, y) = -x^2 + 52x - y^2 + 44y + 256.$$

(a) The shelter orders 100 pounds of dog food. Use Lagrange multipliers to find the number of pounds of each brand of dog food that should be in the order so that the maximum number of dogs can be fed.

(b) What is the maximum number of dogs that can be fed?

**46. Nutrition**   The number of grams of your favorite ice cream can be modeled by
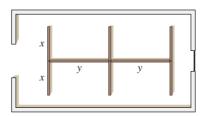
$$G(x, y, z) = 0.05x^2 + 0.16xy + 0.25z^2$$

where $x$ is the number of fat grams, $y$ is the number of carbohydrate grams, and $z$ is the number of protein grams. Use Lagrange multipliers to find the maximum number of grams of ice cream you can eat without consuming more than 400 calories. Assume that there are 9 calories per fat gram, 4 calories per carbohydrate gram, and 4 calories per protein gram.

**47. Construction**   A rancher plans to use an existing stone wall and the side of a barn as a boundary for two adjacent rectangular corrals. Fencing for the perimeter costs \$10 per foot. To separate the corrals, a fence that costs \$4 per foot will divide the region. The total area of the two corrals is to be 6000 square feet.

(a) Use Lagrange multipliers to find the dimensions that will minimize the cost of the fencing.

(b) What is the minimum cost?



**48. Office Space**   Partitions will be used in an office to form four equal work areas with a total area of 360 square feet (see figure). The partitions that are $x$ feet long cost \$100 per foot and the partitions that are $y$ feet long cost \$120 per foot.

(a) Use Lagrange multipliers to find the dimensions $x$ and $y$ that will minimize the cost of the partitions.

(b) What is the minimum cost?



**49. Investment Strategy**   An investor is considering three different stocks in which to invest \$300,000. The average annual dividends for the stocks are

| | |
|---|---|
| General Motors (G) | 2.7% |
| Pepsico, Inc. (P) | 1.7% |
| Sara Lee (S) | 2.4%. |

The amount invested in Pepsico, Inc. must follow the equation

$$3000(S) - 3000(G) + P^2 = 0.$$

How much should be invested in each stock to yield a maximum of dividends?

**50. Investment Strategy**   An investor is considering three different stocks in which to invest \$20,000. The average annual dividends for the stocks are

| | |
|---|---|
| General Mills (G) | 2.4% |
| Campbell Soup (C) | 1.8% |
| Kellogg Co. (K) | 1.9%. |

The amount invested in Campbell Soup must follow the equation

$$1000(K) - 1000(G) + C^2 = 0.$$

How much should be invested in each stock to yield a maximum of dividends?

**51. Advertising**   A private golf club is determining how to spend its \$2700 advertising budget. The club knows from prior experience that the number of responses $A$ is given by $A = 0.0001t^2pr^{1.5}$, where $t$ is the number of cable television ads, $p$ is the number of newspaper ads, and $r$ is the number of radio ads. A cable television ad costs \$30, a newspaper ad costs \$12, and a radio ad costs \$15.

(a) How much should be spent on each type of advertising to obtain the maximum number of responses? (Assume the golf club uses each type of advertising.)

(b) What is the maximum number of responses expected?

# Least Squares Regression Analysis

- Find the sum of the squared errors for mathematical models.
- Find the least squares regression lines for data.
- Find the least squares regression quadratics for data.

## Measuring the Accuracy of a Mathematical Model

When seeking a mathematical model to fit real-life data, you should try to find a model that is both as *simple* and as *accurate* as possible. For instance, a simple linear model for the points shown in Figure 7.37(a) is

$$f(x) = 1.8566x - 5.0246. \qquad \text{Linear model}$$
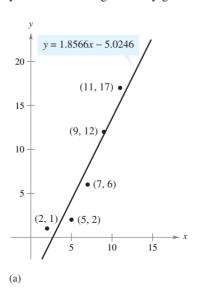
However, Figure 7.37(b) shows that by choosing a slightly more complicated quadratic model

$$g(x) = 0.1996x^2 - 0.7281x + 1.3749 \qquad \text{Quadratic model}$$

you can obtain significantly greater accuracy.



(a)                                           (b)
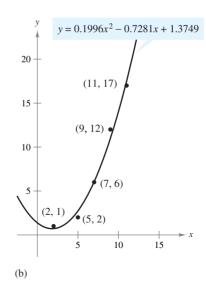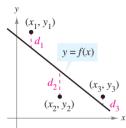
**FIGURE 7.37**

To measure how well the model $y = f(x)$ fits a collection of points, sum the squares of the differences between the actual $y$-values and the model's $y$-values. This sum is called the **sum of the squared errors** and is denoted by $S$. Graphically, $S$ can be interpreted as the sum of the squares of the vertical distances between the graph of $f$ and the given points in the plane, as shown in Figure 7.38. If the model is a perfect fit, then $S = 0$. However, when a perfect fit is not feasible, you should use a model that minimizes $S$.



Sum of the squared errors:
$$S = d_1^2 + d_2^2 + d_3^2$$

**FIGURE 7.38**

**Definition of the Sum of the Squared Errors**

The **sum of the squared errors** for the model $y = f(x)$ with respect to the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is given by

$$S = [f(x_1) - y_1]^2 + [f(x_2) - y_2]^2 + \cdots + [f(x_n) - y_n]^2.$$

*Example 1*   **Finding the Sum of the Squared Errors**

Find the sum of the squared errors for the linear and quadratic models

$$f(x) = 1.8566x - 5.0246 \qquad \text{Linear model}$$
$$g(x) = 0.1996x^2 - 0.7281x + 1.3749 \qquad \text{Quadratic model}$$

(see Figure 7.37) with respect to the points

$$(2, 1), (5, 2), (7, 6), (9, 12), (11, 17).$$

**SOLUTION**   Begin by evaluating each model at the given $x$-values, as shown in the table.

| $x$ | 2 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|
| Actual $y$-values | 1 | 2 | 6 | 12 | 17 |
| Linear model, $f(x)$ | $-1.3114$ | 4.2584 | 7.9716 | 11.6848 | 15.3980 |
| Quadratic model, $g(x)$ | 0.7171 | 2.7244 | 6.0586 | 10.9896 | 17.5174 |

For the linear model $f$, the sum of the squared errors is

$$S = (-1.3114 - 1)^2 + (4.2584 - 2)^2 + (7.9716 - 6)^2$$
$$+ (11.6848 - 12)^2 + (15.3980 - 17)^2$$
$$\approx 16.9959.$$

Similarly, the sum of the squared errors for the quadratic model $g$ is

$$S = (0.7171 - 1)^2 + (2.7244 - 2)^2 + (6.0586 - 6)^2$$
$$+ (10.9896 - 12)^2 + (17.5174 - 17)^2$$
$$\approx 1.8968.$$

**STUDY TIP**

In Example 1, note that the sum of the squared errors for the quadratic model is less than the sum of the squared errors for the linear model, which confirms that the quadratic model is a better fit.

✓ **CHECKPOINT 1**

Find the sum of the squared errors for the linear and quadratic models

$$f(x) = 2.85x - 6.1$$
$$g(x) = 0.1964x^2 + 0.4929x - 0.6$$

with respect to the points $(2, 1), (4, 5), (6, 9), (8, 16), (10, 24)$. Then decide which model is a better fit. ■

## Least Squares Regression Line

The sum of the squared errors can be used to determine which of several models is the best fit for a collection of data. In general, if the sum of the squared errors of $f$ is less than the sum of the squared errors of $g$, then $f$ is said to be a better fit for the data than $g$. In regression analysis, you consider all possible models of a certain type. The one that is defined to be the best-fitting model is the one with the least sum of the squared errors. Example 2 shows how to use the optimization techniques described in Section 7.5 to find the best-fitting linear model for a collection of data.

### Example 2   Finding the Best Linear Model

Find the values of $a$ and $b$ such that the linear model

$$f(x) = ax + b$$

has a minimum sum of the squared errors for the points

$$(-3, 0), (-1, 1), (0, 2), (2, 3).$$

**SOLUTION**    The sum of the squared errors is

$$
\begin{aligned}
S &= [f(x_1) - y_1]^2 + [f(x_2) - y_2]^2 + [f(x_3) - y_3]^2 + [f(x_4) - y_4]^2 \\
&= (-3a + b - 0)^2 + (-a + b - 1)^2 + (b - 2)^2 + (2a + b - 3)^2 \\
&= 14a^2 - 4ab + 4b^2 - 10a - 12b + 14.
\end{aligned}
$$

To find the values of $a$ and $b$ for which $S$ is a minimum, you can use the techniques described in Section 7.5. That is, find the partial derivatives of $S$.

$$\frac{\partial S}{\partial a} = 28a - 4b - 10 \qquad \text{Differentiate with respect to } a.$$

$$\frac{\partial S}{\partial b} = -4a + 8b - 12 \qquad \text{Differentiate with respect to } b.$$

Next, set each partial derivative equal to zero.

$$28a - 4b - 10 = 0 \qquad \text{Set } \partial S/\partial a \text{ equal to 0.}$$
$$-4a + 8b - 12 = 0 \qquad \text{Set } \partial S/\partial b \text{ equal to 0.}$$
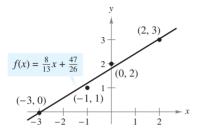
The solution of this system of linear equations is

$$a = \frac{8}{13} \quad \text{and} \quad b = \frac{47}{26}.$$

So, the best-fitting linear model for the given points is

$$f(x) = \frac{8}{13}x + \frac{47}{26}.$$

The graph of this model is shown in Figure 7.39.

### Algebra Review

For help in solving the system of equations in Example 2, see Example 2(b) in the *Chapter 7 Algebra Review*, on page 562.



$f(x) = \frac{8}{13}x + \frac{47}{26}$

**FIGURE 7.39**

### ✓ CHECKPOINT 2

Find the values of $a$ and $b$ such that the linear model $f(x) = ax + b$ has a minimum sum of the squared errors for the points $(-2, 0)$, $(0, 2)$, $(2, 5)$, $(4, 7)$. ■

The line in Example 2 is called the **least squares regression line** for the given data. The solution shown in Example 2 can be generalized to find a formula for the least squares regression line, as shown below. Consider the linear model

$$f(x) = ax + b$$

and the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. The sum of the squared errors is

$$S = \sum_{i=1}^{n} [f(x_i) - y_i]^2 = \sum_{i=1}^{n} (ax_i + b - y_i)^2.$$

To minimize $S$, set the partial derivatives $\partial S / \partial a$ and $\partial S / \partial b$ equal to zero and solve for $a$ and $b$. The results are summarized below.

---

**The Least Squares Regression Line**

The **least squares regression line** for the points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

is $y = ax + b$, where

$$a = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2} \quad \text{and} \quad b = \frac{1}{n} \left( \sum_{i=1}^{n} y_i - a \sum_{i=1}^{n} x_i \right).$$

---

In the formula for the least squares regression line, note that if the $x$-values are symmetrically spaced about zero, then

$$\sum_{i=1}^{n} x_i = 0$$

and the formulas for $a$ and $b$ simplify to

$$a = \frac{n \sum_{i=1}^{n} x_i y_i}{n \sum_{i=1}^{n} x_i^2} \quad \text{and} \quad b = \frac{1}{n} \sum_{i=1}^{n} y_i.$$

Note also that only the *development* of the least squares regression line involves partial derivatives. The *application* of this formula is simply a matter of computing the values of $a$ and $b$—a task that is performed much more simply on a calculator or a computer than by hand.

---

### DISCOVERY

Graph the three points $(2, 2)$, $(2, 1)$, and $(2.1, 1.5)$ and visually estimate the least squares regression line for these data. Now use the formulas on this page or a graphing utility to show that the equation of the line is actually $y = 1.5$. In general, the least squares regression line for "nearly vertical" data can be quite unusual. Show that by interchanging the roles of $x$ and $y$, you can obtain a better linear approximation.

*Example 3* **Modeling Hourly Wages** (R)

The average hourly wages $y$ (in dollars per hour) for production workers in manufacturing industries from 1998 through 2006 are shown in the table. Find the least squares regression line for the data and use the result to estimate the average hourly wage in 2010. *(Source: U.S. Bureau of Labor Statistics)*
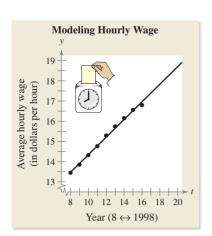
| Year | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 |
|------|------|------|------|------|------|------|------|------|------|
| $y$ | 13.45 | 13.85 | 14.32 | 14.76 | 15.29 | 15.74 | 16.15 | 16.56 | 16.80 |

**SOLUTION** Let $t$ represent the year, with $t = 8$ corresponding to 1998. Then, you need to find the linear model that best fits the points

(8, 13.45), (9, 13.85), (10, 14.32), (11, 14.76), (12, 15.29), (13, 15.74), (14, 16.15), (15, 16.56), (16, 16.80).

Using a calculator with a built-in least squares regression program, you can determine that the best-fitting line is $y = 9.98 + 0.436t$. With this model, you can estimate the 2010 average hourly wage, using $t = 20$, to be

$$y = 9.98 + 0.436(20) = \$18.70 \text{ per hour.}$$

This result is shown graphically in Figure 7.40.



**Modeling Hourly Wage**

Average hourly wage (in dollars per hour)

Year (8 ↔ 1998)

**FIGURE 7.40**

✓**CHECKPOINT 3**

The numbers of cellular phone subscribers $y$ (in thousands) for the years 2001 through 2005 are shown in the table. Find the least squares regression line for the data and use the result to estimate the number of subscribers in 2010. Let $t$ represent the year, with $t = 1$ corresponding to 2001. *(Source: Cellular Telecommunications & Internet Association)*

| Year | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|
| $y$ | 128,375 | 140,767 | 158,722 | 182,140 | 207,896 |

■

**TECHNOLOGY**

(T) Most graphing utilities and spreadsheet software programs have a built-in linear regression program. When you run such a program, the "$r$-value" gives a measure of how well the model fits the data. The closer the value of $|r|$ is to 1, the better the fit. For the data in Example 3, $r \approx 0.998$, which implies that the model is a very good fit. Use a graphing utility or a spreadsheet software program to find the least squares regression line and compare your results with those in Example 3. (Consult the user's manual of a graphing utility or a spreadsheet software program for specific instructions.)*

———————

*Specific calculator keystroke instructions for operations in this and other technology boxes can be found at *college.hmco.com/info/larsonapplied*.

## Least Squares Regression Quadratic

When using regression analysis to model data, remember that the least squares regression line provides only the best *linear* model for a set of data. It does not necessarily provide the best possible model. For instance, in Example 1, you saw that the quadratic model was a better fit than the linear model.

Regression analysis can be performed with many different types of models, such as exponential or logarithmic models. The following development shows how to find the best-fitting quadratic model for a collection of data points. Consider a quadratic model of the form

$$f(x) = ax^2 + bx + c.$$

The sum of the squared errors for this model is

$$S = \sum_{i=1}^{n} [f(x_i) - y_i]^2 = \sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)^2.$$

To find the values of $a$, $b$, and $c$ that minimize $S$, set the three partial derivatives, $\partial S / \partial a$, $\partial S / \partial b$, and $\partial S / \partial c$, equal to zero.

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} 2x_i^2 (ax_i^2 + bx_i + c - y_i) = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} 2x_i (ax_i^2 + bx_i + c - y_i) = 0$$

$$\frac{\partial S}{\partial c} = \sum_{i=1}^{n} 2(ax_i^2 + bx_i + c - y_i) = 0$$

By expanding this system, you obtain the result given in the summary below.

---

### Least Squares Regression Quadratic

The **least squares regression quadratic** for the points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

is $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are the solutions of the system of equations below.

$$a \sum_{i=1}^{n} x_i^4 + b \sum_{i=1}^{n} x_i^3 + c \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i$$

$$a \sum_{i=1}^{n} x_i^3 + b \sum_{i=1}^{n} x_i^2 + c \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

$$a \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i$$

---

### TECHNOLOGY
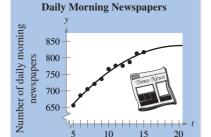
Ⓣ Most graphing utilities have a built-in program for finding the least squares regression quadratic. This program works just like the program for the least squares line. You should use this program to verify your solutions to the exercises.

*Example 4* **Modeling Numbers of Newspapers** Ⓡ

The numbers $y$ of daily morning newspapers in the United States from 1995 through 2005 are shown in the table. Find the least squares regression quadratic for the data and use the result to estimate the number of daily morning newspapers in 2010. *(Source: Editor & Publisher Co.)*

| Year | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|------|------|------|------|------|------|
| $y$ | 656 | 686 | 705 | 721 | 736 | 766 | 776 | 776 | 787 | 813 | 817 |

**Daily Morning Newspapers**



**FIGURE 7.41**

**SOLUTION** Let $t$ represent the year, with $t = 5$ corresponding to 1995. Then, you need to find the quadratic model that best fits the points

$(5, 656)$, $(6, 686)$, $(7, 705)$, $(8, 721)$, $(9, 736)$, $(10, 766)$, $(11, 776)$, $(12, 776)$, $(13, 787)$, $(14, 813)$, $(15, 817)$.

Using a calculator with a built-in least squares regression program, you can determine that the best-fitting quadratic is $y = -0.76t^2 + 30.8t + 525$. With this model, you can estimate the number of daily morning newspapers in 2010, using $t = 20$, to be

$y = -0.76(20)^2 + 30.8(20) + 525 = 837.$

This result is shown graphically in Figure 7.41.

✓**CHECKPOINT 4**

The per capita expenditures $y$ for health services and supplies in dollars in the United States for selected years are listed in the table. Find the least squares regression quadratic for the data and use the result to estimate the per capita expenditure for health care in 2010. Let $t$ represent the year, with $t = 9$ corresponding to 1999. *(Source: U.S. Centers for Medicare and Medicaid Services)*

| Year | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|------|------|
| $y$ | 3818 | 4034 | 4340 | 4652 | 4966 | 5276 | 5598 |

**CONCEPT CHECK**

1. What are the two main goals when seeking a mathematical model to fit real-life data?
2. What does *S*, the sum of the squared errors, measure?
3. Describe how to find the least squares regression line for a given set of data.
4. Describe how to find the least squares regression quadratic for a given set of data.

**Skills Review 7.7** The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 7.4.

In Exercises 1 and 2, evaluate the expression.

**1.** $(2.5 - 1)^2 + (3.25 - 2)^2 + (4.1 - 3)^2$

**2.** $(1.1 - 1)^2 + (2.08 - 2)^2 + (2.95 - 3)^2$

In Exercises 3 and 4, find the partial derivatives of $S$.

**3.** $S = a^2 + 6b^2 - 4a - 8b - 4ab + 6$

**4.** $S = 4a^2 + 9b^2 - 6a - 4b - 2ab + 8$

In Exercises 5–10, evaluate the sum.

**5.** $\displaystyle\sum_{i=1}^{5} i$

**6.** $\displaystyle\sum_{i=1}^{6} 2i$

**7.** $\displaystyle\sum_{i=1}^{4} \frac{1}{i}$

**8.** $\displaystyle\sum_{i=1}^{3} i^2$

**9.** $\displaystyle\sum_{i=1}^{6} (2 - i)^2$
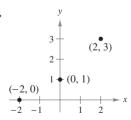
**10.** $\displaystyle\sum_{i=1}^{5} (30 - i^2)$
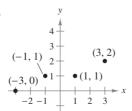
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## Exercises 7.7
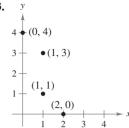
In Exercises 1–4, (a) find the least squares regression line and (b) calculate $S$, the sum of the squared errors. Use the regression capabilities of a graphing utility or a spreadsheet to verify your results.

**1.**



**2.**



**3.**



**4.**
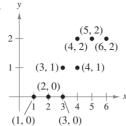


(T)(S) In Exercises 5–8, find the least squares regression line for the points. Use the regression capabilities of a graphing utility or a spreadsheet to verify your results. Then plot the points and graph the regression line.

**5.** $(-2, -1), (0, 0), (2, 3)$

**6.** $(-3, 0), (-1, 1), (1, 1), (3, 2)$

**7.** $(-2, 4), (-1, 1), (0, -1), (1, -3)$

**8.** $(-5, -3), (-4, -2), (-2, -1), (-1, 1)$

(T)(S) In Exercises 9–18, use the regression capabilities of a graphing utility or a spreadsheet to find the least squares regression line for the given points.

**9.** $(-2, 0), (-1, 1), (0, 1), (1, 2), (2, 3)$

**10.** $(-4, -1), (-2, 0), (2, 4), (4, 5)$

**11.** $(-2, 2), (2, 6), (3, 7)$

**12.** $(-5, 1), (1, 3), (2, 3), (2, 5)$

**13.** $(-3, 4), (-1, 2), (1, 1), (3, 0)$

**14.** $(-10, 10), (-5, 8), (3, 6), (7, 4), (5, 0)$

**15.** $(0, 0), (1, 1), (3, 4), (4, 2), (5, 5)$

**16.** $(1, 0), (3, 3), (5, 6)$

**17.** $(0, 6), (4, 3), (5, 0), (8, -4), (10, -5)$

**18.** $(6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8)$

(T)(S) In Exercises 19–22, use the regression capabilities of a graphing utility or a spreadsheet to find the least squares regression quadratic for the given points. Then plot the points and graph the least squares regression quadratic.

**19.** $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

**20.** $(-4, 5), (-2, 6), (2, 6), (4, 2)$

**21.** $(0, 0), (2, 2), (3, 6), (4, 12)$

**22.** $(0, 10), (1, 9), (2, 6), (3, 0)$

(T) (S) In Exercises 23–26, use the regression capabilities of a graphing utility or a spreadsheet to find linear and quadratic models for the data. State which model best fits the data.

**23.** $(-4, 1), (-3, 2), (-2, 2), (-1, 4), (0, 6), (1, 8), (2, 9)$

**24.** $(-1, -4), (0, -3), (1, -3), (2, 0), (4, 5), (6, 9), (9, 3)$

**25.** $(0, 769), (1, 677), (2, 601), (3, 543), (4, 489), (5, 411)$

**26.** $(1, 10.3), (2, 14.2), (3, 18.9), (4, 23.7), (5, 29.1), (6, 35)$

**27. Demand**   A store manager wants to know the demand $y$ for an energy bar as a function of price $x$. The daily sales for three different prices of the product are listed in the table.

| Price, $x$ | $1.00 | $1.25 | $1.50 |
|---|---|---|---|
| Demand, $y$ | 450 | 375 | 330 |

(T) (S) (a) Use the regression capabilities of a graphing utility or a spreadsheet to find the least squares regression line for the data.

(b) Estimate the demand when the price is $1.40.

(c) What price will create a demand of 500 energy bars?

**28. Demand**   A hardware retailer wants to know the demand $y$ for a tool as a function of price $x$. The monthly sales for four different prices of the tool are listed in the table.

| Price, $x$ | $25 | $30 | $35 | $40 |
|---|---|---|---|---|
| Demand, $y$ | 82 | 75 | 67 | 55 |

(T) (S) (a) Use the regression capabilities of a graphing utility or a spreadsheet to find the least squares regression line for the data.

(b) Estimate the demand when the price is $32.95.

(c) What price will create a demand of 83 tools?

**29. Agriculture**   An agronomist used four test plots to determine the relationship between the wheat yield $y$ (in bushels per acre) and the amount of fertilizer $x$ (in hundreds of pounds per acre). The results are shown in the table.

| Fertilizer, $x$ | 1.0 | 1.5 | 2.0 | 2.5 |
|---|---|---|---|---|
| Yield, $y$ | 35 | 44 | 50 | 56 |

(T) (S) (a) Use the regression capabilities of a graphing utility or a spreadsheet to find the least squares regression line for the data.

(b) Estimate the yield for a fertilizer application of 160 pounds per acre.

(T) (S) **30. Finance: Median Income**   In the table below are the median income levels for various age levels in the United States. Use the regression capabilities of a graphing utility or a spreadsheet to find the least squares regression quadratic for the data and use the resulting model to estimate the median income for someone who is 28 years old.   *(Source: U.S. Census Bureau)*

| Age level, $x$ | 20 | 30 | 40 |
|---|---|---|---|
| Median income, $y$ | 28,800 | 47,400 | 58,100 |

| Age level, $x$ | 50 | 60 | 70 |
|---|---|---|---|
| Median income, $y$ | 62,400 | 52,300 | 26,000 |

(T) (S) **31. Infant Mortality**   To study the numbers $y$ of infant deaths per 1000 live births in the United States, a medical researcher obtains the data listed in the table.   *(Source: U.S. National Center for Health Statistics)*

| Year | 1980 | 1985 | 1990 | 1995 | 2000 | 2005 |
|---|---|---|---|---|---|---|
| Deaths, $y$ | 12.6 | 10.6 | 9.2 | 7.6 | 6.9 | 6.8 |

(a) Use the regression capabilities of a graphing utility or a spreadsheet to find the least squares regression line for the data and use this line to estimate the number of infant deaths in 2010. Let $t = 0$ represent 1980.

(b) Use the regression capabilities of a graphing utility or a spreadsheet to find the least squares regression quadratic for the data and use the model to estimate the number of infant deaths in 2010.

(T) (S) **32. Population Growth**   The table gives the approximate world populations $y$ (in billions) for six different years.   *(Source: U.S. Census Bureau)*

| Year | 1800 | 1850 | 1900 | 1950 | 1990 | 2005 |
|---|---|---|---|---|---|---|
| Time, $t$ | $-2$ | $-1$ | 0 | 1 | 1.8 | 2.1 |
| Population, $y$ | 0.8 | 1.1 | 1.6 | 2.4 | 5.3 | 6.5 |

(a) During the 1800s, population growth was almost linear. Use the regression capabilities of a graphing utility or a spreadsheet to find a least squares regression line for those years and use the line to estimate the population in 1875.

(b) Use the regression capabilities of a graphing utility or a spreadsheet to find a least squares regression quadratic for the data from 1850 through 2005 and use the model to estimate the population in the year 2010.

(c) Even though the rate of growth of the population has begun to decline, most demographers believe the population size will pass the 8 billion mark sometime in the next 25 years. What do you think?

(T)(S) **33.** *MAKE A DECISION: REVENUE* The revenues $y$ (in millions of dollars) for Earthlink from 2000 through 2006 are shown in the table. *(Source: Earthlink, Inc.)*
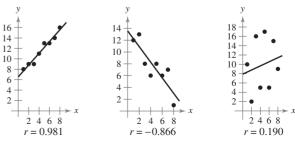
| Year | 2000 | 2001 | 2002 | 2003 |
|------|------|------|------|------|
| Revenue, $y$ | 986.6 | 1244.9 | 1357.4 | 1401.9 |

| Year | 2004 | 2005 | 2006 |
|------|------|------|------|
| Revenue, $y$ | 1382.2 | 1290.1 | 1301.3 |

(a) Use a graphing utility or a spreadsheet to create a scatter plot of the data. Let $t = 0$ represent the year 2000.

(b) Use the regression capabilities of a graphing utility or a spreadsheet to find an appropriate model for the data.

(c) Explain why you chose the type of model that you created in part (b).

(T)(S) **34.** *MAKE A DECISION: COMPUTERS AND INTERNET USERS* The global numbers of personal computers $x$ (in millions) and Internet users $y$ (in millions) from 1999 through 2005 are shown in the table. *(Source: International Telecommunication Union)*

| Year | 1999 | 2000 | 2001 | 2002 |
|------|------|------|------|------|
| Personal computers, $x$ | 394.1 | 465.4 | 526.7 | 575.5 |
| Internet users, $y$ | 275.5 | 390.3 | 489.9 | 618.4 |

| Year | 2003 | 2004 | 2005 |
|------|------|------|------|
| Personal computers, $x$ | 636.6 | 776.6 | 808.7 |
| Internet users, $y$ | 718.8 | 851.8 | 982.5 |

(a) Use a graphing utility or a spreadsheet to create a scatter plot of the data.

(b) Use the regression capabilities of a graphing utility or a spreadsheet to find an appropriate model for the data.

(c) Explain why you chose the type of model that you created in part (b).

(T)(S) In Exercises 35–38, use the regression capabilities of a graphing utility or a spreadsheet to find any model that best fits the data points.

**35.** $(1, 13)$, $(2, 16.5)$, $(4, 24)$, $(5, 28)$, $(8, 39)$, $(11, 50.25)$, $(17, 72)$, $(20, 85)$

**36.** $(1, 5.5)$, $(3, 7.75)$, $(6, 15.2)$, $(8, 23.5)$, $(11, 46)$, $(15, 110)$

**37.** $(1, 1.5)$, $(2.5, 8.5)$, $(5, 13.5)$, $(8, 16.7)$, $(9, 18)$, $(20, 22)$

**38.** $(0, 0.5)$, $(1, 7.6)$, $(3, 60)$, $(4.2, 117)$, $(5, 170)$, $(7.9, 380)$

In Exercises 39–44, plot the points and determine whether the data have positive, negative, or no linear correlation (see figures below). Then use a graphing utility to find the value of $r$ and confirm your result. The number $r$ is called the *correlation coefficient*. It is a measure of how well the model fits the data. Correlation coefficients vary between $-1$ and 1, and the closer $|r|$ is to 1, the better the model.



$r = 0.981$  $r = -0.866$  $r = 0.190$
Positive correlation   Negative correlation   No correlation

**39.** $(1, 4)$, $(2, 6)$, $(3, 8)$, $(4, 11)$, $(5, 13)$, $(6, 15)$

**40.** $(1, 7.5)$, $(2, 7)$, $(3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4.9)$

**41.** $(1, 3)$, $(2, 6)$, $(3, 2)$, $(4, 3)$, $(5, 9)$, $(6, 1)$

**42.** $(0.5, 2)$, $(0.75, 1.75)$, $(1, 3)$, $(1.5, 3.2)$, $(2, 3.7)$, $(2.6, 4)$

**43.** $(1, 36)$, $(2, 10)$, $(3, 0)$, $(4, 4)$, $(5, 16)$, $(6, 36)$

**44.** $(0.5, 9)$, $(1, 8.5)$, $(1.5, 7)$, $(2, 5.5)$, $(2.5, 5)$, $(3, 3.5)$

**True or False?** In Exercises 45–50, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**45.** Data that are modeled by $y = 3.29x - 4.17$ have a negative correlation.

**46.** Data that are modeled by $y = -0.238x + 25$ have a negative correlation.

**47.** If the correlation coefficient is $r \approx -0.98781$, the model is a good fit.

**48.** A correlation coefficient of $r \approx 0.201$ implies that the data have no correlation.

**49.** A linear regression model with a positive correlation will have a slope that is greater than 0.

**50.** If the correlation coefficient for a linear regression model is close to $-1$, the regression line cannot be used to describe the data.

**51. Extended Application** To work an extended application analyzing the earnings per share, sales, and shareholder's equity of PepsiCo from 1999 through 2006, visit this text's website at *college.hmco.com*. *(Data Source: PepsiCo, Inc.)*

# Double Integrals and Area in the Plane

■ Evaluate double integrals.
■ Use double integrals to find the areas of regions.

## Double Integrals

In Section 7.4, you learned that it is meaningful to differentiate functions of several variables by differentiating with respect to one variable at a time while holding the other variable(s) constant. It should not be surprising to learn that you can *integrate* functions of two or more variables using a similar procedure. For instance, if you are given the partial derivative

$$f_x(x, y) = 2xy \qquad \text{Partial with respect to } x$$

then, by holding $y$ constant, you can integrate with respect to $x$ to obtain

$$\int f_x(x, y)\, dx = f(x, y) + C(y)$$
$$= x^2y + C(y).$$

This procedure is called **partial integration with respect to $x$.** Note that the "constant of integration" $C(y)$ is assumed to be a function of $y$, because $y$ is fixed during integration with respect to $x$. Similarly, if you are given the partial derivative

$$f_y(x, y) = x^2 + 2 \qquad \text{Partial with respect to } y$$

then, by holding $x$ constant, you can integrate with respect to $y$ to obtain

$$\int f_y(x, y)\, dy = f(x, y) + C(x)$$
$$= x^2y + 2y + C(x).$$

In this case, the "constant of integration" $C(x)$ is assumed to be a function of $x$, because $x$ is fixed during integration with respect to $y$.

To evaluate a definite integral of a function of two or more variables, you can apply the Fundamental Theorem of Calculus to one variable while holding the other variable(s) constant, as shown.

$$\int_1^{2y} 2xy\, dx = x^2y \Big]_1^{2y} = (2y)^2 y - (1)^2 y$$

$x$ is the variable of integration and $y$ is fixed.

Replace $x$ by the limits of integration.

$$= 4y^3 - y.$$

The result is a function of $y$.

Note that you omit the constant of integration, just as you do for a definite integral of a function of one variable.

> **Example 1** **Finding Partial Integrals**

Find each partial integral.

**a.** $\displaystyle\int_1^x (2x^2y^{-2} + 2y)\, dy$ **b.** $\displaystyle\int_y^{5y} \sqrt{x - y}\, dx$

**SOLUTION**

**a.** $\displaystyle\int_1^x (2x^2y^{-2} + 2y)\, dy = \left[\frac{-2x^2}{y} + y^2\right]_1^x$     Hold $x$ constant.

$$= \left(\frac{-2x^2}{x} + x^2\right) - \left(\frac{-2x^2}{1} + 1\right)$$

$$= 3x^2 - 2x - 1$$

**b.** $\displaystyle\int_y^{5y} \sqrt{x - y}\, dx = \left[\frac{2}{3}(x - y)^{3/2}\right]_y^{5y}$     Hold $y$ constant.

$$= \frac{2}{3}\left[(5y - y)^{3/2} - (y - y)^{3/2}\right] = \frac{16}{3}y^{3/2}$$

✓ **CHECKPOINT 1**

Find each partial integral.

**a.** $\displaystyle\int_1^x (4xy + y^3)\, dy$

**b.** $\displaystyle\int_y^{y^2} \frac{1}{x + y}\, dx$ ■

In Example 1(a), note that the definite integral defines a function of $x$ and can *itself* be integrated. An "integral of an integral" is called a **double integral.** With a function of two variables, there are two types of double integrals.

**STUDY TIP**

Notice that the difference between the two types of double integrals is the order in which the integration is performed, $dy\, dx$ or $dx\, dy$.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\right] dx$$

$$\int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y)\, dx\, dy = \int_a^b \left[\int_{g_1(y)}^{g_2(y)} f(x, y)\, dx\right] dy$$

**TECHNOLOGY**

(T) A symbolic integration utility can be used to evaluate double integrals. To do this, you need to enter the integrand, then integrate twice—once with respect to one of the variables and then with respect to the other variable. Use a symbolic integration utility to evaluate the double integral in Example 2.

> **Example 2** **Evaluating a Double Integral**

Evaluate $\displaystyle\int_1^2 \int_0^x (2xy + 3)\, dy\, dx$.

**SOLUTION**

$$\int_1^2 \int_0^x (2xy + 3)\, dy\, dx = \int_1^2 \left[\int_0^x (2xy + 3)\, dy\right] dx$$

$$= \int_1^2 \left[xy^2 + 3y\right]_0^x dx$$

$$= \int_1^2 (x^3 + 3x)\, dx$$

$$= \left[\frac{x^4}{4} + \frac{3x^2}{2}\right]_1^2$$

$$= \left(\frac{2^4}{4} + \frac{3(2^2)}{2}\right) - \left(\frac{1^4}{4} + \frac{3(1^2)}{2}\right) = \frac{33}{4}$$

✓ **CHECKPOINT 2**

Evaluate the double integral.

$$\int_1^2 \int_0^x (5x^2y - 2)\, dy\, dx$$ ■

## Finding Area with a Double Integral

One of the simplest applications of a double integral is finding the area of a plane region. For instance, consider the region $R$ that is bounded by

$$a \le x \le b \quad \text{and} \quad g_1(x) \le y \le g_2(x).$$

Using the techniques described in Section 5.5, you know that the area of $R$ is

$$\int_a^b [g_2(x) - g_1(x)]\, dx.$$
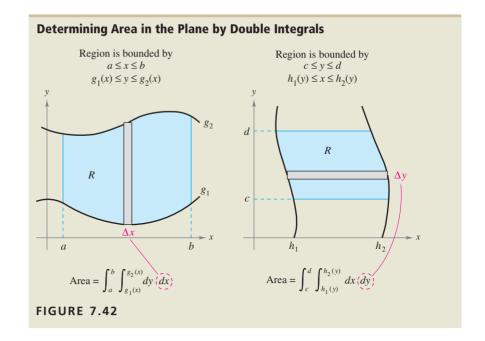
This same area is also given by the double integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx$$

because

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx = \int_a^b \left[ y \right]_{g_1(x)}^{g_2(x)} dx = \int_a^b [g_2(x) - g_1(x)]\, dx.$$

Figure 7.42 shows the two basic types of plane regions whose areas can be determined by a double integral.

**Determining Area in the Plane by Double Integrals**

Region is bounded by
$$a \le x \le b$$
$$g_1(x) \le y \le g_2(x)$$



$$\text{Area} = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx$$

Region is bounded by
$$c \le y \le d$$
$$h_1(y) \le x \le h_2(y)$$



$$\text{Area} = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy$$

**FIGURE 7.42**

### STUDY TIP

In Figure 7.42, note that the horizontal or vertical orientation of the narrow rectangle indicates the order of integration. The "outer" variable of integration always corresponds to the width of the rectangle. Notice also that the outer limits of integration for a double integral are constant, whereas the inner limits may be functions of the outer variable.

$R$: $1 \le x \le 5$
$2 \le y \le 4$

$\text{Area} = \int_1^5 \int_2^4 dy\, dx$

**FIGURE 7.43**

**Example 3**  Finding Area with a Double Integral

Use a double integral to find the area of the rectangular region shown in Figure 7.43.

**SOLUTION**   The bounds for $x$ are $1 \le x \le 5$ and the bounds for $y$ are $2 \le y \le 4$. So, the area of the region is

$$\int_1^5 \int_2^4 dy\, dx = \int_1^5 \left[ y \right]_2^4 dx \qquad \text{Integrate with respect to } y.$$

$$= \int_1^5 (4 - 2)\, dx \qquad \text{Apply Fundamental Theorem of Calculus.}$$
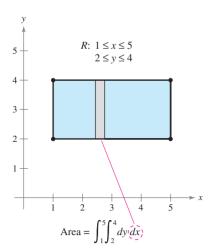
$$= \int_1^5 2\, dx \qquad \text{Simplify.}$$

$$= \left[ 2x \right]_1^5 \qquad \text{Integrate with respect to } x.$$

$$= 10 - 2 \qquad \text{Apply Fundamental Theorem of Calculus.}$$

$$= 8 \text{ square units.} \qquad \text{Simplify.}$$

You can confirm this by noting that the rectangle measures two units by four units.

✓ **CHECKPOINT 3**

Use a double integral to find the area of the rectangular region shown in Example 3 by integrating with respect to $x$ and then with respect to $y$. ■



$R$: $0 \le x \le 1$
$x^3 \le y \le x^2$

$(1, 1)$

$y = x^2$

$y = x^3$

$\text{Area} = \int_0^1 \int_{x^3}^{x^2} dy\, dx$

**FIGURE 7.44**

**Example 4**  Finding Area with a Double Integral

Use a double integral to find the area of the region bounded by the graphs of $y = x^2$ and $y = x^3$.

**SOLUTION**   As shown in Figure 7.44, the two graphs intersect when $x = 0$ and $x = 1$. Choosing $x$ to be the outer variable, the bounds for $x$ are $0 \le x \le 1$ and the bounds for $y$ are $x^3 \le y \le x^2$. This implies that the area of the region is

$$\int_0^1 \int_{x^3}^{x^2} dy\, dx = \int_0^1 \left[ y \right]_{x^3}^{x^2} dx \qquad \text{Integrate with respect to } y.$$

$$= \int_0^1 (x^2 - x^3)\, dx \qquad \text{Apply Fundamental Theorem of Calculus.}$$

$$= \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 \qquad \text{Integrate with respect to } x.$$

$$= \frac{1}{3} - \frac{1}{4} \qquad \text{Apply Fundamental Theorem of Calculus.}$$

$$= \frac{1}{12} \text{ square unit.} \qquad \text{Simplify.}$$
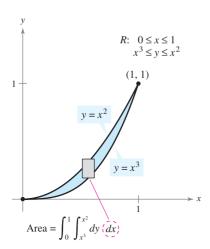
✓ **CHECKPOINT 4**

Use a double integral to find the area of the region bounded by the graphs of $y = 2x$ and $y = x^2$. ■

In setting up double integrals, the most difficult task is likely to be determining the correct limits of integration. This can be simplified by making a sketch of the region $R$ and identifying the appropriate bounds for $x$ and $y$.

## Example 5    Changing the Order of Integration

For the double integral

$$\int_0^2 \int_{y^2}^4 dx\, dy$$

a. sketch the region $R$ whose area is represented by the integral,

b. rewrite the integral so that $x$ is the outer variable, and

c. show that both orders of integration yield the same value.

### SOLUTION
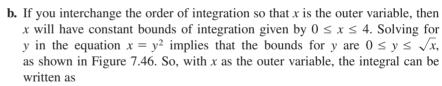
a. From the limits of integration, you know that

$$y^2 \leq x \leq 4 \qquad \text{Variable bounds for } x$$

which means that the region $R$ is bounded on the left by the parabola $x = y^2$ and on the right by the line $x = 4$. Furthermore, because

$$0 \leq y \leq 2 \qquad \text{Constant bounds for } y$$

you know that the region lies above the $x$-axis, as shown in Figure 7.45.

b. If you interchange the order of integration so that $x$ is the outer variable, then $x$ will have constant bounds of integration given by $0 \leq x \leq 4$. Solving for $y$ in the equation $x = y^2$ implies that the bounds for $y$ are $0 \leq y \leq \sqrt{x}$, as shown in Figure 7.46. So, with $x$ as the outer variable, the integral can be written as

$$\int_0^4 \int_0^{\sqrt{x}} dy\, dx.$$

c. Both integrals yield the same value.

$$\int_0^2 \int_{y^2}^4 dx\, dy = \int_0^2 \Big[x\Big]_{y^2}^4 dy = \int_0^2 (4 - y^2)\, dy = \left[4y - \frac{y^3}{3}\right]_0^2 = \frac{16}{3}$$

$$\int_0^4 \int_0^{\sqrt{x}} dy\, dx = \int_0^4 \Big[y\Big]_0^{\sqrt{x}} dx = \int_0^4 \sqrt{x}\, dx = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{16}{3}$$

### ✓ CHECKPOINT 5

For the double integral $\int_0^2 \int_{2y}^4 dx\, dy$,

a. sketch the region $R$ whose area is represented by the integral,

b. rewrite the integral so that $x$ is the outer variable, and

c. show that both orders of integration yield the same result. ■



**FIGURE 7.45**

$R: \quad 0 \leq y \leq 2$
$\quad\quad y^2 \leq x \leq 4$

$x = y^2$

$(4, 2)$

$\Delta y$

Area $= \int_0^2 \int_{y^2}^4 dx\, (dy)$



**FIGURE 7.46**

$R: \quad 0 \leq x \leq 4$
$\quad\quad 0 \leq y \leq \sqrt{x}$

$y = \sqrt{x}$

$(4, 2)$

$\Delta x$

Area $= \int_0^4 \int_0^{\sqrt{x}} dy\, (dx)$
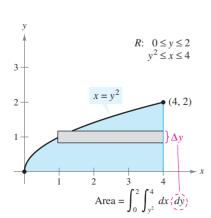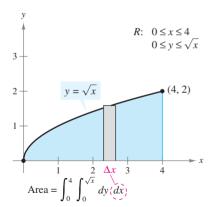
### STUDY TIP

To designate a double integral or an area of a region without specifying a particular order of integration, you can use the symbol

$$\int_R \int dA$$

where $dA = dx\, dy$ or $dA = dy\, dx$.

$R: 0 \le x \le 2$
$x^2 - x \le y \le x$

(2, 2)

$y = x$

$y = x^2 - x$

$\Delta x$

Area $= \int_0^2 \int_{x^2-x}^x dy\,(dx)$

**FIGURE 7.47**

✓ **CHECKPOINT 6**

Use a double integral to calculate the area denoted by $\int_R\int dA$ where $R$ is the region bounded by $y = 2x + 3$ and $y = x^2$. ■

*Example 6*  **Finding Area with a Double Integral**

Use a double integral to calculate the area denoted by

$$\int_R\int dA$$

where $R$ is the region bounded by $y = x$ and $y = x^2 - x$.

**SOLUTION**   Begin by sketching the region $R$, as shown in Figure 7.47. From the sketch, you can see that vertical rectangles of width $dx$ are more convenient than horizontal ones. So, $x$ is the outer variable of integration and its constant bounds are $0 \le x \le 2$. This implies that the bounds for $y$ are $x^2 - x \le y \le x$, and the area is given by

$$\int_R\int dA = \int_0^2 \int_{x^2-x}^x dy\,dx \qquad \text{Substitute bounds for region.}$$

$$= \int_0^2 \Big[y\Big]_{x^2-x}^x dx \qquad \text{Integrate with respect to } y.$$

$$= \int_0^2 [x - (x^2 - x)]\,dx \qquad \text{Apply Fundamental Theorem of Calculus.}$$

$$= \int_0^2 (2x - x^2)\,dx \qquad \text{Simplify.}$$

$$= \Big[x^2 - \frac{x^3}{3}\Big]_0^2 \qquad \text{Integrate with respect to } x.$$

$$= 4 - \frac{8}{3} \qquad \text{Apply Fundamental Theorem of Calculus.}$$

$$= \frac{4}{3} \text{ square units.} \qquad \text{Simplify.}$$

As you are working the exercises for this section, you should be aware that the primary uses of double integrals will be discussed in Section 7.9. Double integrals by way of areas in the plane have been introduced so that you can gain practice in finding the limits of integration. When setting up a double integral, remember that your first step should be to sketch the region $R$. After doing this, you have two choices of integration orders: $dx\,dy$ or $dy\,dx$.

**CONCEPT CHECK**

1. What is an "integral of an integral" called?

2. In the double integral $\int_0^2 \int_0^1 dy\,dx$, in what order is the integration performed? (Do not integrate.)

3. True or false: Changing the order of integration will sometimes change the value of a double integral.

4. To designate a double integral or an area of a region without specifying a particular order of integration, what symbol can you use?

**Skills Review 7.8**   The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 5.2–5.5.

In Exercises 1–12, evaluate the definite integral.

**1.** $\displaystyle\int_0^1 dx$

**2.** $\displaystyle\int_0^2 3\,dy$

**3.** $\displaystyle\int_1^4 2x^2\,dx$

**4.** $\displaystyle\int_0^1 2x^3\,dx$

**5.** $\displaystyle\int_1^2 (x^3 - 2x + 4)\,dx$

**6.** $\displaystyle\int_0^2 (4 - y^2)\,dy$

**7.** $\displaystyle\int_1^2 \frac{2}{7x^2}\,dx$

**8.** $\displaystyle\int_1^4 \frac{2}{\sqrt{x}}\,dx$

**9.** $\displaystyle\int_0^2 \frac{2x}{x^2 + 1}\,dx$

**10.** $\displaystyle\int_2^e \frac{1}{y - 1}\,dy$

**11.** $\displaystyle\int_0^2 xe^{x^2 + 1}\,dx$

**12.** $\displaystyle\int_0^1 e^{-2y}\,dy$

In Exercises 13–16, sketch the region bounded by the graphs of the equations.

**13.** $y = x$, $y = 0$, $x = 3$

**14.** $y = x$, $y = 3$, $x = 0$

**15.** $y = 4 - x^2$, $y = 0$, $x = 0$

**16.** $y = x^2$, $y = 4x$

---

## Exercises 7.8

In Exercises 1–10, evaluate the partial integral.

**1.** $\displaystyle\int_0^x (2x - y)\,dy$

**2.** $\displaystyle\int_x^{x^2} \frac{y}{x}\,dy$

**3.** $\displaystyle\int_1^{2y} \frac{y}{x}\,dx$

**4.** $\displaystyle\int_0^{e^y} y\,dx$

**5.** $\displaystyle\int_0^{\sqrt{4-x^2}} x^2 y\,dy$

**6.** $\displaystyle\int_{x^2}^{\sqrt{x}} (x^2 + y^2)\,dy$

**7.** $\displaystyle\int_1^{e^y} \frac{y\ln x}{x}\,dx$

**8.** $\displaystyle\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2)\,dx$

**9.** $\displaystyle\int_0^x ye^{xy}\,dy$

**10.** $\displaystyle\int_y^3 \frac{xy}{\sqrt{x^2 + 1}}\,dx$

In Exercises 11–24, evaluate the double integral.

**11.** $\displaystyle\int_0^1\int_0^2 (x + y)\,dy\,dx$

**12.** $\displaystyle\int_0^2\int_0^2 (6 - x^2)\,dy\,dx$

**13.** $\displaystyle\int_0^4\int_0^3 xy\,dy\,dx$

**14.** $\displaystyle\int_0^1\int_0^x \sqrt{1 - x^2}\,dy\,dx$

**15.** $\displaystyle\int_0^1\int_0^y (x + y)\,dx\,dy$

**16.** $\displaystyle\int_0^2\int_{3y^2 - 6y}^{2y - y^2} 3y\,dx\,dy$

**17.** $\displaystyle\int_1^2\int_0^4 (3x^2 - 2y^2 + 1)\,dx\,dy$

**18.** $\displaystyle\int_0^1\int_y^{2y} (1 + 2x^2 + 2y^2)\,dx\,dy$

**19.** $\displaystyle\int_0^2\int_0^{\sqrt{1-y^2}} -5xy\,dx\,dy$

**20.** $\displaystyle\int_0^4\int_0^x \frac{2}{x^2 + 1}\,dy\,dx$

**21.** $\displaystyle\int_0^2\int_0^{6x^2} x^3\,dy\,dx$

**22.** $\displaystyle\int_{-1}^1\int_{-2}^2 (x^2 - y^2)\,dy\,dx$

**23.** $\displaystyle\int_0^\infty\int_0^\infty e^{-(x+y)/2}\,dy\,dx$

**24.** $\displaystyle\int_0^\infty\int_0^\infty xye^{-(x^2 + y^2)}\,dx\,dy$

In Exercises 25–32, sketch the region $R$ whose area is given by the double integral. Then change the order of integration and show that both orders yield the same area.

**25.** $\displaystyle\int_0^1 \int_0^2 dy\, dx$

**26.** $\displaystyle\int_1^2 \int_2^4 dx\, dy$

**27.** $\displaystyle\int_0^1 \int_{2y}^2 dx\, dy$

**28.** $\displaystyle\int_0^4 \int_0^{\sqrt{x}} dy\, dx$

**29.** $\displaystyle\int_0^2 \int_{x/2}^1 dy\, dx$

**30.** $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 dy\, dx$

**31.** $\displaystyle\int_0^1 \int_{y^2}^{\sqrt[3]{y}} dx\, dy$

**32.** $\displaystyle\int_{-2}^2 \int_0^{4-y^2} dx\, dy$
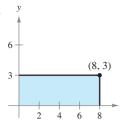
In Exercises 33 and 34, evaluate the double integral. Note that it is necessary to change the order of integration.

**33.** $\displaystyle\int_0^3 \int_y^3 e^{x^2} dx\, dy$
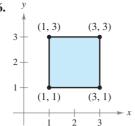
**34.** $\displaystyle\int_0^2 \int_x^2 e^{-y^2} dy\, dx$

In Exercises 35–40, use a double integral to find the area of the specified region.

**35.**



**36.**



(1, 3)    (3, 3)

(1, 1)    (3, 1)

**37.**



$y = 4 - x^2$

**38.**



$y = 4 - x^2$

$y = x + 2$

**39.**



$\sqrt{x} + \sqrt{y} = 2$

**40.**



$y = \dfrac{1}{\sqrt{x-1}}$

$2 \le x \le 5$
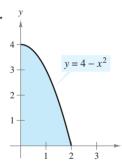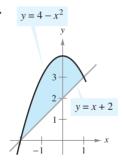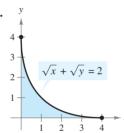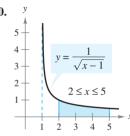
In Exercises 41–46, use a double integral to find the area of the region bounded by the graphs of the equations.

**41.** $y = 9 - x^2, \ y = 0$

**42.** $y = x^{3/2}, \ y = x$

**43.** $2x - 3y = 0, \ x + y = 5, \ y = 0$

**44.** $xy = 9, \ y = x, \ y = 0, \ x = 9$

**45.** $y = x, \ y = 2x, \ x = 2$

**46.** $y = x^2 + 2x + 1, \ y = 3(x + 1)$

(T) In Exercises 47–54, use a symbolic integration utility to evaluate the double integral.

**47.** $\displaystyle\int_0^1 \int_0^2 e^{-x^2 - y^2} dx\, dy$

**48.** $\displaystyle\int_0^2 \int_{x^2}^{2x} (x^3 + 3y^2)\, dy\, dx$

**49.** $\displaystyle\int_1^2 \int_0^x e^{xy}\, dy\, dx$

**50.** $\displaystyle\int_1^2 \int_y^{2y} \ln(x + y)\, dx\, dy$

**51.** $\displaystyle\int_0^1 \int_x^1 \sqrt{1 - x^2}\, dy\, dx$

**52.** $\displaystyle\int_0^3 \int_0^{x^2} \sqrt{x}\sqrt{1 + x}\, dy\, dx$

**53.** $\displaystyle\int_0^2 \int_{\sqrt{4-x^2}}^{4-x^2/4} \frac{xy}{x^2 + y^2 + 1}\, dy\, dx$

**54.** $\displaystyle\int_0^4 \int_0^y \frac{2}{(x + 1)(y + 1)}\, dx\, dy$

**True or False?**   In Exercises 55 and 56, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**55.** $\displaystyle\int_{-1}^1 \int_{-2}^2 y\, dy\, dx = \int_{-1}^1 \int_{-2}^2 y\, dx\, dy$

**56.** $\displaystyle\int_2^5 \int_1^6 x\, dy\, dx = \int_1^6 \int_2^5 x\, dx\, dy$

## Section 7.9

# Applications of Double Integrals

- Use double integrals to find the volumes of solids.
- Use double integrals to find the average values of real-life models.
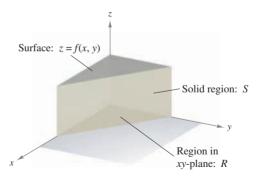
## Volume of a Solid Region

In Section 7.8, you used double integrals as an alternative way to find the area of a plane region. In this section, you will study the primary uses of double integrals: to find the volume of a solid region and to find the average value of a function.

Consider a function $z = f(x, y)$ that is continuous and nonnegative over a region $R$. Let $S$ be the solid region that lies between the $xy$-plane and the surface

$$z = f(x, y) \qquad \text{Surface lying above the } xy\text{-plane}$$

directly above the region $R$, as shown in Figure 7.48. You can find the volume of $S$ by integrating $f(x, y)$ over the region $R$.



Surface: $z = f(x, y)$

Solid region: $S$

Region in
$xy$-plane: $R$

**FIGURE 7.48**

---

**Determining Volume with Double Integrals**

If $R$ is a bounded region in the $xy$-plane and $f$ is continuous and nonnegative over $R$, then the **volume of the solid** region between the surface $z = f(x, y)$ and $R$ is given by the double integral

$$\int_R \int f(x, y) \, dA$$
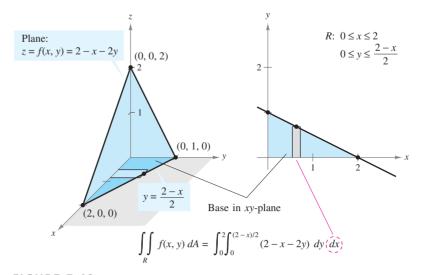
where $dA = dx\,dy$ or $dA = dy\,dx$.

**Example 1** Finding the Volume of a Solid

Find the volume of the solid region bounded in the first octant by the plane

$$z = 2 - x - 2y.$$

**SOLUTION**



$$\iint_R f(x, y)\, dA = \int_0^2 \int_0^{(2-x)/2} (2 - x - 2y)\, dy\, dx$$

**FIGURE 7.49**

To set up the double integral for the volume, it is helpful to sketch both the solid region and the plane region $R$ in the $xy$-plane. In Figure 7.49, you can see that the region $R$ is bounded by the lines $x = 0$, $y = 0$, and $y = \frac{1}{2}(2 - x)$. One way to set up the double integral is to choose $x$ as the outer variable. With that choice, the constant bounds for $x$ are $0 \le x \le 2$ and the variable bounds for $y$ are $0 \le y \le \frac{1}{2}(2 - x)$. So, the volume of the solid region is
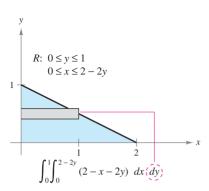
$$
\begin{aligned}
V &= \int_0^2 \int_0^{(2-x)/2} (2 - x - 2y)\, dy\, dx \\
&= \int_0^2 \left[ (2 - x)y - y^2 \right]_0^{(2-x)/2} dx \\
&= \int_0^2 \left\{ (2 - x)\left(\frac{1}{2}\right)(2 - x) - \left[\frac{1}{2}(2 - x)\right]^2 \right\} dx \\
&= \frac{1}{4} \int_0^2 (2 - x)^2\, dx \\
&= \left[ -\frac{1}{12}(2 - x)^3 \right]_0^2 \\
&= \frac{2}{3} \text{ cubic unit.}
\end{aligned}
$$

$$\int_0^1 \int_0^{2-2y} (2 - x - 2y)\, dx\, dy$$

**FIGURE 7.50**

✓ **CHECKPOINT 1**
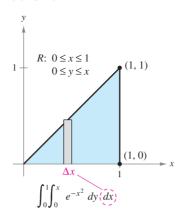
Find the volume of the solid region bounded in the first octant by the plane $z = 4 - 2x - y$. ∎
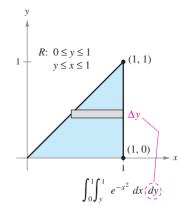
In Example 1, the order of integration was arbitrary. Although the example used $x$ as the outer variable, you could just as easily have used $y$ as the outer variable. The next example describes a situation in which one order of integration is more convenient than the other.

### *Example 2* Comparing Different Orders of Integration

Find the volume under the surface $f(x, y) = e^{-x^2}$ bounded by the $xz$-plane and the planes $y = x$ and $x = 1$, as shown in Figure 7.51.

**SOLUTION**

Surface:
$f(x, y) = e^{-x^2}$
$y = 0$
$x = 1$
$y = x$

**FIGURE 7.51**

$R:\ 0 \le x \le 1$
$\quad\ 0 \le y \le x$
$(1, 1)$
$(1, 0)$
$\Delta x$

$$\int_0^1 \int_0^x e^{-x^2}\, dy\, dx$$

$R:\ 0 \le y \le 1$
$\quad\ y \le x \le 1$
$(1, 1)$
$(1, 0)$
$\Delta y$

$$\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy$$

**FIGURE 7.52**

In the $xy$-plane, the bounds of region $R$ are the lines $y = 0$, $x = 1$, and $y = x$. The two possible orders of integration are indicated in Figure 7.52. If you attempt to evaluate the two double integrals shown in the figure, you will discover that the one on the right involves finding the antiderivative of $e^{-x^2}$, which you know is not an elementary function. The integral on the left, however, can be evaluated more easily, as shown.
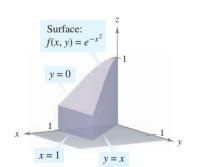
$$V = \int_0^1 \int_0^x e^{-x^2}\, dy\, dx$$

$$= \int_0^1 \left[ e^{-x^2} y \right]_0^x dx$$

$$= \int_0^1 x e^{-x^2}\, dx$$

$$= \left[ -\frac{1}{2} e^{-x^2} \right]_0^1$$

$$= -\frac{1}{2}\left( \frac{1}{e} - 1 \right) \approx 0.316 \text{ cubic unit}$$

---

**TECHNOLOGY**

Ⓣ Use a symbolic integration utility to evaluate the double integral in Example 2.

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✓**CHECKPOINT 2**

Find the volume under the surface $f(x, y) = e^{x^2}$, bounded by the $xz$-plane and the planes $y = 2x$ and $x = 1$. ∎

**Guidelines for Finding the Volume of a Solid**

**1.** Write the equation of the surface in the form $z = f(x, y)$ and sketch the solid region.

**2.** Sketch the region $R$ in the $xy$-plane and determine the order and limits of integration.

**3.** Evaluate the double integral

$$\int_R \int f(x, y) \, dA$$

using the order and limits determined in the second step.

The first step above suggests that you sketch the three-dimensional solid region. This is a good suggestion, but it is not always feasible and is not as important as making a sketch of the two-dimensional region $R$.

**Example 3**    **Finding the Volume of a Solid**

Find the volume of the solid bounded above by the surface

$$f(x, y) = 6x^2 - 2xy$$

and below by the plane region $R$ shown in Figure 7.53.

**SOLUTION**    Because the region $R$ is bounded by the parabola $y = 3x - x^2$ and the line $y = x$, the limits for $y$ are $x \le y \le 3x - x^2$. The limits for $x$ are $0 \le x \le 2$, and the volume of the solid is

$$V = \int_0^2 \int_x^{3x - x^2} (6x^2 - 2xy) \, dy \, dx$$
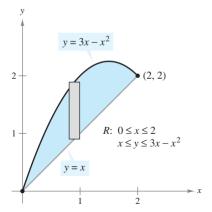
$$= \int_0^2 \left[ 6x^2 y - xy^2 \right]_x^{3x - x^2} dx$$

$$= \int_0^2 \left[ (18x^3 - 6x^4 - 9x^3 + 6x^4 - x^5) - (6x^3 - x^3) \right] dx$$

$$= \int_0^2 (4x^3 - x^5) \, dx$$

$$= \left[ x^4 - \frac{x^6}{6} \right]_0^2$$

$$= \frac{16}{3} \text{ cubic units.}$$



**FIGURE 7.53**

✓ **CHECKPOINT 3**

Find the volume of the solid bounded above by the surface $f(x, y) = 4x^2 + 2xy$ and below by the plane region bounded by $y = x^2$ and $y = 2x$. ∎

A *population density function* $p = f(x, y)$ is a model that describes density (in people per square unit) of a region. To find the population of a region $R$, evaluate the double integral

$$\int_R \int f(x, y) \, dA.$$

### Example 4
**MAKE A DECISION**   **Finding the Population of a Region**   Ⓡ

The population density (in people per square mile) of the city shown in Figure 7.54 can be modeled by

$$f(x, y) = \frac{50{,}000}{x + |y| + 1}$$

where $x$ and $y$ are measured in miles. Approximate the city's population. Will the city's average population density be less than 10,000 people per square mile?

**SOLUTION**   Because the model involves the absolute value of $y$, it follows that the population density is symmetrical about the $x$-axis. So, the population in the first quadrant is equal to the population in the fourth quadrant. This means that you can find the total population by doubling the population in the first quadrant.

$$
\begin{aligned}
\text{Population} &= 2 \int_0^4 \int_0^5 \frac{50{,}000}{x + y + 1} \, dy \, dx \\
&= 100{,}000 \int_0^4 \left[ \ln(x + y + 1) \right]_0^5 dx \\
&= 100{,}000 \int_0^4 \left[ \ln(x + 6) - \ln(x + 1) \right] dx \\
&= 100{,}000 \Big[ (x + 6) \ln(x + 6) - (x + 6) - \\
&\qquad\qquad (x + 1) \ln(x + 1) + (x + 1) \Big]_0^4 \\
&= 100{,}000 \Big[ (x + 6) \ln(x + 6) - (x + 1) \ln(x + 1) - 5 \Big]_0^4 \\
&= 100{,}000 [10 \ln(10) - 5 \ln(5) - 5 - 6 \ln(6) + 5] \\
&\approx 422{,}810 \text{ people}
\end{aligned}
$$

So, the city's population is about 422,810. Because the city covers a region 4 miles wide and 10 miles long, its area is 40 square miles. So, the average population density is

$$\text{Average population density} = \frac{422{,}810}{40}$$

$$\approx 10{,}570 \text{ people per square mile.}$$

No, the city's average population density is not less than 10,000 people per square mile.



*y*   $R:\ 0 \le x \le 4$
     $-5 \le y \le 5$

Ocean   City
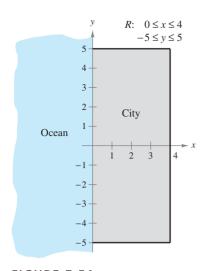
**FIGURE 7.54**

**✓ CHECKPOINT 4**

In Example 4, what integration technique was used to integrate

$$\int \left[ \ln(x + 6) - \ln(x + 1) \right] dx? \ \blacksquare$$

## Average Value of a Function over a Region

**Average Value of a Function Over a Region**

If $f$ is integrable over the plane region $R$ with area $A$, then its **average value** over $R$ is

$$\text{Average value} = \frac{1}{A} \int_R \int f(x, y)\, dA.$$

**Example 5**  **Finding Average Profit**  ®

A manufacturer determines that the profit for selling $x$ units of one product and $y$ units of a second product is modeled by

$$P = -(x - 200)^2 - (y - 100)^2 + 5000.$$

The weekly sales for product 1 vary between 150 and 200 units, and the weekly sales for product 2 vary between 80 and 100 units. Estimate the average weekly profit for the two products.

**SOLUTION**   Because $150 \le x \le 200$ and $80 \le y \le 100$, you can estimate the weekly profit to be the average of the profit function over the rectangular region shown in Figure 7.55. Because the area of this rectangular region is $(50)(20) = 1000$, it follows that the average profit $V$ is

$$V = \frac{1}{1000} \int_{150}^{200} \int_{80}^{100} \left[ -(x - 200)^2 - (y - 100)^2 + 5000 \right] dy\, dx$$

$$= \frac{1}{1000} \int_{150}^{200} \left[ -(x - 200)^2 y - \frac{(y - 100)^3}{3} + 5000y \right]_{80}^{100} dx$$

$$= \frac{1}{1000} \int_{150}^{200} \left[ -20(x - 200)^2 - \frac{292{,}000}{3} \right] dx$$

$$= \frac{1}{3000} \left[ -20(x - 200)^3 + 292{,}000x \right]_{150}^{200}$$

$$\approx \$4033.$$

$R$: $150 \le x \le 200$
$80 \le y \le 100$
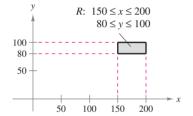
**FIGURE 7.55**

✓**CHECKPOINT 5**

Find the average value of $f(x, y) = 4 - \frac{1}{2}x - \frac{1}{2}y$ over the region $0 \le x \le 2$ and $0 \le y \le 2$. ∎

**CONCEPT CHECK**

1. Complete the following: The double integral $\int_R \int f(x, y)\, dA$ gives the _____ of the solid region between the surface $z = f(x, y)$ and the bounded region in the $xy$-plane $R$.

2. Give the guidelines for finding the volume of a solid.

3. What does a population density function describe?

4. What is the average value of $f(x, y)$ over the plane region $R$?

**Skills Review 7.9**  The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 5.4 and 7.8.

In Exercises 1–4, sketch the region that is described.

**1.** $0 \le x \le 2$, $0 \le y \le 1$

**2.** $1 \le x \le 3$, $2 \le y \le 3$

**3.** $0 \le x \le 4$, $0 \le y \le 2x - 1$

**4.** $0 \le x \le 2$, $0 \le y \le x^2$

In Exercises 5–10, evaluate the double integral.

**5.** $\displaystyle\int_0^1 \int_1^2 dy\, dx$

**6.** $\displaystyle\int_0^3 \int_1^3 dx\, dy$

**7.** $\displaystyle\int_0^1 \int_0^x x\, dy\, dx$

**8.** $\displaystyle\int_0^4 \int_1^y y\, dx\, dy$

**9.** $\displaystyle\int_1^3 \int_x^{x^2} 2\, dy\, dx$

**10.** $\displaystyle\int_0^1 \int_x^{-x^2+2} dy\, dx$

## Exercises 7.9

In Exercises 1–8, sketch the region of integration and evaluate the double integral.

**1.** $\displaystyle\int_0^2 \int_0^1 (3x + 4y)\, dy\, dx$

**2.** $\displaystyle\int_0^3 \int_0^1 (2x + 6y)\, dy\, dx$

**3.** $\displaystyle\int_0^1 \int_y^{\sqrt{y}} x^2 y^2\, dx\, dy$

**4.** $\displaystyle\int_0^6 \int_{y/2}^3 (x + y)\, dx\, dy$

**5.** $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} y\, dy\, dx$

**6.** $\displaystyle\int_0^2 \int_0^{4-x^2} xy^2\, dy\, dx$

**7.** $\displaystyle\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$

**8.** $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx$

In Exercises 9–12, set up the integral for both orders of integration and use the more convenient order to evaluate the integral over the region R.

**9.** $\displaystyle\int_R \int xy\, dA$

R: rectangle with vertices at $(0, 0)$, $(0, 5)$, $(3, 5)$, $(3, 0)$

**10.** $\displaystyle\int_R \int x\, dA$

R: semicircle bounded by $y = \sqrt{25 - x^2}$ and $y = 0$
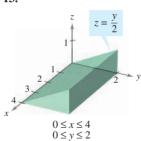
**11.** $\displaystyle\int_R \int \frac{y}{x^2 + y^2}\, dA$

R: triangle bounded by $y = x$, $y = 2x$, $x = 2$
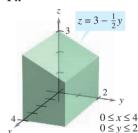
**12.** $\displaystyle\int_R \int \frac{y}{1 + x^2}\, dA$

R: region bounded by $y = 0$, $y = \sqrt{x}$, $x = 4$
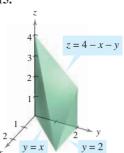
In Exercises 13–22, use a double integral to find the volume of the specified solid.

**13.**



$z = \dfrac{y}{2}$

$0 \le x \le 4$
$0 \le y \le 2$

**14.**



$z = 3 - \tfrac{1}{2}y$

$0 \le x \le 4$
$0 \le y \le 2$

**15.**



$z = 4 - x - y$

$y = x$   $y = 2$

**16.**



$z = 4$

$y = x$

$x = 2$

**17.**



$2x + 3y + 4z = 12$

**18.**



$x + y + z = 2$

**19.**



$z = 1 - xy$
$y = x$
$y = 1$

**20.**



$z = 4 - y^2$
$y = x$
$y = 2$

**21.**



$z = 4 - x^2 - y^2$

$-1 \le x \le 1$
$-1 \le y \le 1$

**22.**



$x^2 + z^2 = 1$
$x = 1$
$y = x$
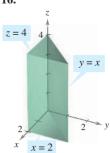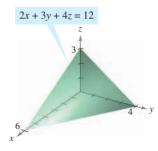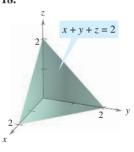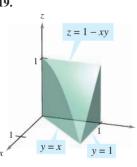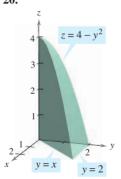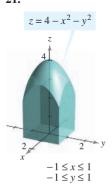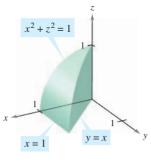
In Exercises 23–26, use a double integral to find the volume of the solid bounded by the graphs of the equations.

**23.** $z = xy$, $z = 0$, $y = 0$, $y = 4$, $x = 0$, $x = 1$

**24.** $z = x$, $z = 0$, $y = x$, $y = 0$, $x = 0$, $x = 4$

**25.** $z = x^2$, $z = 0$, $x = 0$, $x = 2$, $y = 0$, $y = 4$

**26.** $z = x + y$, $x^2 + y^2 = 4$ (first octant)

**27. Population Density**   The population density (in people per square mile) for a coastal town can be modeled by

$$f(x, y) = \frac{120{,}000}{(2 + x + y)^3}$$

where $x$ and $y$ are measured in miles. What is the population inside the rectangular area defined by the vertices $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(2, 2)$?

**28. Population Density**   The population density (in people per square mile) for a coastal town on an island can be modeled by

$$f(x, y) = \frac{5000xe^y}{1 + 2x^2}$$

where $x$ and $y$ are measured in miles. What is the population inside the rectangular area defined by the vertices $(0, 0)$, $(4, 0)$, $(0, -2)$, and $(4, -2)$?

In Exercises 29–32, find the average value of $f(x, y)$ over the region $R$.

**29.** $f(x, y) = x$
    $R$: rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, $(0, 2)$

**30.** $f(x, y) = xy$
    $R$: rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, $(0, 2)$

**31.** $f(x, y) = x^2 + y^2$
    $R$: square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$

**32.** $f(x, y) = e^{x+y}$
    $R$: triangle with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$

**33. Average Revenue**   A company sells two products whose demand functions are given by

$$x_1 = 500 - 3p_1 \quad \text{and} \quad x_2 = 750 - 2.4p_2.$$

So, the total revenue is given by

$$R = x_1 p_1 + x_2 p_2.$$

Estimate the average revenue if the price $p_1$ varies between \$50 and \$75 and the price $p_2$ varies between \$100 and \$150.

**34. Average Revenue**   After 1 year, the company in Exercise 33 finds that the demand functions for its two products are given by

$$x_1 = 500 - 2.5p_1 \quad \text{and} \quad x_2 = 750 - 3p_2.$$

Repeat Exercise 33 using these demand functions.

**35. Average Weekly Profit**   A firm's weekly profit in marketing two products is given by

$$P = 192x_1 + 576x_2 - x_1^2 - 5x_2^2 - 2x_1x_2 - 5000$$

where $x_1$ and $x_2$ represent the numbers of units of each product sold weekly. Estimate the average weekly profit if $x_1$ varies between 40 and 50 units and $x_2$ varies between 45 and 50 units.

**36. Average Weekly Profit**   After a change in marketing, the weekly profit of the firm in Exercise 35 is given by

$$P = 200x_1 + 580x_2 - x_1^2 - 5x_2^2 - 2x_1x_2 - 7500.$$

Estimate the average weekly profit if $x_1$ varies between 55 and 65 units and $x_2$ varies between 50 and 60 units.

**37. Average Production**   The Cobb-Douglas production function for an automobile manufacturer is

$$f(x, y) = 100x^{0.6}y^{0.4}$$

where $x$ is the number of units of labor and $y$ is the number of units of capital. Estimate the average production level if the number of units of labor $x$ varies between 200 and 250 and the number of units of capital $y$ varies between 300 and 325.

**38. Average Production**   Repeat Exercise 37 for the production function given by

$$f(x, y) = x^{0.25}y^{0.75}.$$

# Algebra Review

## Solving Systems of Equations

*Nonlinear System in Two Variables*

$$\begin{cases} 4x + 3y = 6 \\ x^2 - y = 4 \end{cases}$$

*Linear System in Three Variables*

$$\begin{cases} -x + 2y + 4z = 2 \\ 2x - \phantom{2}y + \phantom{2}z = 0 \\ 6x \phantom{-2y} + 2z = 3 \end{cases}$$

Three of the sections in this chapter (7.5, 7.6, and 7.7) involve solutions of systems of equations. These systems can be linear or nonlinear, as shown at the left.

There are many techniques for solving a system of linear equations. Two of the more common ones are listed here.

1. *Substitution*: Solve for one of the variables in one of the equations and substitute the value into another equation.

2. *Elimination*: Add multiples of one equation to a second equation to eliminate a variable in the second equation.

### Example 1 | Solving Systems of Equations

Solve each system of equations.

**a.** $\begin{cases} y - x^3 = 0 \\ x - y^3 = 0 \end{cases}$

**b.** $\begin{cases} -400p_1 + 300p_2 = -25 \\ \phantom{-}300p_1 - 360p_2 = -535 \end{cases}$

**SOLUTION**

**a.** Example 3, page 519

$$\begin{cases} y - x^3 = 0 \\ x - y^3 = 0 \end{cases}$$   Equation 1 / Equation 2

$$y = x^3$$   Solve for $y$ in Equation 1.

$$x - (x^3)^3 = 0$$   Substitute $x^3$ for $y$ in Equation 2.

$$x - x^9 = 0$$   $(x^m)^n = x^{mn}$

$$x(x - 1)(x + 1)(x^2 + 1)(x^4 + 1) = 0$$   Factor.

$$x = 0$$   Set factors equal to zero.

$$x = 1$$   Set factors equal to zero.

$$x = -1$$   Set factors equal to zero.

**b.** Example 4, page 520

$$\begin{cases} -400p_1 + 300p_2 = -25 \\ \phantom{-}300p_1 - 360p_2 = -535 \end{cases}$$   Equation 1 / Equation 2

$$p_2 = \tfrac{1}{12}(16p_1 - 1)$$   Solve for $p_2$ in Equation 1.

$$300p_1 - 360\left(\tfrac{1}{12}\right)(16p_1 - 1) = -535$$   Substitute for $p_2$ in Equation 2.

$$300p_1 - 30(16p_1 - 1) = -535$$   Multiply factors.

$$-180p_1 = -565$$   Combine like terms.

$$p_1 = \tfrac{113}{36} \approx 3.14$$   Divide each side by $-180$.

$$p_2 = \tfrac{1}{12}\left[16\left(\tfrac{113}{36}\right) - 1\right]$$   Find $p_2$ by substituting $p_1$.

$$p_2 \approx 4.10$$   Solve for $p_2$.

*Example 2* **Solving Systems of Equations**

Solve each system of equations.

**a.** $\begin{cases} y(24 - 12x - 4y) = 0 \\ x(24 - 6x - 8y) = 0 \end{cases}$ **b.** $\begin{cases} 28a - 4b = 10 \\ -4a + 8b = 12 \end{cases}$

**SOLUTION**

**a.** Example 5, page 521

Before solving this system of equations, factor 4 out of the first equation and factor 2 out of the second equation.

$$\begin{cases} y(24 - 12x - 4y) = 0 & \text{Original Equation 1} \\ x(24 - 6x - 8y) = 0 & \text{Original Equation 2} \end{cases}$$
$$\begin{cases} y(4)(6 - 3x - y) = 0 & \text{Factor 4 out of Equation 1.} \\ x(2)(12 - 3x - 4y) = 0 & \text{Factor 2 out of Equation 2.} \end{cases}$$
$$\begin{cases} y(6 - 3x - y) = 0 & \text{Equation 1} \\ x(12 - 3x - 4y) = 0 & \text{Equation 2} \end{cases}$$

In each equation, either factor can be 0, so you obtain four different linear systems. For the first system, substitute $y = 0$ into the second equation to obtain $x = 4$.

$$\begin{cases} y = 0 \\ 12 - 3x - 4y = 0 \end{cases} \qquad (4, 0) \text{ is a solution.}$$

You can solve the second system by the method of elimination.

$$\begin{cases} 6 - 3x - y = 0 \\ 12 - 3x - 4y = 0 \end{cases} \qquad \left(\tfrac{4}{3}, 2\right) \text{ is a solution.}$$

The third system is already solved.

$$\begin{cases} y = 0 \\ x = 0 \end{cases} \qquad (0, 0) \text{ is a solution.}$$

You can solve the last system by substituting $x = 0$ into the first equation to obtain $y = 6$.

$$\begin{cases} 6 - 3x - y = 0 \\ x = 0 \end{cases} \qquad (0, 6) \text{ is a solution.}$$

**b.** Example 2, page 537

$$\begin{cases} 28a - 4b = 10 & \text{Equation 1} \\ -4a + 8b = 12 & \text{Equation 2} \end{cases}$$
$$\begin{aligned} -2a + 4b &= 6 & \text{Divide Equation 2 by 2.} \\ 26a &= 16 & \text{Add new equation to Equation 1.} \\ a &= \tfrac{8}{13} & \text{Divide each side by 26.} \\ 28\left(\tfrac{8}{13}\right) - 4b &= 10 & \text{Substitute for } a \text{ in Equation 1.} \\ b &= \tfrac{47}{26} & \text{Solve for } b. \end{aligned}$$

# Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 565.
Answers to odd-numbered Review Exercises are given in the back of the text.*

| Section 7.1 | Review Exercises |
|---|---|
| ■ Plot points in space. | 1, 2 |
| ■ Find the distance between two points in space. | 3, 4 |
| $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ | |
| ■ Find the midpoints of line segments in space. | 5, 6 |
| $\text{Midpoint} = \left( \dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2} \right)$ | |
| ■ Write the standard forms of the equations of spheres. | 7–10 |
| $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$ | |
| ■ Find the centers and radii of spheres. | 11, 12 |
| ■ Sketch the coordinate plane traces of spheres. | 13, 14 |

| Section 7.2 | |
|---|---|
| ■ Sketch planes in space. | 15–18 |
| ■ Classify quadric surfaces in space. | 19–26 |

| Section 7.3 | |
|---|---|
| ■ Evaluate functions of several variables. | 27, 28, 62 |
| ■ Find the domains and ranges of functions of several variables. | 29, 30 |
| ■ Sketch the level curves of functions of two variables. | 31–34 |
| ■ Use functions of several variables to answer questions about real-life situations. | 35–40 |

| Section 7.4 | |
|---|---|
| ■ Find the first partial derivatives of functions of several variables. | 41–50 |
| $\dfrac{\partial z}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \qquad \dfrac{\partial z}{\partial y} = \lim\limits_{\Delta y \to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$ | |
| ■ Find the slopes of surfaces in the $x$- and $y$-directions. | 51–54 |
| ■ Find the second partial derivatives of functions of several variables. | 55–58 |
| ■ Use partial derivatives to answer questions about real-life situations. | 59–61 |

| Section 7.5 | |
|---|---|
| ■ Find the relative extrema of functions of two variables. | 63–70 |
| ■ Use relative extrema to answer questions about real-life situations. | 71, 72 |

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\* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

## Section 7.6

## Section 7.7

$$a = \left[ n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \right] \bigg/ \left[ n\sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2 \right], \quad b = \frac{1}{n}\left( \sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i \right)$$

## Section 7.8

## Section 7.9

$$\text{Volume} = \int_R\int f(x, y)\, dA$$

$$\text{Average value} = \frac{1}{A}\int_R\int f(x, y)\, dA$$

## Study Strategies

■ **Comparing Two Dimensions with Three Dimensions**   Many of the formulas and techniques in this chapter are generalizations of formulas and techniques used in earlier chapters in the text. Here are several examples.

| Two-Dimensional Coordinate System | Three-Dimensional Coordinate System |
|---|---|
| *Distance Formula* <br> $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ | *Distance Formula* <br> $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ |
| *Midpoint Formula* <br> $\text{Midpoint} = \left( \dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right)$ | *Midpoint Formula* <br> $\text{Midpoint} = \left( \dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2} \right)$ |
| *Equation of Circle* <br> $(x - h)^2 + (y - k)^2 = r^2$ | *Equation of Sphere* <br> $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$ |
| *Equation of Line* <br> $ax + by = c$ | *Equation of Plane* <br> $ax + by + cz = d$ |
| *Derivative of $y = f(x)$* <br> $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ | *Partial Derivative of $z = f(x, y)$* <br> $\dfrac{\partial z}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ |
| *Area of Region* <br> $A = \displaystyle\int_a^b f(x)\, dx$ | *Volume of Region* <br> $V = \displaystyle\int_R\int f(x, y)\, dA$ |

# Review Exercises

In Exercises 1 and 2, plot the points.

**1.** $(2, -1, 4), (-1, 3, -3)$

**2.** $(1, -2, -3), (-4, -3, 5)$

In Exercises 3 and 4, find the distance between the two points.

**3.** $(0, 0, 0), (2, 5, 9)$          **4.** $(-4, 1, 5), (1, 3, 7)$

In Exercises 5 and 6, find the midpoint of the line segment joining the two points.

**5.** $(2, 6, 4), (-4, 2, 8)$          **6.** $(5, 0, 7), (-1, -2, 9)$

In Exercises 7–10, find the standard form of the equation of the sphere.

**7.** Center: $(0, 1, 0)$;  radius: 5

**8.** Center: $(4, -5, 3)$;  radius: 10

**9.** Diameter endpoints: $(0, 0, 4), (4, 6, 0)$

**10.** Diameter endpoints:  $(3, 4, 0), (5, 8, 2)$

In Exercises 11 and 12, find the center and radius of the sphere.

**11.** $x^2 + y^2 + z^2 + 4x - 2y - 8z + 5 = 0$

**12.** $x^2 + y^2 + z^2 + 4y - 10z - 7 = 0$

In Exercises 13 and 14, sketch the *xy*-trace of the sphere.

**13.** $(x + 2)^2 + (y - 1)^2 + (z - 3)^2 = 25$

**14.** $(x - 1)^2 + (y + 3)^2 + (z - 6)^2 = 72$

In Exercises 15–18, find the intercepts and sketch the graph of the plane.

**15.** $x + 2y + 3z = 6$

**16.** $2y + z = 4$

**17.** $3x - 6z = 12$

**18.** $4x - y + 2z = 8$

In Exercises 19–26, identify the surface.

**19.** $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$

**20.** $16x^2 + 16y^2 - 9z^2 = 0$

**21.** $x^2 + \dfrac{y^2}{16} + \dfrac{z^2}{9} = 1$

**22.** $x^2 - \dfrac{y^2}{16} - \dfrac{z^2}{9} = 1$

**23.** $z = \dfrac{x^2}{9} + y^2$

**24.** $-4x^2 + y^2 + z^2 = 4$

**25.** $z = \sqrt{x^2 + y^2}$

**26.** $z = 9x + 3y - 5$

In Exercises 27 and 28, find the function values.

**27.** $f(x, y) = xy^2$

(a) $f(2, 3)$          (b) $f(0, 1)$

(c) $f(-5, 7)$          (d) $f(-2, -4)$

**28.** $f(x, y) = \dfrac{x^2}{y}$

(a) $f(6, 9)$          (b) $f(8, 4)$

(c) $f(t, 2)$          (d) $f(r, r)$

In Exercises 29 and 30, describe the region $R$ in the *xy*-plane that corresponds to the domain of the function. Then find the range of the function.

**29.** $f(x, y) = \sqrt{1 - x^2 - y^2}$

**30.** $f(x, y) = \dfrac{1}{x + y}$

In Exercises 31–34, describe the level curves of the function. Sketch the level curves for the given *c*-values.

**31.** $z = 10 - 2x - 5y, \ c = 0, 2, 4, 5, 10$

**32.** $z = \sqrt{9 - x^2 - y^2}, \ c = 0, 1, 2, 3$

**33.** $z = (xy)^2, \ c = 1, 4, 9, 12, 16$

**34.** $z = y - x^2, c = 0, \pm 1, \pm 2$

**35.** **Meteorology**   The contour map shown below represents the average yearly precipitation for Iowa.   *(Source: U.S. National Oceanic and Atmospheric Administration)*

(a) Discuss the use of color to represent the level curves.

(b) Which part of Iowa receives the most precipitation?

(c) Which part of Iowa receives the least precipitation?

**36. Population Density** The contour map below represents the population density of New York. *(Source: U.S. Bureau of Census)*

(a) Discuss the use of color to represent the level curves.

(b) Do the level curves correspond to equally spaced population densities?

(c) Describe how to obtain a more detailed contour map.



**37. Chemistry** The acidity of rainwater is measured in units called pH, and smaller pH values are increasingly acidic. The map shows the curves of equal pH and gives evidence that downwind of heavily industrialized areas, the acidity has been increasing. Using the level curves on the map, determine the direction of the prevailing winds in the northeastern United States.



**38. Sales** The table gives the sales $x$ (in billions of dollars), the shareholder's equity $y$ (in billions of dollars), and the earnings per share $z$ (in dollars) for Johnson & Johnson for the years 2000 through 2005. *(Source: Johnson & Johnson)*

| Year | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
|------|------|------|------|------|------|------|
| $x$  | 29.1 | 33.0 | 36.3 | 41.9 | 47.3 | 50.5 |
| $y$  | 18.8 | 24.2 | 22.7 | 26.9 | 31.8 | 37.9 |
| $z$  | 1.70 | 1.91 | 2.23 | 2.70 | 3.10 | 3.50 |

A model for these data is

$$z = f(x, y) = 0.078x + 0.008y - 0.767.$$

Ⓣ (a) Use a graphing utility and the model to approximate $z$ for the given values of $x$ and $y$.

(b) Which of the two variables in this model has the greater influence on shareholder's equity?

(c) Simplify the expression for $f(x, 45)$ and interpret its meaning in the context of the problem.

**39. Equation of Exchange** Economists use an equation of exchange to express the relation among money, prices, and business transactions. This equation can be written as

$$P = \frac{MV}{T}$$

where $M$ is the money supply, $V$ is the velocity of circulation, $T$ is the total number of transactions, and $P$ is the price level. Find $P$ when $M = \$2500$, $V = 6$, and $T = 6000$.

**40. Biomechanics** The Froude number $F$, defined as

$$F = \frac{v^2}{gl}$$

where $v$ represents velocity, $g$ represents gravitational acceleration, and $l$ represents stride length, is an example of a "similarity criterion." Find the Froude number of a rabbit for which velocity is 2 meters per second, gravitational acceleration is 3 meters per second squared, and stride length is 0.75 meter.

In Exercises 41–50, find all first partial derivatives.

**41.** $f(x, y) = x^2y + 3xy + 2x - 5y$

**42.** $f(x, y) = 4xy + xy^2 - 3x^2y$

**43.** $z = \dfrac{x^2}{y^2}$

**44.** $z = (xy + 2x + 4y)^2$

**45.** $f(x, y) = \ln(2x + 3y)$

**46.** $f(x, y) = \ln\sqrt{2x + 3y}$

**47.** $f(x, y) = xe^y + ye^x$

**48.** $f(x, y) = x^2e^{-2y}$

**49.** $w = xyz^2$

**50.** $w = 3xy - 5xz + 2yz$

In Exercises 51–54, find the slope of the surface at the indicated point in (a) the $x$-direction and (b) the $y$-direction.

**51.** $z = 3x - 4y + 9$, $(3, 2, 10)$

**52.** $z = 4x^2 - y^2$, $(2, 4, 0)$

**53.** $z = 8 - x^2 - y^2$, $(1, 2, 3)$

**54.** $z = x^2 - y^2$, $(5, -4, 9)$

In Exercises 55–58, find all second partial derivatives.

**55.** $f(x, y) = 3x^2 - xy + 2y^3$

**56.** $f(x, y) = \dfrac{y}{x + y}$

**57.** $f(x, y) = \sqrt{1 + x + y}$

**58.** $f(x, y) = x^2e^{-y^2}$

**59. Marginal Cost** A company manufactures two models of skis: cross-country skis and downhill skis. The cost function for producing $x$ pairs of cross-country skis and $y$ pairs of downhill skis is given by

$$C = 15(xy)^{1/3} + 99x + 139y + 2293.$$

Find the marginal costs when $x = 500$ and $y = 250$.

**60. Marginal Revenue** At a baseball stadium, souvenir caps are sold at two locations. If $x_1$ and $x_2$ are the numbers of baseball caps sold at location 1 and location 2, respectively, then the total revenue for the caps is modeled by

$$R = 15x_1 + 16x_2 - \frac{1}{10}x_1^2 - \frac{1}{10}x_2^2 - \frac{1}{100}x_1x_2.$$

Given that $x_1 = 50$ and $x_2 = 40$, find the marginal revenues at location 1 and at location 2.

**61. Medical Science** The surface area $A$ of an average human body in square centimeters can be approximated by the model $A(w, h) = 101.4w^{0.425}h^{0.725}$, where $w$ is the weight in pounds and $h$ is the height in inches.

(a) Determine the partial derivatives of $A$ with respect to $w$ and with respect to $h$.

(b) Evaluate $\partial A/\partial w$ at $(180, 70)$. Explain your result.

**62. Medicine** In order to treat a certain bacterial infection, a combination of two drugs is being tested. Studies have shown that the duration $D$ (in hours) of the infection in laboratory tests can be modeled by

$$D(x, y) = x^2 + 2y^2 - 18x - 24y + 2xy + 120$$

where $x$ is the dosage in hundreds of milligrams of the first drug and $y$ is the dosage in hundreds of milligrams of the second drug. Evaluate $D(5, 2.5)$ and $D(7.5, 8)$ and interpret your results.

**In Exercises 63–70, find any critical points and relative extrema of the function.**

**63.** $f(x, y) = x^2 + 2y^2$

**64.** $f(x, y) = x^3 - 3xy + y^2$

**65.** $f(x, y) = 1 - (x + 2)^2 + (y - 3)^2$

**66.** $f(x, y) = e^x - x + y^2$

**67.** $f(x, y) = x^3 + y^2 - xy$

**68.** $f(x, y) = y^2 + xy + 3y - 2x + 5$

**69.** $f(x, y) = x^3 + y^3 - 3x - 3y + 2$

**70.** $f(x, y) = -x^2 - y^2$

**71. Revenue** A company manufactures and sells two products. The demand functions for the products are given by

$$p_1 = 100 - x_1 \quad \text{and} \quad p_2 = 200 - 0.5x_2.$$

(a) Find the total revenue function for $x_1$ and $x_2$.

(b) Find $x_1$ and $x_2$ such that the revenue is maximized.

(c) What is the maximum revenue?

**72. Profit** A company manufactures a product at two different locations. The costs of manufacturing $x_1$ units at plant 1 and $x_2$ units at plant 2 are modeled by $C_1 = 0.03x_1^2 + 4x_1 + 300$ and $C_2 = 0.05x_2^2 + 7x_2 + 175$, respectively. If the product sells for \$10 per unit, find $x_1$ and $x_2$ such that the profit, $P = 10(x_1 + x_2) - C_1 - C_2$, is maximized.

**In Exercises 73–78, locate any extrema of the function by using Lagrange multipliers.**

**73.** $f(x, y) = x^2y$
   Constraint: $x + 2y = 2$

**74.** $f(x, y) = x^2 + y^2$
   Constraint: $x + y = 4$

**75.** $f(x, y, z) = xyz$
   Constraint: $x + 2y + z - 4 = 0$

**76.** $f(x, y, z) = x^2z + yz$
   Constraint: $2x + y + z = 5$

**77.** $f(x, y, z) = x^2 + y^2 + z^2$
   Constraints: $x + z = 6, y + z = 8$

**78.** $f(x, y, z) = xyz$
   Constraints: $x + y + z = 32, x - y + z = 0$

**In Exercises 79 and 80, use a spreadsheet to find the indicated extremum. In each case, assume that $x$, $y$, and $z$ are nonnegative.**

**79.** Maximize $f(x, y, z) = xy$
   Constraints: $x^2 + y^2 = 16, x - 2z = 0$

**80.** Minimize $f(x, y, z) = x^2 + y^2 + z^2$
   Constraints: $x - 2z = 4, x + y = 8$

**81. Maximum Production Level** The production function for a manufacturer is given by $f(x, y) = 4x + xy + 2y$. Assume that the total amount available for labor $x$ and capital $y$ is \$2000 and that units of labor and capital cost \$20 and \$4, respectively. Find the maximum production level for this manufacturer.

**82. Minimum Cost** A manufacturer has an order for 1000 units of wooden benches that can be produced at two locations. Let $x_1$ and $x_2$ be the numbers of units produced at the two locations. Find the number that should be produced at each location to meet the order and minimize cost if the cost function is given by

$$C = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2.$$

**In Exercises 83 and 84, (a) use the method of least squares to find the least squares regression line and (b) calculate the sum of the squared errors.**

**83.** $(-2, -3), (-1, -1), (1, 2), (3, 2)$

**84.** $(-3, -1), (-2, -1), (0, 0), (1, 1), (2, 1)$

**85. Agriculture**  An agronomist used four test plots to deter-
mine the relationship between the wheat yield $y$ (in bushels
per acre) and the amount of fertilizer $x$ (in hundreds of
pounds per acre). The results are listed in the table.

| Fertilizer, $x$ | 1.0 | 1.5 | 2.0 | 2.5 |
|---|---|---|---|---|
| Yield, $y$ | 32 | 41 | 48 | 53 |

Ⓣ (a) Use the regression capabilities of a graphing utility or
Ⓢ    a spreadsheet to find the least squares regression line
     for the data.

(b) Estimate the yield for a fertilizer application of
    20 pounds per acre.

**86. Work Force**  The table gives the percents $x$ and numbers
$y$ (in millions) of women in the work force for selected
years.  *(Source: U.S. Bureau of Labor Statistics)*

| Year | 1970 | 1975 | 1980 | 1985 |
|---|---|---|---|---|
| Percent, $x$ | 43.3 | 46.3 | 51.5 | 54.5 |
| Number, $y$ | 31.5 | 37.5 | 45.5 | 51.1 |

| Year | 1990 | 1995 | 2000 | 2005 |
|---|---|---|---|---|
| Percent, $x$ | 57.5 | 58.9 | 59.9 | 59.3 |
| Number, $y$ | 56.8 | 60.9 | 66.3 | |

Ⓣ (a) Use the regression capabilities of a graphing utility or
Ⓢ    a spreadsheet to find the least squares regression line
     for the data.

(b) According to this model, approximately how many
    women enter the labor force for each one-point increase
    in the percent of women in the labor force?

Ⓣ In Exercises 87 and 88, use the regression capabilities
Ⓢ of a graphing utility or a spreadsheet to find the least
squares regression quadratic for the given points. Plot
the points and graph the least squares regression
quadratic.

**87.** $(-1, 9), (0, 7), (1, 5), (2, 6), (4, 23)$

**88.** $(0, 10), (2, 9), (3, 7), (4, 4), (5, 0)$

In Exercises 89–92, evaluate the double integral.
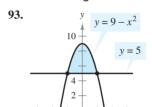
**89.** $\displaystyle\int_0^1 \int_0^{1+x} (4x - 2y)\, dy\, dx$
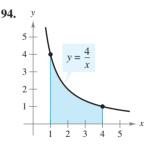
**90.** $\displaystyle\int_{-3}^3 \int_0^4 (x - y^2)\, dx\, dy$
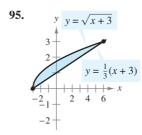
**91.** $\displaystyle\int_1^2 \int_1^{2y} \frac{x}{y^2}\, dx\, dy$
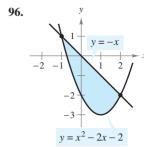
**92.** $\displaystyle\int_0^4 \int_0^{\sqrt{16-x^2}} 2x\, dy\, dx$

In Exercises 93–96, use a double integral to find the
area of the region.

**93.**



**94.**



**95.**



**96.**
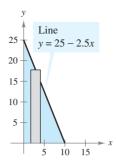


**97.** Find the volume of the solid bounded by the graphs of
$z = (xy)^2$, $z = 0$, $y = 0$, $y = 4$, $x = 0$, and $x = 4$.

**98.** Find the volume of the solid bounded by the graphs of
$z = x + y$, $z = 0$, $x = 0$, $x = 3$, $y = x$, and $y = 0$.

**99. Average Elevation**  In a triangular coastal area, the
elevation in miles above sea level at the point $(x, y)$ is
modeled by

$$f(x, y) = 0.25 - 0.025x - 0.01y$$

where $x$ and $y$ are measured in miles (see figure). Find the
average elevation of the triangular area.
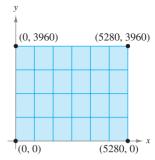


Figure for 99



Figure for 100

**100. Real Estate**  The value of real estate (in dollars per
square foot) for a rectangular section of a city is given by

$$f(x, y) = 0.003x^{2/3}y^{3/4}$$

where $x$ and $y$ are measured in feet (see figure). Find the
average value of real estate for this section.

## Chapter Test

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–3, (a) plot the points on a three-dimensional coordinate system, (b) find the distance between the points, and (c) find the coordinates of the midpoint of the line segment joining the points.

**1.** $(1, -3, 0), (3, -1, 0)$    **2.** $(-2, 2, 3), (-4, 0, 2)$    **3.** $(3, -7, 2), (5, 11, -6)$

**4.** Find the center and radius of the sphere whose equation is

$$x^2 + y^2 + z^2 - 20x + 10y - 10z + 125 = 0.$$

In Exercise 5–7, identify the surface.

**5.** $3x - y - z = 0$    **6.** $36x^2 + 9y^2 - 4z^2 = 0$    **7.** $4x^2 - y^2 - 16z = 0$

In Exercises 8–10, find $f(3, 3)$ and $f(1, 1)$.

**8.** $f(x, y) = x^2 + xy + 1$    **9.** $f(x, y) = \dfrac{x + 2y}{3x - y}$    **10.** $f(x, y) = xy \ln \dfrac{x}{y}$

In Exercises 11 and 12, find $f_x$ and $f_y$ and evaluate each at the point $(10, -1)$.

**11.** $f(x, y) = 3x^2 + 9xy^2 - 2$    **12.** $f(x, y) = x\sqrt{x + y}$

In Exercises 13 and 14, find any critical points, relative extrema, and saddle points of the function.

**13.** $f(x, y) = 3x^2 + 4y^2 - 6x + 16y - 4$

**14.** $f(x, y) = 4xy - x^4 - y^4$

| Exposure, $x$ | Mortality, $y$ |
|---------------|----------------|
| 1.35 | 118.5 |
| 2.67 | 135.2 |
| 3.93 | 167.3 |
| 5.14 | 197.6 |
| 7.43 | 204.7 |

Table for 16

**15.** The production function for a manufacturer can be modeled by

$$f(x, y) = 60x^{0.7}y^{0.3}$$

where $x$ is the number of units of labor and $y$ is the number of units of capital. Each unit of labor costs \$42 and each unit of capital costs \$144. The total cost of labor and capital is limited to \$240,000.

(a) Find the numbers of units of labor and capital needed to maximize production.

(b) Find the maximum production level for this manufacturer.

**16.** After contamination by a carcinogen, people in different geographic regions were assigned an exposure index to represent the degree of contamination. The table shows the exposure index $x$ and the corresponding mortality $y$ (per 100,000 people). Use the regression capabilities of a graphing utility or a spreadsheet to find the least squares regression quadratic for the data.

In Exercises 17 and 18, evaluate the double integral.

**17.** $\displaystyle\int_0^1 \int_x^1 (30x^2 y - 1) \, dy \, dx$    **18.** $\displaystyle\int_0^{\sqrt{e-1}} \int_0^{2y} \dfrac{1}{y^2 + 1} \, dx \, dy$

**19.** Use a double integral to find the area of the region bounded by the graphs of $y = 3$ and $y = x^2 - 2x + 3$ (see figure).

**20.** Find the average value of $f(x, y) = x^2 + y$ over the region defined by a rectangle with vertices $(0, 0), (1, 0), (1, 3),$ and $(0, 3)$.

Figure for 19

# 8 Trigonometric Functions



The amounts of precipitation during certain times of the year can be modeled by trigonometric functions. (See Section 8.5, Exercise 57.)

## Applications
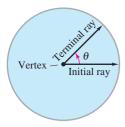
Trigonometric functions have many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Sprinkler System, Exercise 53, page 578
- Empire State Building, Exercise 69, page 589
- Make a Decision: Construction Workers, Exercise 79, page 598
- Consumer Trends, Exercise 53, page 617
- Inventory, Exercise 55, page 617

## Section 8.1

# Radian Measure of Angles

- Find coterminal angles.
- Convert from degree to radian measure and from radian to degree measure.
- Use formulas relating to triangles.
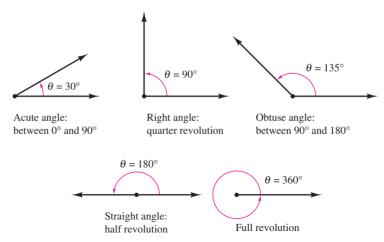


**FIGURE 8.1**  Standard Position of an Angle

## Angles and Degree Measure

As shown in Figure 8.1, an **angle** has three parts: an **initial ray,** a **terminal ray,** and a **vertex.** An angle is in **standard position** if its initial ray coincides with the positive *x*-axis and its vertex is at the origin.

Figure 8.2 shows the degree measures of several common angles. Note that $\theta$ (the lowercase Greek letter theta) is used to represent an angle and its measure. Angles whose measures are between 0° and 90° are **acute,** and angles whose measures are between 90° and 180° are **obtuse.** An angle whose measure is 90° is a **right** angle, and an angle whose measure is 180° is a **straight** angle.



**FIGURE 8.2**



**FIGURE 8.3**  Coterminal Angles

Positive angles are measured *counterclockwise* beginning with the initial ray. Negative angles are measured *clockwise*. For instance, Figure 8.3 shows an angle whose measure is −45°.

Merely knowing where an angle's initial and terminal rays are located does not allow you to assign a measure to the angle. To measure an angle, you must know how the terminal ray was revolved. For example, Figure 8.3 shows that the angle measuring −45° has the same terminal ray as the angle measuring 315°. Such angles are called **coterminal.**

**FIGURE 8.4**

Although it may seem strange to consider angle measures that are larger than 360°, such angles have very useful applications in trigonometry. An angle that is larger than 360° is one whose terminal ray has revolved more than one full revolution counterclockwise. Figure 8.4 shows two angles measuring more than 360°. In a similar way, you can generate an angle whose measure is less than −360° by revolving a terminal ray more than one full revolution clockwise.

### *Example 1*  Finding Coterminal Angles

For each angle, find a coterminal angle $\theta$ such that $0° \leq \theta < 360°$.

**a.** 450°

**b.** 750°

**c.** −160°

**d.** −390°

**SOLUTION**

**a.** To find an angle coterminal to 450°, subtract 360°, as shown in Figure 8.5(a).

$$\theta = 450° - 360° = 90°$$

**b.** To find an angle that is coterminal to 750°, subtract 2(360°), as shown in Figure 8.5(b).
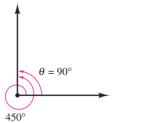
$$\theta = 750° - 2(360°) = 750° - 720° = 30°$$

**c.** To find an angle coterminal to −160°, add 360°, as shown in Figure 8.5(c).

$$\theta = -160° + 360° = 200°$$
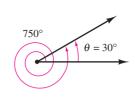
**d.** To find an angle that is coterminal to −390°, add 2(360°), as shown in Figure 8.5(d).
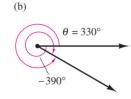
$$\theta = -390° + 2(360°) = -390° + 720° = 330°$$



(a)

(b)



(c)

(d)

**FIGURE 8.5**

### ✓ CHECKPOINT 1

For each angle, find a coterminal angle $\theta$ such that $0° \leq \theta < 360°$.

**a.** $\theta = -210°$

**b.** $\theta = -330°$

**c.** $\theta = 495°$

**d.** $\theta = 390°$ ■

## Radian Measure



The arc length of the sector is the radian measure of $\theta$.

**FIGURE 8.6**
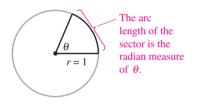
A second way to measure angles is in terms of radians. To assign a radian measure to an angle $\theta$, consider $\theta$ to be the central angle of a circular sector of radius 1, as shown in Figure 8.6. The radian measure of $\theta$ is then defined to be the length of the arc of the sector. Recall that the circumference of a circle is given by

$$\text{Circumference} = (2\pi)(\text{radius}).$$

So, the circumference of a circle of radius 1 is simply $2\pi$, and you can conclude that the radian measure of an angle measuring $360°$ is $2\pi$. In other words

$$360° = 2\pi \text{ radians}$$

or

$$180° = \pi \text{ radians}.$$

Figure 8.7 gives the radian measures of several common angles.



$$30° = \frac{\pi}{6} \qquad 45° = \frac{\pi}{4} \qquad 60° = \frac{\pi}{3}$$

$$90° = \frac{\pi}{2} \qquad 180° = \pi$$

$$360° = 2\pi$$
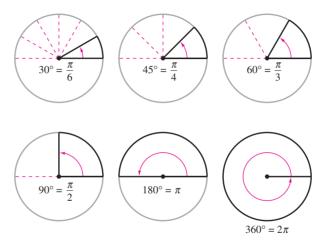
**FIGURE 8.7** Radian Measures of Several Common Angles

It is important for you to be able to convert back and forth between the degree and radian measures of an angle. You should remember the conversions for the common angles shown in Figure 8.7. For other conversions, you can use the conversion rule below.

**Angle Measure Conversion Rule**

The degree measure and radian measure of an angle are related by the equation

$$180° = \pi \text{ radians}.$$

Conversions between degrees and radians can be done as follows.

1. To convert degrees to radians, multiply degrees by $\dfrac{\pi \text{ radians}}{180°}$.

2. To convert radians to degrees, multiply radians by $\dfrac{180°}{\pi \text{ radians}}$.

$Example\ 2$  **Converting from Degrees to Radians**

Convert each degree measure to radian measure.

**a.** $135°$  **b.** $40°$  **c.** $540°$  **d.** $-270°$

**SOLUTION**  To convert from degree measure to radian measure, multiply the degree measure by $(\pi\ \text{radians})/180°$.

**a.** $135° = (135\ \text{degrees})\left(\dfrac{\pi\ \text{radians}}{180\ \text{degrees}}\right) = \dfrac{3\pi}{4}\ \text{radians}$

**b.** $40° = (40\ \text{degrees})\left(\dfrac{\pi\ \text{radians}}{180\ \text{degrees}}\right) = \dfrac{2\pi}{9}\ \text{radian}$

**c.** $540° = (540\ \text{degrees})\left(\dfrac{\pi\ \text{radians}}{180\ \text{degrees}}\right) = 3\pi\ \text{radians}$

**d.** $-270° = (-270\ \text{degrees})\left(\dfrac{\pi\ \text{radians}}{180\ \text{degrees}}\right) = -\dfrac{3\pi}{2}\ \text{radians}$

✓ **CHECKPOINT 2**

Convert each degree measure to radian measure.

**a.** $225°$

**b.** $-45°$

**c.** $240°$

**d.** $150°$ ■

Although it is common to list radian measure in multiples of $\pi$, this is not necessary. For instance, if the degree measure of an angle is $79.3°$, the radian measure is

$$79.3° = (79.3\ \text{degrees})\left(\dfrac{\pi\ \text{radians}}{180\ \text{degrees}}\right) \approx 1.384\ \text{radians}.$$

**TECHNOLOGY**

(T) Most calculators and graphing utilities have both degree and radian modes. You should learn how to use your calculator to convert from degrees to radians, and vice versa. Use a calculator or graphing utility to verify the results of Examples 2 and 3.*

$Example\ 3$  **Converting from Radians to Degrees**

Convert each radian measure to degree measure.

**a.** $-\dfrac{\pi}{2}$  **b.** $\dfrac{7\pi}{4}$  **c.** $\dfrac{11\pi}{6}$  **d.** $\dfrac{9\pi}{2}$

**SOLUTION**  To convert from radian measure to degree measure, multiply the radian measure by $180°/(\pi\ \text{radians})$.

**a.** $-\dfrac{\pi}{2}\ \text{radians} = \left(-\dfrac{\pi}{2}\ \text{radians}\right)\left(\dfrac{180\ \text{degrees}}{\pi\ \text{radians}}\right) = -90°$

**b.** $\dfrac{7\pi}{4}\ \text{radians} = \left(\dfrac{7\pi}{4}\ \text{radians}\right)\left(\dfrac{180\ \text{degrees}}{\pi\ \text{radians}}\right) = 315°$

**c.** $\dfrac{11\pi}{6}\ \text{radians} = \left(\dfrac{11\pi}{6}\ \text{radians}\right)\left(\dfrac{180\ \text{degrees}}{\pi\ \text{radians}}\right) = 330°$

**d.** $\dfrac{9\pi}{2}\ \text{radians} = \left(\dfrac{9\pi}{2}\ \text{radians}\right)\left(\dfrac{180\ \text{degrees}}{\pi\ \text{radians}}\right) = 810°$

✓ **CHECKPOINT 3**

Convert each radian measure to degree measure.

**a.** $\dfrac{5\pi}{3}$  **b.** $\dfrac{7\pi}{6}$
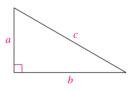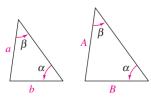
**c.** $\dfrac{3\pi}{2}$  **d.** $-\dfrac{3\pi}{4}$ ■

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*Specific calculator keystroke instructions for operations in this and other technology boxes can be found at *college.hmco.com/info/larsonapplied*.
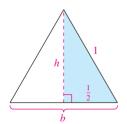
## Triangles



**FIGURE 8.8**   $a^2 + b^2 + c^2$



**FIGURE 8.9**   $\dfrac{a}{b} = \dfrac{A}{B}$

**A Summary of Rules About Triangles**

1. The sum of the angles of a triangle is 180°.

2. The sum of the two acute angles of a right triangle is 90°.

3. **Pythagorean Theorem**  The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse, as shown in Figure 8.8.

4. **Similar Triangles**  If two triangles are similar (have the same angle measures), then the ratios of the corresponding sides are equal, as shown in Figure 8.9.

5. The area of a triangle is equal to one-half the base times the height. That is, $A = \frac{1}{2}bh$.

6. Each angle of an equilateral triangle measures 60°.

7. Each acute angle of an isosceles right triangle measures 45°.

8. The altitude of an equilateral triangle bisects its base.

---

*Example 4*   **Finding the Area of a Triangle**

Find the area of an equilateral triangle with one-foot sides.

**SOLUTION**   To use the formula $A = \frac{1}{2}bh$, you must first find the height of the triangle, as shown in Figure 8.10. To do this, apply the Pythagorean Theorem to the shaded portion of the triangle.

$$h^2 + \left(\frac{1}{2}\right)^2 = 1^2 \qquad \text{Pythagorean Theorem}$$

$$h^2 = \frac{3}{4} \qquad \text{Simplify.}$$

$$h = \frac{\sqrt{3}}{2} \qquad \text{Solve for } h.$$



**FIGURE 8.10**

✓ **CHECKPOINT 4**

Find the area of an isosceles right triangle with a hypotenuse of $\sqrt{2}$ feet. ■

So, the area of the triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}(1)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4} \text{ square foot.}$$

**CONCEPT CHECK**

1. The measure of an angle is 35°. Is the angle obtuse or acute?

2. Is the angle whose measure is 45° coterminal to an angle whose measure is 315°?

3. What is the measure of a right angle? What is the measure of a straight angle?

4. Name the three parts of an angle.

| **Skills Review 8.1** | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 1.1. |

In Exercises 1 and 2, find the area of the triangle.

**1.** Base: 10 cm; height: 7 cm          **2.** Base: 4 in.; height: 6 in.

In Exercises 3–6, let $a$ and $b$ represent the lengths of the legs, and let $c$ represent the length of the hypotenuse, of a right triangle. Solve for the missing side length.

**3.** $a = 5, b = 12$          **4.** $a = 3, c = 5$          **5.** $a = 8, c = 17$          **6.** $b = 8, c = 10$

In Exercises 7–10, let $a$, $b$, and $c$ represent the side lengths of a triangle. Use the information below to determine whether the figure is a right triangle, an isosceles triangle, or an equilateral triangle.

**7.** $a = 4, b = 4, c = 4$          **8.** $a = 3, b = 3, c = 4$

**9.** $a = 12, b = 16, c = 20$          **10.** $a = 1, b = 1, c = \sqrt{2}$
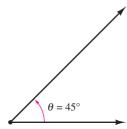
# Exercises 8.1
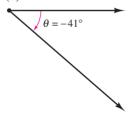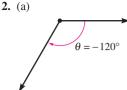
In Exercises 1–4, determine two coterminal angles (one positive and one negative) for each angle. Give the answers in degrees.

**1.** (a)          (b)



$\theta = 45°$          $\theta = -41°$

**2.** (a)          (b)



$\theta = -120°$          $\theta = 390°$

**3.** (a)          (b)



$\theta = 300°$          $\theta = 740°$

**4.** (a)          (b)



$\theta = -420°$          $\theta = 230°$
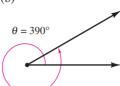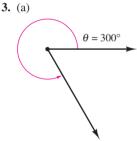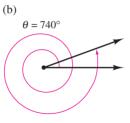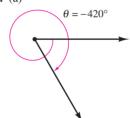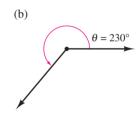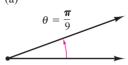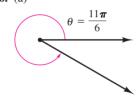
In Exercises 5–8, determine two coterminal angles (one positive and one negative) for each angle. Give the answers in radians.

**5.** (a)          (b)



$\theta = \dfrac{\pi}{9}$          $\theta = \dfrac{2\pi}{3}$

**6.** (a)          (b)



$\theta = \dfrac{11\pi}{6}$          $\theta = -\dfrac{7\pi}{6}$

**7.** (a)

$$\theta = -\frac{9\pi}{4}$$

(b)

$$\theta = -\frac{2\pi}{15}$$

**8.** (a)

$$\theta = \frac{8\pi}{9}$$

(b)

$$\theta = \frac{8\pi}{45}$$
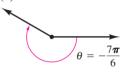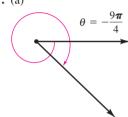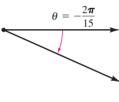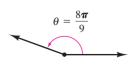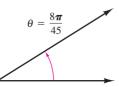
In Exercises 9–20, express the angle in radian measure as a multiple of $\pi$. Use a calculator to verify your result.

**9.** $30°$

**10.** $60°$

**11.** $270°$

**12.** $210°$

**13.** $315°$

**14.** $120°$

**15.** $-20°$

**16.** $-240°$

**17.** $-270°$

**18.** $-315°$

**19.** $330°$

**20.** $405°$

In Exercises 21–30, express the angle in degree measure. Use a calculator to verify your result.

**21.** $\dfrac{5\pi}{2}$

**22.** $\dfrac{5\pi}{4}$

**23.** $\dfrac{7\pi}{3}$

**24.** $\dfrac{\pi}{9}$

**25.** $-\dfrac{\pi}{12}$

**26.** $-\dfrac{7\pi}{4}$

**27.** $\dfrac{9\pi}{4}$

**28.** $-\dfrac{3\pi}{2}$

**29.** $\dfrac{19\pi}{6}$

**30.** $\dfrac{8\pi}{3}$

In Exercises 31–34, find the indicated measure of the angle. Express radian measure as a multiple of $\pi$.

| Degree Measure | Radian Measure |
|---|---|
| **31.** $-270°$ | |
| **32.** | $\dfrac{\pi}{9}$ |
| **33.** $144°$ | |
| **34.** | $-\dfrac{7\pi}{12}$ |
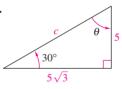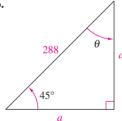
In Exercises 35–42, solve the triangle for the indicated side and/or angle.
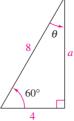
**35.**
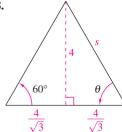
**36.**

**37.**

**38.**
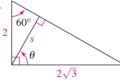
**39.**
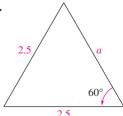
**40.**

**41.**

**42.**

In Exercises 43–46, find the area of the equilateral triangle with sides of length *s*.
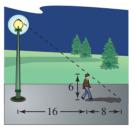
**43.** $s = 4$ in.

**44.** $s = 8$ m
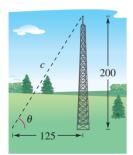
**45.** $s = 5$ ft

**46.** $s = 12$ cm
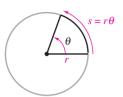
**47.** **Height**   A person 6 feet tall standing 16 feet from a streetlight casts a shadow 8 feet long (see figure). What is the height of the streetlight?

**48. Length**    A guy wire is stretched from a broadcasting tower at a point 200 feet above the ground to an anchor 125 feet from the base (see figure). How long is the wire?



**⑤ 49. Arc Length**    Let $r$ represent the radius of a circle, $\theta$ the central angle (measured in radians), and $s$ the length of the arc intercepted by the angle (see figure). Use the relationship $\theta = s/r$ and a spreadsheet to complete the table.
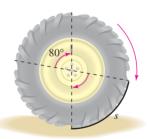


| $r$ | 8 ft | 15 in. | 85 cm | | |
|---|---|---|---|---|---|
| $s$ | 12 ft | | | 96 in. | 8642 mi |
| $\theta$ | | 1.6 | $\dfrac{3\pi}{4}$ | 4 | $\dfrac{2\pi}{3}$ |

**50. Arc Length**    The minute hand on a clock is $3\frac{1}{2}$ inches long (see figure). Through what distance does the tip of the minute hand move in 25 minutes?
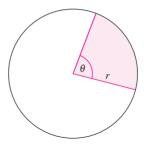


**51. Distance**    A tractor tire that is 5 feet in diameter $d$ is partially filled with a liquid ballast for additional traction. To check the air pressure, the tractor operator rotates the tire until the valve stem is at the top so that the liquid will not enter the gauge. On a given occasion, the operator notes that the tire must be rotated 80° to have the stem in the proper position (see figure).

(a) Find the radian measure of this rotation.

(b) How far must the tractor be moved to get the valve stem in the proper position?



**52. Speed of Revolution**    A compact disc can have an angular speed up to 3142 radians per minute.

(a) At this angular speed, how many revolutions per minute would the CD make?

(b) How long would it take the CD to make 10,000 revolutions?

**Area of Sector of a Circle**    In Exercises 53 and 54, use the following information. A sector of a circle is the region bounded by two radii of the circle and their intercepted arc (see figure).



For a circle of radius $r$, the area $A$ of a sector of the circle with central angle $\theta$ (measured in radians) is given by $A = \frac{1}{2}r^2\theta$.

**53. Sprinkler System**    A sprinkler system on a farm is set to spray water over a distance of 70 feet and rotates through an angle of 120°. Find the area of the region.

**54. Windshield Wiper**    A car's rear windshield wiper rotates 125°. The wiper mechanism has a total length of 25 inches and wipes the windshield over a distance of 14 inches. Find the area covered by the wiper.

**True or False?**    In Exercises 55–58, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**55.** An angle whose measure is 75° is obtuse.

**56.** $\theta = -35°$ is coterminal to 325°.

**57.** A right triangle can have one angle whose measure is 89°.

**58.** An angle whose measure is $\pi$ radians is a straight angle.

## Section 8.2

# The Trigonometric Functions

- ■ Recognize trigonometric functions.
- ■ Use trigonometric identities.
- ■ Evaluate trigonometric functions and solve right triangles.
- ■ Solve trigonometric equations.

## The Trigonometric Functions

There are two common approaches to the study of trigonometry. In one case the trigonometric functions are defined as ratios of two sides of a right triangle. In the other case these functions are defined in terms of a point on the terminal side of an arbitrary angle. The first approach is the one generally used in surveying, navigation, and astronomy, where a typical problem involves a triangle, three of whose six parts (sides and angles) are known and three of which are to be determined. The second approach is the one normally used in science and economics, where the periodic nature of the trigonometric functions is emphasized. In the definitions below, the six trigonometric functions are defined from both viewpoints.



**FIGURE 8.11**



**FIGURE 8.12**

### Definitions of the Trigonometric Functions

*Right Triangle Definition:* $0 < \theta < \dfrac{\pi}{2}$. (See Figure 8.11.)

$$\sin \theta = \frac{\text{opp.}}{\text{hyp.}} \qquad \csc \theta = \frac{\text{hyp.}}{\text{opp.}}$$

$$\cos \theta = \frac{\text{adj.}}{\text{hyp.}} \qquad \sec \theta = \frac{\text{hyp.}}{\text{adj.}}$$

$$\tan \theta = \frac{\text{opp.}}{\text{adj.}} \qquad \cot \theta = \frac{\text{adj.}}{\text{opp.}}$$
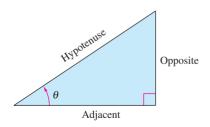
*Circular Function Definition:* $\theta$ is any angle in standard position and $(x, y)$ is a point on the terminal ray of the angle. (See Figure 8.12.)

$$\sin \theta = \frac{y}{r} \qquad \csc \theta = \frac{r}{y}$$
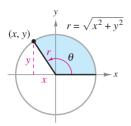
$$\cos \theta = \frac{x}{r} \qquad \sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x} \qquad \cot \theta = \frac{x}{y}$$

The full names of the trigonometric functions are **sine, cosecant, cosine, secant, tangent,** and **cotangent.**
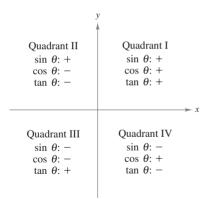
## Trigonometric Identities

In the circular function definition of the six trigonometric functions, the value of $r$ is always positive. From this, it follows that the signs of the trigonometric functions are determined from the signs of $x$ and $y$, as shown in Figure 8.13.

The trigonometric reciprocal identities below are also direct consequences of the definitions.

$$\sin \theta = \frac{1}{\csc \theta} \qquad \cos \theta = \frac{1}{\sec \theta} \qquad \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{1}{\cot \theta}$$

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta}$$

Furthermore, because

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 \\ &= \frac{x^2 + y^2}{r^2} \\ &= \frac{r^2}{r^2} \\ &= 1 \end{aligned}$$

you can obtain the Pythagorean Identity $\sin^2 \theta + \cos^2 \theta = 1$. Other trigonometric identities are listed below. In the list, $\phi$ is the lowercase Greek letter phi.

**FIGURE 8.13**

Quadrant II
$\sin \theta$: +
$\cos \theta$: −
$\tan \theta$: −

Quadrant I
$\sin \theta$: +
$\cos \theta$: +
$\tan \theta$: +

Quadrant III
$\sin \theta$: −
$\cos \theta$: −
$\tan \theta$: +

Quadrant IV
$\sin \theta$: −
$\cos \theta$: +
$\tan \theta$: −

**STUDY TIP**

The symbol $\sin^2 \theta$ is used to represent $(\sin \theta)^2$.

**Trigonometric Identities**

*Pythagorean Identities*

$\sin^2 \theta + \cos^2 \theta = 1$

$\tan^2 \theta + 1 = \sec^2 \theta$

$\cot^2 \theta + 1 = \csc^2 \theta$

*Sum or Difference of Two Angles*

$\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi$

$\cos(\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi$

$\tan(\theta \pm \phi) = \dfrac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi}$

*Double Angle*

$\sin 2\theta = 2 \sin \theta \cos \theta$

$\cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$

*Reduction Formulas*

$\sin(-\theta) = -\sin \theta$

$\cos(-\theta) = \cos \theta$

$\tan(-\theta) = -\tan \theta$

$\sin \theta = -\sin(\theta - \pi)$

$\cos \theta = -\cos(\theta - \pi)$

$\tan \theta = \tan(\theta - \pi)$

*Half Angle*

$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$

$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$

Although an angle can be measured in either degrees or radians, radian measure is preferred in calculus. So, all angles in the remainder of this chapter are assumed to be measured in radians unless otherwise indicated. In other words, $\sin 3$ means the sine of 3 radians, and $\sin 3°$ means the sine of 3 degrees.

## Evaluating Trigonometric Functions

There are two common methods of evaluating trigonometric functions: decimal approximations using a calculator and exact evaluations using trigonometric identities and formulas from geometry. The next three examples illustrate the second method.



FIGURE 8.14

*Example 1*   **Evaluating Trigonometric Functions**

Evaluate the sine, cosine, and tangent of $\pi/3$.

**SOLUTION**   Begin by drawing the angle $\theta = \pi/3$ in standard position, as shown in Figure 8.14. Because $\pi/3$ radians is $60°$, you can imagine an equilateral triangle with sides of length 1 and with $\theta$ as one of its angles. Because the altitude of the triangle bisects its base, you know that $x = \frac{1}{2}$. So, using the Pythagorean Theorem, you have
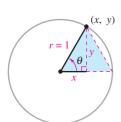
$$y = \sqrt{r^2 - x^2} = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$
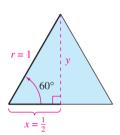
Now, using $x = \frac{1}{2}$, $y = \frac{1}{2}\sqrt{3}$, and $r = 1$, you can find the values of the sine, cosine, and tangent as shown.

$$\sin \frac{\pi}{3} = \frac{y}{r} = \frac{\frac{1}{2}\sqrt{3}}{1} = \frac{\sqrt{3}}{2}$$

$$\cos \frac{\pi}{3} = \frac{x}{r} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\tan \frac{\pi}{3} = \frac{y}{x} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}$$

✓**CHECKPOINT 1**

Evaluate the sine, cosine, and tangent of $\dfrac{\pi}{6}$. ■

The sines, cosines, and tangents of several common angles are listed in the table below. You should remember, or be able to derive, these values.

Trigonometric Values of Common Angles

| $\theta$ (degrees) | 0 | 30° | 45° | 60° | 90° | 180° | 270° |
|---|---|---|---|---|---|---|---|
| $\theta$ (radians) | 0 | $\dfrac{\pi}{6}$ | $\dfrac{\pi}{4}$ | $\dfrac{\pi}{3}$ | $\dfrac{\pi}{2}$ | $\pi$ | $\dfrac{3\pi}{2}$ |
| $\sin \theta$ | 0 | $\dfrac{1}{2}$ | $\dfrac{\sqrt{2}}{2}$ | $\dfrac{\sqrt{3}}{2}$ | 1 | 0 | $-1$ |
| $\cos \theta$ | 1 | $\dfrac{\sqrt{3}}{2}$ | $\dfrac{\sqrt{2}}{2}$ | $\dfrac{1}{2}$ | 0 | $-1$ | 0 |
| $\tan \theta$ | 0 | $\dfrac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ | Undefined | 0 | Undefined |

**STUDY TIP**

Learning the table of values at the right is worth the effort because doing so will increase both your efficiency and your confidence. Here is a pattern for the sine function that may help you remember the values.

| $\theta$ | 0 | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| $\sin \theta$ | $\dfrac{\sqrt{0}}{2}$ | $\dfrac{\sqrt{1}}{2}$ | $\dfrac{\sqrt{2}}{2}$ | $\dfrac{\sqrt{3}}{2}$ | $\dfrac{\sqrt{4}}{2}$ |

Reverse the order to get cosine values of the same angles.

To extend the use of the values in the table on the previous page to angles in quadrants other than the first quadrant, you can use the concept of a reference angle, as shown in Figure 8.15, together with the appropriate quadrant sign. The **reference angle $\theta'$** for an angle $\theta$ is the smallest positive angle between the terminal side of $\theta$ and the $x$-axis. For instance, the reference angle for 135° is 45° and the reference angle for 210° is 30°.



Reference angle: $\boldsymbol{\pi} - \boldsymbol{\theta}$     Reference angle: $\boldsymbol{\theta} - \boldsymbol{\pi}$     Reference angle: $2\boldsymbol{\pi} - \boldsymbol{\theta}$

**FIGURE 8.15**

To find the value of a trigonometric function of any angle $\theta$, first determine the function value for the associated reference angle $\theta'$. Then, depending on the quadrant in which $\theta$ lies, prefix the appropriate sign to the function value.

### Example 2   Evaluating Trigonometric Functions

Evaluate each trigonometric function.

**a.** $\sin \dfrac{3\pi}{4}$     **b.** $\tan 330°$     **c.** $\cos \dfrac{7\pi}{6}$

**SOLUTION**

**a.** Because the reference angle for $3\pi/4$ is $\pi/4$ and the sine is positive in the second quadrant, you can write
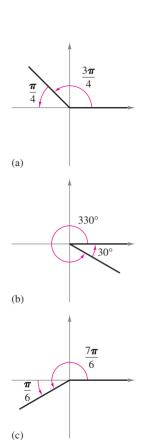
$$\sin \frac{3\pi}{4} = \sin \frac{\pi}{4} \qquad \text{Reference angle}$$

$$= \frac{\sqrt{2}}{2}. \qquad \text{See Figure 8.16(a).}$$

**b.** Because the reference angle for 330° is 30° and the tangent is negative in the fourth quadrant, you can write

$$\tan 330° = -\tan 30° \qquad \text{Reference angle}$$

$$= -\frac{\sqrt{3}}{3}. \qquad \text{See Figure 8.16(b).}$$

**c.** Because the reference angle for $7\pi/6$ is $\pi/6$ and the cosine is negative in the third quadrant, you can write

$$\cos \frac{7\pi}{6} = -\cos \frac{\pi}{6} \qquad \text{Reference angle}$$

$$= -\frac{\sqrt{3}}{2}. \qquad \text{See Figure 8.16(c).}$$



(a)

(b)

(c)

**FIGURE 8.16**

✓ **CHECKPOINT 2**

Evaluate each trigonometric function.

**a.** $\sin \dfrac{5\pi}{6}$

**b.** $\cos 135°$

**c.** $\tan \dfrac{5\pi}{3}$   ■

*Example 3* **Evaluating Trigonometric Functions**

Evaluate each trigonometric function.

**a.** $\sin\left(-\dfrac{\pi}{3}\right)$  **b.** $\sec 60°$  **c.** $\cos 15°$  **d.** $\sin 2\pi$  **e.** $\cot 0°$  **f.** $\tan \dfrac{9\pi}{4}$

**SOLUTION**

**a.** By the reduction formula $\sin(-\theta) = -\sin\theta$,

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

**b.** By the reciprocal formula $\sec\theta = 1/\cos\theta$,

$$\sec 60° = \frac{1}{\cos 60°} = \frac{1}{1/2} = 2.$$

**c.** By the difference formula $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$,

$$\cos 15° = \cos(45° - 30°)$$
$$= (\cos 45°)(\cos 30°) + (\sin 45°)(\sin 30°)$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}.$$

**d.** Because the reference angle for $2\pi$ is 0,

$$\sin 2\pi = \sin 0 = 0.$$

**e.** Using the reciprocal formula $\cot\theta = 1/\tan\theta$ and the fact that $\tan 0° = 0$, you can conclude that $\cot 0°$ is undefined.

**f.** Because the reference angle for $9\pi/4$ is $\pi/4$ and the tangent is positive in the first quadrant,

$$\tan\frac{9\pi}{4} = \tan\frac{\pi}{4} = 1.$$

✓**CHECKPOINT 3**

Evaluate each trigonometric function.

**a.** $\sin\left(-\dfrac{\pi}{6}\right)$  **b.** $\csc 45°$

**c.** $\cos 75°$  **d.** $\cos 2\pi$

**e.** $\sec 0°$  **f.** $\cot \dfrac{13\pi}{4}$ ■

**TECHNOLOGY**

(T) Examples 1, 2, and 3 all involve standard angles such as $\pi/6$ and $\pi/3$. To evaluate trigonometric functions involving nonstandard angles, you should use a calculator. When doing this, remember to set the calculator to the proper mode—either *degree* mode or *radian* mode. Furthermore, most calculators have only three trigonometric functions: sine, cosine, and tangent. To evaluate the other three functions, you should combine these keys with the reciprocal key. For instance, you can evaluate the secant of $\pi/7$ as shown:

| Function | Calculator Steps | Display |
|---|---|---|
| $\sec \dfrac{\pi}{7}$ | (COS) (π) (÷) (7) (  ) (x⁻¹) | 1.109916264 |
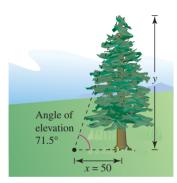
**FIGURE 8.17**

*Example 4*    **Solving a Right Triangle** ®

A surveyor is standing 50 feet from the base of a large tree. The surveyor measures the angle of elevation to the top of the tree as 71.5°. How tall is the tree?

**SOLUTION**    Referring to Figure 8.17, you can see that

$$\tan 71.5° = \frac{y}{x}$$

where $x = 50$ and $y$ is the height of the tree. So, you can determine the height of the tree to be

$$y = (x)(\tan 71.5°) \approx (50)(2.98868) \approx 149.4 \text{ feet.}$$

✓ **CHECKPOINT 4**

Find the height of a building that casts a 75-foot shadow when the angle of elevation of the sun is 35°. ∎



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Some occupations, such as that of a fighter pilot, require excellent vision, including good depth perception and good peripheral vision.

*Example 5*    **Calculating Peripheral Vision** ®
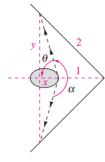
To measure the extent of your peripheral vision, stand 1 foot from the corner of a room, facing the corner. Have a friend move an object along the wall until you can just barely see it. If the object is 2 feet from the corner, as shown in Figure 8.18, what is the total angle of your peripheral vision?
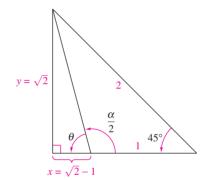
**SOLUTION**    Let $\alpha$ represent the total angle of your peripheral vision. As shown in Figure 8.19, you can model the physical situation with an isosceles right triangle whose legs are $\sqrt{2}$ feet and whose hypotenuse is 2 feet. In the triangle, the angle $\theta$ is given by

$$\tan \theta = \frac{y}{x} = \frac{\sqrt{2}}{\sqrt{2} - 1} \approx 3.414.$$

Using the inverse tangent function of a calculator, you can determine that $\theta \approx 73.7°$. So, $\alpha/2 \approx 180° - 73.7° = 106.3°$, which implies that $\alpha \approx 212.6°$. In other words, the total angle of your peripheral vision is about 212.6°.

✓ **CHECKPOINT 5**

If the object in Example 5 is 4 feet from the corner, find the total angle of your peripheral vision. ∎



**FIGURE 8.18**



**FIGURE 8.19**

## Solving Trigonometric Equations

An important part of the study of trigonometry is learning how to solve trigonometric equations. For example, consider the equation

$$\sin \theta = 0.$$

You know that $\theta = 0$ is one solution. Also, in Example 3(d), you saw that $\theta = 2\pi$ is another solution. But these are not the only solutions. In fact, this equation has infinitely many solutions. Any one of the values of $\theta$ shown below will work.

$$\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$$

To simplify the situation, the search for solutions is usually restricted to the interval $0 \le \theta \le 2\pi$, as shown in Example 6.

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### Algebra Review

For more examples of the algebra involved in solving trigonometric equations, see the *Chapter 8 Algebra Review*, on pages 619 and 620.
</div>

> **Example 6**  **Solving Trigonometric Equations**

Solve for $\theta$ in each equation. Assume $0 \le \theta \le 2\pi$.

**a.** $\sin \theta = -\dfrac{\sqrt{3}}{2}$  **b.** $\cos \theta = 1$  **c.** $\tan \theta = 1$

**SOLUTION**

**a.** To solve the equation $\sin \theta = -\sqrt{3}/2$, first remember that

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Because the sine is negative in the third and fourth quadrants, it follows that you are seeking values of $\theta$ in these quadrants that have a reference angle of $\pi/3$. The two angles fitting these criteria are

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \quad \text{and} \quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

as indicated in Figure 8.20.

**b.** To solve $\cos \theta = 1$, remember that $\cos 0 = 1$ and note that in the interval $[0, 2\pi]$, the only angles whose reference angles are zero are zero, $\pi$, and $2\pi$. Of these, zero and $2\pi$ have cosines of 1. (The cosine of $\pi$ is $-1$.) So, the equation has two solutions:

$$\theta = 0 \quad \text{and} \quad \theta = 2\pi.$$

**c.** Because $\tan \pi/4 = 1$ and the tangent is positive in the first and third quadrants, it follows that the two solutions are

$$\theta = \frac{\pi}{4} \quad \text{and} \quad \theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}.$$

Reference angle: $\dfrac{\pi}{3}$

$\theta = \dfrac{4\pi}{3}$

Quadrant III

Reference angle: $\dfrac{\pi}{3}$

$\theta = \dfrac{5\pi}{3}$

Quadrant IV

**FIGURE 8.20**

### ✓ CHECKPOINT 6

Solve for $\theta$ in each equation. Assume $0 \le \theta \le 2\pi$.

**a.** $\cos \theta = \dfrac{\sqrt{2}}{2}$  **b.** $\tan \theta = -\sqrt{3}$  **c.** $\sin \theta = -\dfrac{1}{2}$ ■

*Example 7*  **Solving a Trigonometric Equation**

Solve the equation for $\theta$.

$$\cos 2\theta = 2 - 3 \sin \theta, \quad 0 \leq \theta \leq 2\pi$$

**SOLUTION**  You can use the double-angle identity $\cos 2\theta = 1 - 2 \sin^2\theta$ to rewrite the original equation, as shown.

$$\cos 2\theta = 2 - 3 \sin \theta$$
$$1 - 2 \sin^2\theta = 2 - 3 \sin \theta$$
$$0 = 2 \sin^2\theta - 3 \sin \theta + 1$$
$$0 = (2 \sin \theta - 1)(\sin \theta - 1)$$

For $2 \sin \theta - 1 = 0$, you have $\sin \theta = \frac{1}{2}$, which has solutions of

$$\theta = \frac{\pi}{6} \quad \text{and} \quad \theta = \frac{5\pi}{6}.$$

For $\sin \theta - 1 = 0$, you have $\sin \theta = 1$, which has a solution of

$$\theta = \frac{\pi}{2}.$$

So, for $0 \leq \theta \leq 2\pi$, the three solutions are

$$\theta = \frac{\pi}{6}, \quad \frac{\pi}{2}, \quad \text{and} \quad \frac{5\pi}{6}.$$

✓ **CHECKPOINT 7**

Solve the equation for $\theta$.

$$\sin 2\theta + \sin \theta = 0, \quad 0 \leq \theta \leq 2\pi \quad \blacksquare$$

---

**STUDY TIP**

In Example 7, note that the expression $2 \sin^2\theta - 3 \sin \theta + 1$ is a quadratic in $\sin \theta$, and as such can be factored. For instance, if you let $x = \sin \theta$, then the quadratic factors as

$$2x^2 - 3x + 1 = (2x - 1)(x - 1).$$

---

**CONCEPT CHECK**

1. Relative to the angle $\theta$, name the three sides of a right triangle.

2. In the right triangle definition of trigonometric functions, $\sin \theta$ is equal to what?

3. In the circular function definition of trigonometric functions, $\cos \theta$ is equal to what?

4. The smallest positive angle between the terminal side of an angle $\theta$ and the x-axis is denoted $\theta'$. What is the angle $\theta'$ called?

| **Skills Review 8.2** | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4 and 8.1. |
|---|---|

In Exercises 1–8, convert the angle to radian measure.

**1.** $135°$       **2.** $315°$       **3.** $-210°$       **4.** $-300°$

**5.** $-120°$       **6.** $-225°$       **7.** $540°$       **8.** $390°$

In Exercises 9–16, solve for $x$.

**9.** $x^2 - x = 0$       **10.** $2x^2 + x = 0$       **11.** $2x^2 - x = 1$       **12.** $x^2 - 2x = 3$

**13.** $x^2 - 2x = -1$       **14.** $2x^2 + x = 1$       **15.** $x^2 - 5x = -6$       **16.** $x^2 + x = 2$

In Exercises 17–20, solve for $t$.

**17.** $\frac{2\pi}{24}(t - 4) = \frac{\pi}{2}$       **18.** $\frac{2\pi}{12}(t - 2) = \frac{\pi}{4}$       **19.** $\frac{2\pi}{365}(t - 10) = \frac{\pi}{4}$       **20.** $\frac{2\pi}{12}(t - 4) = \frac{\pi}{2}$
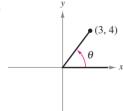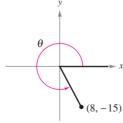
## Exercises 8.2
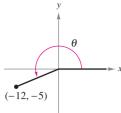
In Exercises 1–6, determine all six trigonometric functions for the angle $\theta$.

**1.**



**2.**
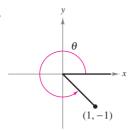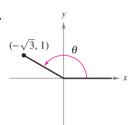


**3.**
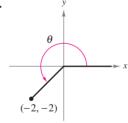


**4.**



**5.**
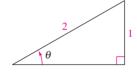


**6.**
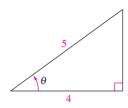


In Exercises 7–12, find the indicated trigonometric function from the given function.

**7.** Given $\sin \theta = \frac{1}{2}$, find $\csc \theta$.
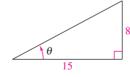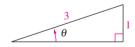


**8.** Given $\sin \theta = \frac{1}{3}$, find $\tan \theta$.



**9.** Given $\cos \theta = \frac{4}{5}$, find $\cot \theta$.



**10.** Given $\sec \theta = \frac{13}{5}$, find $\cot \theta$.



**11.** Given $\cot \theta = \frac{15}{8}$, find $\sec \theta$.



**12.** Given $\tan \theta = \frac{1}{2}$, find $\sin \theta$.
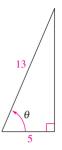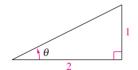
In Exercises 13–18, sketch a right triangle corresponding to the trigonometric function of the angle $\theta$ and find the other five trigonometric functions of $\theta$.

**13.** $\sin \theta = \frac{1}{3}$        **14.** $\cot \theta = 5$

**15.** $\sec \theta = 2$        **16.** $\cos \theta = \frac{5}{7}$

**17.** $\tan \theta = 3$        **18.** $\csc \theta = 4.25$

In Exercises 19–24, determine the quadrant in which $\theta$ lies.

**19.** $\sin \theta < 0, \cos \theta > 0$      **20.** $\sin \theta > 0, \cos \theta < 0$

**21.** $\sin \theta > 0, \sec \theta > 0$      **22.** $\cot \theta < 0, \cos \theta > 0$

**23.** $\csc \theta > 0, \tan \theta < 0$      **24.** $\cos \theta > 0, \tan \theta < 0$

In Exercises 25–30, construct an appropriate triangle to complete the table. ($0 \le \theta \le 90°$, $0 \le \theta \le \pi/2$)

| | Function | $\theta$ (deg) | $\theta$ (rad) | Function Value |
|---|---|---|---|---|
| **25.** | sin | 30° | | |
| **26.** | cos | 45° | | |
| **27.** | tan | | $\pi/3$ | |
| **28.** | sec | | $\pi/4$ | |
| **29.** | cot | | | 1 |
| **30.** | tan | | | $\sqrt{3}/3$ |

In Exercises 31–38, evaluate the sines, cosines, and tangents of the angles *without* using a calculator.

**31.** (a) $60°$        (b) $-\dfrac{2\pi}{3}$

**32.** (a) $\dfrac{\pi}{4}$        (b) $\dfrac{5\pi}{4}$

**33.** (a) $-\dfrac{\pi}{6}$        (b) $150°$

**34.** (a) $-\dfrac{\pi}{2}$        (b) $\dfrac{\pi}{2}$

**35.** (a) $225°$        (b) $-225°$

**36.** (a) $300°$        (b) $330°$

**37.** (a) $750°$        (b) $510°$

**38.** (a) $\dfrac{10\pi}{3}$        (b) $\dfrac{17\pi}{3}$

In Exercises 39–46, use a calculator to evaluate the trigonometric functions to four decimal places.

**39.** (a) $\sin 10°$        (b) $\csc 10°$

**40.** (a) $\sec 225°$        (b) $\sec 135°$

**41.** (a) $\tan \dfrac{\pi}{9}$        (b) $\tan \dfrac{10\pi}{9}$

**42.** (a) $\cot 4.5$        (b) $\tan 4.5$

**43.** (a) $\cos(-110°)$        (b) $\cos 250°$

**44.** (a) $\tan 240°$        (b) $\cot 210°$

**45.** (a) $\csc 2.62$        (b) $\csc 150°$

**46.** (a) $\sin(-0.65)$        (b) $\sin 5.63$

In Exercises 47–52, find two values of $\theta$ corresponding to each function. List the measure of $\theta$ in radians ($0 \le \theta \le 2\pi$). Do not use a calculator.

**47.** (a) $\sin \theta = \frac{1}{2}$        (b) $\sin \theta = -\frac{1}{2}$

**48.** (a) $\cos \theta = \dfrac{\sqrt{2}}{2}$      (b) $\cos \theta = -\dfrac{\sqrt{2}}{2}$

**49.** (a) $\csc \theta = \dfrac{2\sqrt{3}}{3}$      (b) $\cot \theta = -1$

**50.** (a) $\sec \theta = 2$        (b) $\sec \theta = -2$

**51.** (a) $\tan \theta = 1$        (b) $\cot \theta = -\sqrt{3}$

**52.** (a) $\sin \theta = \dfrac{\sqrt{3}}{2}$      (b) $\sin \theta = -\dfrac{\sqrt{3}}{2}$

In Exercises 53–62, solve the equation for $\theta$ ($0 \le \theta \le 2\pi$). For some of the equations you should use the trigonometric identities listed in this section. Use the *trace* feature of a graphing utility to verify your results.

**53.** $2 \sin^2 \theta = 1$        **54.** $\tan^2 \theta = 3$

**55.** $\tan^2 \theta - \tan \theta = 0$      **56.** $2 \cos^2 \theta - \cos \theta = 1$

**57.** $\sin 2\theta - \cos \theta = 0$      **58.** $\cos 2\theta + 3 \cos \theta + 2 = 0$

**59.** $\sin \theta = \cos \theta$        **60.** $\sec \theta \csc \theta = 2 \csc \theta$
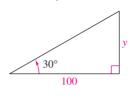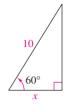
**61.** $\cos^2 \theta + \sin \theta = 1$      **62.** $\cos \dfrac{\theta}{2} - \cos \theta = 1$

In Exercises 63–68, solve for *x*, *y*, or *r* as indicated.

**63.** Solve for *y*.        **64.** Solve for *x*.
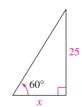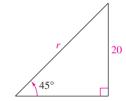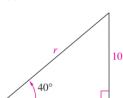
**65.** Solve for *x*.        **66.** Solve for *r*.
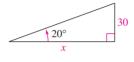
**67.** Solve for *r*.
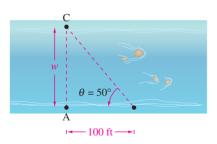


**68.** Solve for *x*.
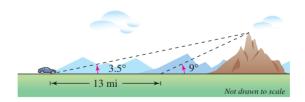


**69. Empire State Building** You are standing 45 meters from the base of the Empire State Building. You estimate that the angle of elevation to the top of the 86th floor is 82°. If the total height of the building is another 123 meters above the 86th floor, what is the approximate height of the building? One of your friends is on the 86th floor. What is the distance between you and your friend?

**70. Height** A six-foot person walks from the base of a broadcasting tower directly toward the tip of the shadow cast by the tower. When the person is 132 feet from the tower and 3 feet from the tip of the shadow, the person's shadow starts to appear beyond the tower's shadow.

(a) Draw the right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the tower.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) What is the height of the tower?

**71. Length** A 20-foot ladder leaning against the side of a house makes a 75° angle with the ground (see figure). How far up the side of the house does the ladder reach?



**72. Width of a River** A biologist wants to know the width *w* of a river in order to set instruments to study the pollutants in the water. From point A the biologist walks downstream 100 feet and sights to point C. From this sighting it is determined that $\theta = 50°$ (see figure). How wide is the river?



**73. Height of a Mountain** In traveling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is 3.5°. After you drive 13 miles closer to the mountain, the angle of elevation is 9°. Approximate the height of the mountain.



Not drawn to scale

**74. Distance** From a 150-foot observation tower on the coast, a Coast Guard officer sights a boat in difficulty. The angle of depression of the boat is 3° (see figure). How far is the boat from the shoreline?



Not drawn to scale

**75. Medicine** The temperature *T* in degrees Fahrenheit of a patient *t* hours after arriving at the emergency room of a hospital at 10:00 P.M. is given by

$$T(t) = 98.6 + 4 \cos \frac{\pi t}{36}, \quad 0 \le t \le 18.$$

Find the patient's temperature at each time.

(a) 10:00 P.M.    (b) 4:00 A.M.    (c) 10:00 A.M.

At what time do you expect the patient's temperature to return to normal? Explain your reasoning.

**(T) 76. Sales** A company that produces a window and door insulating kit forecasts monthly sales over the next 2 years to be

$$S = 23.1 + 0.442t + 4.3 \sin \frac{\pi t}{6}$$

where *S* is measured in thousands of units and *t* is the time in months, with $t = 1$ corresponding to January 2008. Use a graphing utility to estimate sales for each month.

(a) February 2008        (b) February 2009
(c) September 2008       (d) September 2009

**(T)(S)** In Exercises 77 and 78, use a graphing utility or a spreadsheet to complete the table. Then graph the function.

| *x*    | 0 | 2 | 4 | 6 | 8 | 10 |
|--------|---|---|---|---|---|----|
| $f(x)$ |   |   |   |   |   |    |

**77.** $f(x) = \frac{2}{5}x + 2 \sin \frac{\pi x}{5}$   **78.** $f(x) = \frac{1}{2}(5 - x) + 3 \cos \frac{\pi x}{5}$

**Section 8.3**

# Graphs of Trigonometric Functions

■ Sketch graphs of trigonometric functions.
■ Evaluate limits of trigonometric functions.
■ Use trigonometric functions to model real-life situations.

## Graphs of Trigonometric Functions

When you are sketching the graph of a trigonometric function, it is common to use $x$ (rather than $\theta$) as the independent variable. On the simplest level, you can sketch the graph of a function such as

$$f(x) = \sin x$$

by constructing a table of values, plotting the resulting points, and connecting them with a smooth curve, as shown in Figure 8.21. Some examples of values are shown in the table below.

| $x$ | 0 | $\dfrac{\pi}{6}$ | $\dfrac{\pi}{4}$ | $\dfrac{\pi}{3}$ | $\dfrac{\pi}{2}$ | $\dfrac{2\pi}{3}$ | $\dfrac{3\pi}{4}$ | $\dfrac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin x$ | 0.00 | 0.50 | 0.71 | 0.87 | 1.00 | 0.87 | 0.71 | 0.50 | 0.00 |

In Figure 8.21, note that the maximum value of $\sin x$ is 1 and the minimum value is $-1$. The **amplitude** of the sine function (or the cosine function) is defined to be half of the difference between its maximum and minimum values. So, the amplitude of $f(x) = \sin x$ is 1.

The periodic nature of the sine function becomes evident when you observe that as $x$ increases beyond $2\pi$, the graph repeats itself over and over, continuously oscillating about the $x$-axis. The **period** of the function is the distance (on the $x$-axis) between successive cycles. So, the period of $f(x) = \sin x$ is $2\pi$.



**FIGURE 8.21**

**DISCOVERY**

When the real number line is wrapped around the unit circle, each real number $t$ corresponds with a point $(x, y) = (\cos t, \sin t)$ on the circle. You can visualize this graphically by setting your graphing utility to *simultaneous* mode. For instance, using *radian* and *parametric* modes as well, let

    X1T = cos(T)
    Y1T = sin(T)
    X2T = T
    Y2T = sin(T).

Use the viewing window settings shown below.

    Tmin = 0
    Tmax = 6.3
    Tstep = .1
    Xmin = –2
    Xmax = 7
    Xscl = 1
    Ymin = –3
    Ymax = 3
    Yscl = 1

Now graph the functions. Notice how the graphing utility traces out the unit circle and the sine function simultaneously.

    Try changing Y2T to cos T or tan T.

Figure 8.22 shows the graphs of at least one cycle of all six trigonometric functions.

Domain: all reals
Range: $[-1, 1]$
Period: $2\pi$

$y = \sin x$

Domain: all reals
Range: $[-1, 1]$
Period: $2\pi$

$y = \cos x$

Domain: all $x \ne \dfrac{\pi}{2} + n\pi$
Range: $(-\infty, \infty)$
Period: $\pi$

$y = \tan x$

Domain: all $x \ne n\pi$
Range: $(-\infty, -1] \cup [1, \infty)$
Period: $2\pi$

$y = \csc x = \dfrac{1}{\sin x}$

Domain: all $x \ne \dfrac{\pi}{2} + n\pi$
Range: $(-\infty, -1] \cup [1, \infty)$
Period: $2\pi$

$y = \sec x = \dfrac{1}{\cos x}$

Domain: all $x \ne n\pi$
Range: $(-\infty, \infty)$
Period: $\pi$

$y = \cot x = \dfrac{1}{\tan x}$
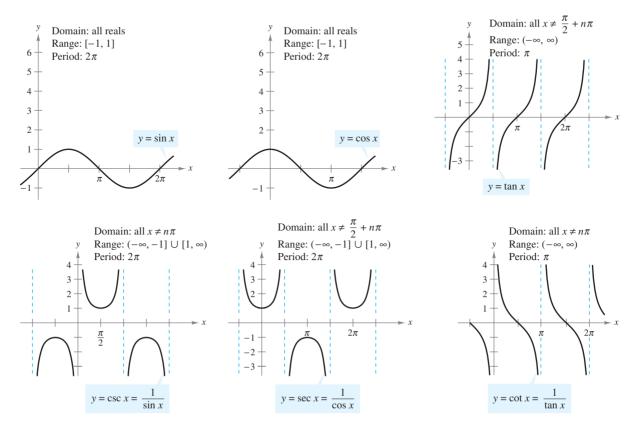
**FIGURE 8.22** Graphs of the Six Trigonometric Functions

Familiarity with the graphs of the six basic trigonometric functions allows you to sketch graphs of more general functions such as

$$y = a \sin bx$$

and
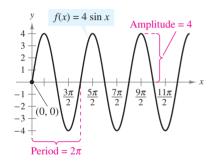
$$y = a \cos bx.$$

Note that the function $y = a \sin bx$ oscillates between $-a$ and $a$ and so has an amplitude of

$$|a|. \qquad \text{Amplitude of } y = a \sin bx$$

Furthermore, because $bx = 0$ when $x = 0$ and $bx = 2\pi$ when $x = 2\pi/b$, it follows that the function $y = a \sin bx$ has a period of

$$\frac{2\pi}{|b|}. \qquad \text{Period of } y = a \sin bx$$

**FIGURE 8.23**
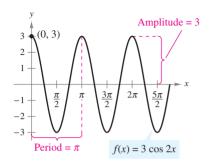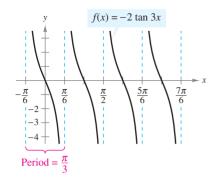
### Example 1    Graphing a Trigonometric Function

Sketch the graph of $f(x) = 4 \sin x$.

**SOLUTION**    The graph of $f(x) = 4 \sin x$ has the characteristics below.

Amplitude:  4

Period:  $2\pi$

Three cycles of the graph are shown in Figure 8.23, starting with the point $(0, 0)$.

### ✓ CHECKPOINT 1

Sketch the graph of $g(x) = 2 \cos x$.  ■



**FIGURE 8.24**

### Example 2    Graphing a Trigonometric Function

Sketch the graph of $f(x) = 3 \cos 2x$.

**SOLUTION**    The graph of $f(x) = 3 \cos 2x$ has the characteristics below.

Amplitude:  3

Period:  $\dfrac{2\pi}{2} = \pi$

Almost three cycles of the graph are shown in Figure 8.24, starting with the maximum point $(0, 3)$.

### ✓ CHECKPOINT 2

Sketch the graph of $g(x) = 2 \sin 4x$.  ■



**FIGURE 8.25**

### Example 3    Graphing a Trigonometric Function

Sketch the graph of $f(x) = -2 \tan 3x$.

**SOLUTION**    The graph of this function has a period of $\pi/3$. The vertical asymptotes of this tangent function occur at

$$x = \ldots, -\frac{\pi}{6}, \underbrace{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}}_{\text{Period} = \frac{\pi}{3}}, \ldots$$

Several cycles of the graph are shown in Figure 8.25, starting with the vertical asymptote $x = -\pi/6$.

### ✓ CHECKPOINT 3

Sketch the graph of $g(x) = \tan 4x$.  ■

## Limits of Trigonometric Functions

The sine and cosine functions are continuous over the entire real line. So, you can use direct substitution to evaluate a limit such as

$$\lim_{x \to 0} \sin x = \sin 0 = 0.$$

When direct substitution with a trigonometric limit yields an indeterminate form, such as $0/0$, you can rely on technology to help evaluate the limit. The next example examines the limit of a function that you will encounter again in Section 8.4.

### Example 4  Evaluating a Trigonometric Limit  (T)

Use a calculator to evaluate the function

$$f(x) = \frac{\sin x}{x}$$
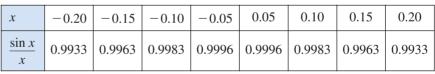
at several $x$-values near $x = 0$. Then use the result to estimate

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Use a graphing utility (set in *radian* mode) to confirm your result.

**SOLUTION**    The table shows several values of the function at $x$-values near zero. (Note that the function is undefined when $x = 0$.)



$$f(x) = \frac{\sin x}{x}$$

**FIGURE 8.26**
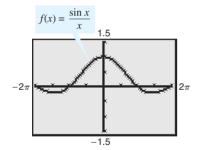
| $x$ | $-0.20$ | $-0.15$ | $-0.10$ | $-0.05$ | 0.05 | 0.10 | 0.15 | 0.20 |
|---|---|---|---|---|---|---|---|---|
| $\dfrac{\sin x}{x}$ | 0.9933 | 0.9963 | 0.9983 | 0.9996 | 0.9996 | 0.9983 | 0.9963 | 0.9933 |

From the table, it appears that the limit is 1. That is

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Figure 8.26 shows the graph of $f(x) = (\sin x)/x$. From this graph, it appears that $f(x)$ gets closer and closer to 1 as $x$ approaches zero (from either side).

### ✓CHECKPOINT 4

Use a calculator to evaluate the function

$$f(x) = \frac{1 - \cos x}{x}$$

at several $x$-values near $x = 0$. Then use the result to estimate

$$\lim_{x \to 0} \frac{1 - \cos x}{x}. \quad ■$$

### DISCOVERY

Try using the technique illustrated in Example 4 to evaluate

$$\lim_{x \to 0} \frac{\sin 5x}{5x}.$$

Can you hypothesize the limit of the general form

$$\lim_{x \to 0} \frac{\sin nx}{nx}$$

where $n$ is a positive integer?
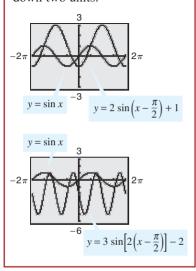
**Graphing Trigonometric Functions**

(T) A graphing utility allows you to explore the effects of the constants $a$, $b$, $c$, and $d$ on the graph of a function of the form

$$f(x) = a \sin[b(x - c)] + d.$$

After trying several values for these constants, you can see that $a$ determines the amplitude, $b$ determines the period, $c$ determines the horizontal shift, and $d$ determines the vertical shift. Two examples are shown below. In each case, the graph of $f$ is compared with the graph of $y = \sin x$. For instance, in the first graph, notice that relative to the graph of $y = \sin x$ the graph of $f$ is shifted $\pi/2$ units to the right, stretched vertically by a factor of 2, and shifted up one unit. Similarly, in the second graph, notice that relative to the graph of $y = \sin x$, the graph of $f$ is shifted $\pi/2$ units to the right, stretched horizontally by a factor of $\frac{1}{2}$, stretched vertically by a factor of 3, and shifted down two units.

$y = \sin x$

$y = 2 \sin\left(x - \frac{\pi}{2}\right) + 1$

$y = \sin x$

$y = 3 \sin\left[2\left(x - \frac{\pi}{2}\right)\right] - 2$

## Applications

There are many examples of periodic phenomena in both business and biology. Many businesses have cyclical sales patterns, and plant growth is affected by the day-night cycle. The next example describes the cyclical pattern followed by many types of predator-prey populations, such as coyotes and rabbits.

**Example 5**    **Modeling Predator-Prey Cycles**    (R)

The population $P$ of a predator at time $t$ (in months) is modeled by
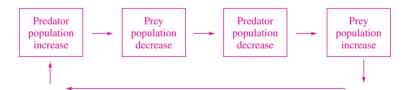
$$P = 10{,}000 + 3000 \sin \frac{2\pi t}{24}, \quad t \geq 0$$

and the population $p$ of its primary food source (its prey) is modeled by

$$p = 15{,}000 + 5000 \cos \frac{2\pi t}{24}, \quad t \geq 0.$$

Graph both models on the same set of axes and explain the oscillations in the size of each population.

**SOLUTION**    Each function has a period of 24 months. The predator population has an amplitude of 3000 and oscillates about the line $y = 10{,}000$. The prey population has an amplitude of 5000 and oscillates about the line $y = 15{,}000$. The graphs of the two models are shown in Figure 8.27. The cycles of this predator-prey population are explained by the diagram below.
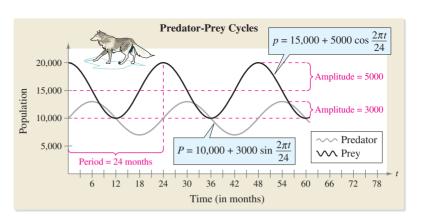


**FIGURE 8.27**

✓ **CHECKPOINT 5**

Write the keystrokes required to graph correctly the predator-prey cycle from Example 5 using a graphing utility. ■

Example 6  Modeling Biorhythms  ®

A popular theory that attempts to explain the ups and downs of everyday life states that each of us has three cycles, which begin at birth. These three cycles can be modeled by sine waves
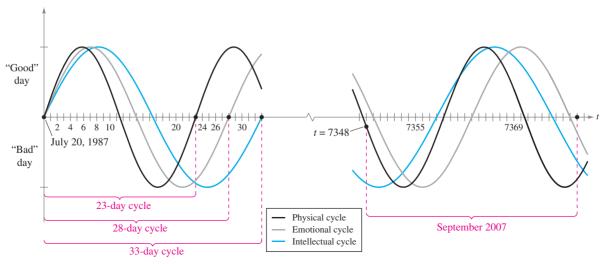
Physical (23 days):    $P = \sin \dfrac{2\pi t}{23}, \quad t \geq 0$

Emotional (28 days):   $E = \sin \dfrac{2\pi t}{28}, \quad t \geq 0$

Intellectual (33 days):  $I = \sin \dfrac{2\pi t}{33}, \quad t \geq 0$

where $t$ is the number of days since birth. Describe the biorhythms during the month of September 2007, for a person who was born on July 20, 1987.

SOLUTION   Figure 8.28 shows the person's biorhythms during the month of September 2007. Note that September 1, 2007 was the 7348th day of the person's life.



FIGURE 8.28

✓ CHECKPOINT 6

Use a graphing utility to describe the biorhythms of the person in Example 6 during the month of January 2007. Assume that January 1, 2007 is the 7105th day of the person's life. ■

CONCEPT CHECK

1. What is the amplitude of $f(x) = \sin x$?
2. What is the period of $f(x) = \cos x$?
3. What does the amplitude of a sine function or a cosine function represent?
4. What does the period of a sine function or a cosine function represent?

In Exercises 1 and 2, find the limit.

**1.** $\lim\limits_{x \to 2} (x^2 + 4x + 2)$

**2.** $\lim\limits_{x \to 3} (x^3 - 2x^2 + 1)$

In Exercises 3–10, evaluate the trigonometric function without using a calculator.

**3.** $\cos \dfrac{\pi}{2}$

**4.** $\sin \pi$

**5.** $\tan \dfrac{5\pi}{4}$

**6.** $\cot \dfrac{2\pi}{3}$

**7.** $\sin \dfrac{11\pi}{6}$

**8.** $\cos \dfrac{5\pi}{6}$

**9.** $\cos \dfrac{5\pi}{3}$

**10.** $\sin \dfrac{4\pi}{3}$
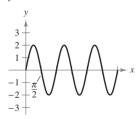
In Exercises 11–18, use a calculator to evaluate the trigonometric function to four decimal places.
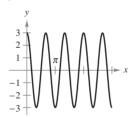
**11.** $\cos 15°$

**12.** $\sin 220°$

**13.** $\sin 275°$

**14.** $\cos 310°$

**15.** $\sin 103°$

**16.** $\cos 72°$

**17.** $\tan 327°$

**18.** $\tan 140°$

# Exercises 8.3

In Exercises 1–14, find the period and amplitude.
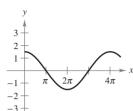
**1.** $y = 2 \sin 2x$

**2.** $y = 3 \cos 3x$





**3.** $y = \dfrac{3}{2} \cos \dfrac{x}{2}$

**4.** $y = -2 \sin \dfrac{x}{3}$





**5.** $y = \dfrac{1}{2} \cos \pi x$

**6.** $y = \dfrac{5}{2} \cos \dfrac{\pi x}{2}$





**7.** $y = -2 \sin x$

**8.** $y = -\cos \dfrac{2x}{3}$





**9.** $y = -2 \sin 10x$

**10.** $y = \dfrac{1}{3} \sin 8x$
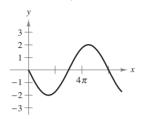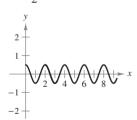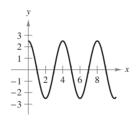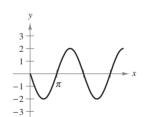
**11.** $y = \dfrac{1}{2} \sin \dfrac{2x}{3}$

**12.** $y = 5 \cos \dfrac{x}{4}$

**13.** $y = 3 \sin 4\pi x$

**14.** $y = \dfrac{2}{3} \cos \dfrac{\pi x}{10}$

In Exercises 15–20, find the period of the function.

**15.** $y = 3 \tan x$

**16.** $y = 7 \tan 2\pi x$
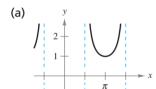
**17.** $y = 3 \sec 5x$
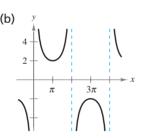
**18.** $y = \csc 4x$
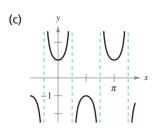
**19.** $y = \cot \dfrac{\pi x}{6}$
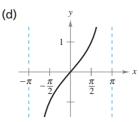
**20.** $y = 5 \tan \dfrac{2\pi x}{3}$
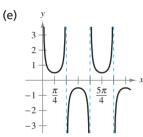
In Exercises 21–26, match the trigonometric function with the correct graph and give the period of the function. [The graphs are labeled (a)–(f).]

(a)



(b)



(c)



(d)



(e)



(f)
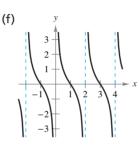


**21.** $y = \sec 2x$

**22.** $y = \frac{1}{2} \csc 2x$

**23.** $y = \cot \frac{\pi x}{2}$

**24.** $y = -\sec x$

**25.** $y = 2 \csc \frac{x}{2}$

**26.** $y = \tan \frac{x}{2}$

In Exercises 27–36, sketch the graph of the function by hand. Use a graphing utility to verify your sketch.

**27.** $y = \sin \frac{x}{2}$

**28.** $y = 4 \sin \frac{x}{3}$

**29.** $y = 2 \cos \frac{\pi x}{3}$

**30.** $y = \frac{3}{2} \cos \frac{2x}{3}$

**31.** $y = -2 \sin 6x$

**32.** $y = -3 \cos 4x$

**33.** $y = \cos 2\pi x$

**34.** $y = \frac{3}{2} \sin \frac{\pi x}{4}$

**35.** $y = 2 \tan x$

**36.** $y = 2 \cot x$

In Exercises 37–46, sketch the graph of the function.

**37.** $y = -\sin \frac{2\pi x}{3}$

**38.** $y = 10 \cos \frac{\pi x}{6}$

**39.** $y = \cot 2x$

**40.** $y = 3 \tan \pi x$

**41.** $y = \csc \frac{2x}{3}$

**42.** $y = \csc \frac{x}{4}$

**43.** $y = 2 \sec 2x$

**44.** $y = \sec \pi x$

**45.** $y = \csc 2\pi x$

**46.** $y = -\tan x$

(T)(S) In Exercises 47–56, complete the table (using a spreadsheet or a graphing utility set in *radian* mode) to estimate $\lim_{x \to 0} f(x)$.

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|-----|--------|---------|----------|---------|--------|-------|
| $f(x)$ | | | | | | |

**47.** $f(x) = \dfrac{\sin 4x}{2x}$

**48.** $f(x) = \dfrac{\sin 2x}{\sin 3x}$

**49.** $f(x) = \dfrac{\sin x}{5x}$

**50.** $f(x) = \dfrac{1 - \cos 2x}{x}$

**51.** $f(x) = \dfrac{3(1 - \cos x)}{x}$

**52.** $f(x) = \dfrac{2 \sin(x/4)}{x}$

**53.** $f(x) = \dfrac{\tan 2x}{x}$

**54.** $f(x) = \dfrac{\tan 4x}{3x}$

**55.** $f(x) = \dfrac{\sin^2 x}{x}$

**56.** $f(x) = \dfrac{1 - \cos^2 x}{2x}$

(T) In Exercises 57–60, use a graphing utility to graph the function $f$ and find $\lim_{x \to 0} f(x)$.

**57.** $f(x) = \dfrac{\sin x}{2x}$
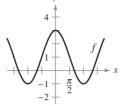
**58.** $f(x) = \dfrac{\sin 5x}{2x}$

**59.** $f(x) = \dfrac{\sin 5x}{\sin 2x}$
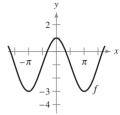
**60.** $f(x) = \dfrac{\tan 2x}{3x}$

**Graphical Reasoning**   In Exercises 61–64, find $a$ and $d$ for $f(x) = a \cos x + d$ such that the graph of $f$ matches the figure.
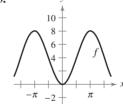
**61.**
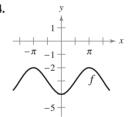


**62.**
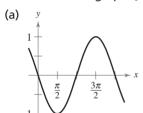


**63.**
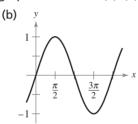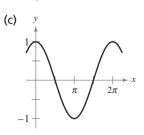


**64.**
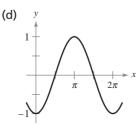
**Phase Shift** In Exercises 65–68, match the function with the correct graph. [The graphs are labeled (a)–(d).]

(a)



(b)



(c)



(d)



**65.** $y = \sin x$

**66.** $y = \sin\left(x - \dfrac{\pi}{2}\right)$

**67.** $y = \sin(x - \pi)$

**68.** $y = \sin\left(x - \dfrac{3\pi}{2}\right)$

**69. Health** For a person at rest, the velocity $v$ (in liters per second) of air flow into and out of the lungs during a respiratory cycle is given by

$$v = 0.9 \sin \frac{\pi t}{3}$$

where $t$ is the time in seconds. Inhalation occurs when $v > 0$, and exhalation occurs when $v < 0$.

(a) Find the time for one full respiratory cycle.

(b) Find the number of cycles per minute.

(T) (c) Use a graphing utility to graph the velocity function.

(T) **70. Health** After exercising for a few minutes, a person has a respiratory cycle for which the velocity of air flow is approximated by

$$y = 1.75 \sin \frac{\pi t}{2}.$$

Use this model to repeat Exercise 69.

**71. Music** When tuning a piano, a technician strikes a tuning fork for the A above middle C and sets up wave motion that can be approximated by $y = 0.001 \sin 880\pi t$, where $t$ is the time in seconds.

(a) What is the period $p$ of this function?

(b) What is the frequency $f$ of this note ($f = 1/p$)?

(T) (c) Use a graphing utility to graph this function.

**72. Health** The function $P = 100 - 20 \cos(5\pi t/3)$ approximates the blood pressure $P$ (in millimeters of mercury) at time $t$ in seconds for a person at rest.

(a) Find the period of the function.

(b) Find the number of heartbeats per minute.

(T) (c) Use a graphing utility to graph the pressure function.

**73. Biology: Predator-Prey Cycle** The population $P$ of a predator at time $t$ (in months) is modeled by

$$P = 8000 + 2500 \sin \frac{2\pi t}{24}$$

and the population $p$ of its prey is modeled by

$$p = 12{,}000 + 4000 \cos \frac{2\pi t}{24}.$$

(T) (a) Use a graphing utility to graph both models in the same viewing window.

(b) Explain the oscillations in the size of each population.

**74. Biology: Predator-Prey Cycle** The population $P$ of a predator at time $t$ (in months) is modeled by

$$P = 5700 + 1200 \sin \frac{2\pi t}{24}$$

and the population $p$ of its prey is modeled by

$$p = 9800 + 2750 \cos \frac{2\pi t}{24}.$$

(T) (a) Use a graphing utility to graph both models in the same viewing window.

(b) Explain the oscillations in the size of each population.

**Sales** In Exercises 75 and 76, sketch the graph of the sales function over 1 year where $S$ is sales in thousands of units and $t$ is the time in months, with $t = 1$ corresponding to January.

**75.** $S = 22.3 - 3.4 \cos \dfrac{\pi t}{6}$     **76.** $S = 74.50 + 43.75 \sin \dfrac{\pi t}{6}$

**77. Biorhythms** For the person born on July 20, 1987, use the biorhythm cycles given in Example 6 to calculate this person's three energy levels on December 31, 2011. Assume this is the 8930th day of the person's life.

(S) **78. Biorhythms** Use your birthday and the concept of biorhythms as given in Example 6 to calculate your three energy levels on December 31, 2011. Use a spreadsheet to calculate the number of days between your birthday and December 31, 2011.

**79. MAKE A DECISION: CONSTRUCTION WORKERS** The number $W$ (in thousands) of construction workers employed in the United States during 2006 can be modeled by

$$W = 7594 + 455.2 \sin(0.41t - 1.713)$$

where $t$ is the time in months, with $t = 1$ corresponding to January 1. *(Source: U.S. Bureau of Labor Statistics)*

(T) (a) Use a graphing utility to graph $W$.

(b) Did the number of construction workers exceed 8 million in 2006? If so, during which month(s)?

**80.** *MAKE A DECISION: SALES*   The snowmobile sales $S$ (in units) at a dealership are modeled by

$$S = 58.3 + 32.5 \cos \frac{\pi t}{6}$$

where $t$ is the time in months, with $t = 1$ corresponding to January 1.
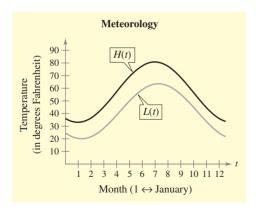
(T) (a) Use a graphing utility to graph $S$.

(b) Will the sales exceed 75 units during any month? If so, during which month(s)?

**81. Meteorology**   The normal monthly high temperatures for Erie, Pennsylvania are approximated by

$$H(t) = 56.94 - 20.86 \cos \frac{\pi t}{6} - 11.58 \sin \frac{\pi t}{6}$$

and the normal monthly low temperatures are approximated by

$$L(t) = 41.80 - 17.13 \cos \frac{\pi t}{6} - 13.39 \sin \frac{\pi t}{6}$$
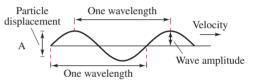
where $t$ is the time in months, with $t = 1$ corresponding to January. *(Source: NOAA)* Use the figure to answer the questions below.

**Meteorology**
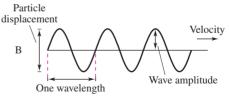


(a) During what part of the year is the difference between the normal high and low temperatures greatest? When is it smallest?

(b) The sun is the farthest north in the sky around June 21, but the graph shows the highest temperatures at a later date. Approximate the lag time of the temperatures relative to the position of the sun.

(B) **82. Finance: Cyclical Stocks**   The term "cyclical stock" describes the stock of a company whose profits are greatly influenced by changes in the economic business cycle. The market prices of cyclical stocks mirror the general state of the economy and reflect the various phases of the business cycle. Give a description and sketch the graph of a given corporation's stock prices during recurrent periods of prosperity and recession. *(Source: Adapted from Garman/Forgue, Personal Finance, Eighth Edition)*

(B) **83. Physics**   Use the graphs below to answer each question.
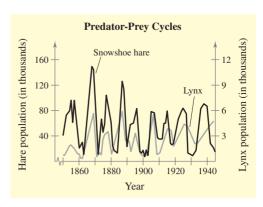


(a) Which graph (A or B) has a longer wavelength, or period?

(b) Which graph (A or B) has a greater amplitude?

(c) The frequency of a graph is the number of oscillations or cycles that occur during a given period of time. Which graph (A or B) has a greater frequency?

(d) Based on the definition of frequency, how are frequency and period related?

*(Source: Adapted from Shipman/Wilson/Todd, An Introduction to Physical Science, Eleventh Edition)*

(B) **84. Biology: Predator-Prey Cycles**   The graph below demonstrates snowshoe hare and lynx population fluctuations. The cycles of each population follow a periodic pattern. Describe several factors that could be contributing to the cyclical patterns. *(Source: Adapted from Levine/Miller, Biology: Discovering Life, Second Edition)*



**True or False?**   In Exercises 85–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**85.** The amplitude of $f(x) = -3 \cos 2x$ is $-3$.

**86.** The period of $f(x) = 5 \cot\left(-\frac{4x}{3}\right)$ is $\frac{3\pi}{2}$.

**87.** $\lim\limits_{x \to 0} \dfrac{\sin 5x}{3x} = \dfrac{5}{3}$

**88.** One solution of $\tan x = 1$ is $\dfrac{5\pi}{4}$.

## Mid-Chapter Quiz

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, express the angle in radian measure as a multiple of $\pi$. Use a calculator to verify your result.

**1.** $15°$  **2.** $105°$  **3.** $-80°$  **4.** $35°$

In Exercises 5–8, express the angle in degree measure. Use a calculator to verify your result.

**5.** $\dfrac{2\pi}{3}$  **6.** $\dfrac{4\pi}{15}$  **7.** $-\dfrac{4\pi}{3}$  **8.** $\dfrac{11\pi}{12}$

In Exercises 9–14, evaluate the trigonometric function *without* using a calculator.

**9.** $\sin\left(-\dfrac{\pi}{4}\right)$  **10.** $\cos 210°$  **11.** $\tan\dfrac{5\pi}{6}$

**12.** $\cot 45°$  **13.** $\sec(-60°)$  **14.** $\csc\dfrac{3\pi}{2}$

In Exercises 15–17, solve the equation for $\theta$ $(0 \le \theta \le 2\pi)$.
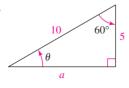
**15.** $\tan\theta - 1 = 0$
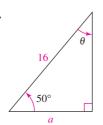
**16.** $\cos^2\theta - 2\cos\theta + 1 = 0$

**17.** $\sin^2\theta = 3\cos^2\theta$
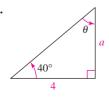
In Exercises 18–20, find the indicated side and/or angle.

**18.**


**19.**


**20.**

Figure for 21

**21.** A map maker needs to determine the distance $d$ across a small lake. The distance from point A to point B is 500 feet and the angle $\theta$ is 35° (see figure). What is $d$?
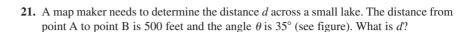
In Exercises 22–24, (a) sketch the graph and (b) determine the period of the function.

**22.** $y = 3\sin\pi x$  **23.** $y = -2\cos 4x$  **24.** $y = \tan\dfrac{\pi x}{3}$

**(T) 25.** A company that produces snowboards forecasts monthly sales for 1 year to be

$$S = 53.5 + 40.5\cos\dfrac{\pi t}{6}$$

where $S$ is the sales (in thousands of dollars) and $t$ is the time in months, with $t = 1$ corresponding to January 1.

(a) Use a graphing utility to graph $S$.

(b) Use the graph to determine the months of maximum and minimum sales.
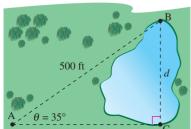
## Section 8.4

# Derivatives of Trigonometric Functions

■ Find derivatives of trigonometric functions.
■ Find the relative extrema of trigonometric functions.
■ Use derivatives of trigonometric functions to answer questions about real-life situations.

## Derivatives of Trigonometric Functions

In Example 4 and Checkpoint 4 in the preceding section, you looked at two important trigonometric limits:

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0.$$

These two limits are used in the development of the derivative of the sine function.

$$\frac{d}{dx}[\sin x] = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left( \cos x \frac{\sin \Delta x}{\Delta x} - \sin x \frac{1 - \cos \Delta x}{\Delta x} \right)$$

$$= \cos x \left( \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left( \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} \right)$$
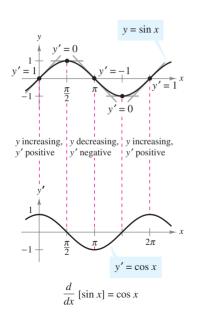
$$= (\cos x)(1) - (\sin x)(0)$$

$$= \cos x.$$

This differentiation rule is illustrated graphically in Figure 8.29. Note that the *slope* of the sine curve determines the *value* of the cosine curve. If $u$ is a function of $x$, the Chain Rule version of this differentiation rule is

$$\frac{d}{dx}[\sin u] = \cos u \frac{du}{dx}.$$

The Chain Rule versions of the differentiation rules for all six trigonometric functions are listed below.



**FIGURE 8.29**

$\frac{d}{dx}[\sin x] = \cos x$

---

**STUDY TIP**

To help you remember these differentiation rules, note that each trigonometric function that begins with a "c" has a negative sign in its derivative.

---

### Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin u] = \cos u \frac{du}{dx} \qquad \frac{d}{dx}[\cos u] = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \sec^2 u \frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = -\csc^2 u \frac{du}{dx}$$

$$\frac{d}{dx}[\sec u] = \sec u \tan u \frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = -\csc u \cot u \frac{du}{dx}$$

**Example 1** **Differentiating Trigonometric Functions**

Differentiate each function.

**a.** $y = \sin 2x$ **b.** $y = \cos(x - 1)$ **c.** $y = \tan 3x$

**SOLUTION**

**a.** Letting $u = 2x$, you obtain

$$\frac{dy}{dx} = \cos u \frac{du}{dx} = \cos 2x \frac{d}{dx}[2x] = (\cos 2x)(2) = 2\cos 2x.$$

**b.** Letting $u = x - 1$, you can see that $u' = 1$. So, the derivative is simply

$$\frac{dy}{dx} = -\sin(x - 1).$$

**c.** Letting $u = 3x$, you have $u' = 3$, which implies that

$$\frac{dy}{dx} = 3\sec^2 3x.$$

✓**CHECKPOINT 1**

Differentiate each function.

**a.** $y = \cos 4x$

**b.** $y = \sin(x^2 - 1)$

**c.** $y = \tan \dfrac{x}{2}$ ∎

**Example 2** **Differentiating a Trigonometric Function**

Differentiate $f(x) = \cos 3x^2$.

**SOLUTION** Letting $u = 3x^2$, you obtain

$$f'(x) = -\sin u \frac{du}{dx} \qquad\qquad \text{Apply Cosine Differentiation Rule.}$$

$$= -\sin 3x^2 \frac{d}{dx}[3x^2] \qquad \text{Substitute } 3x^2 \text{ for } u.$$

$$= -(\sin 3x^2)(6x)$$

$$= -6x \sin 3x^2. \qquad\qquad \text{Simplify.}$$

✓**CHECKPOINT 2**

Differentiate each function.

**a.** $g(x) = \sin \sqrt{x}$

**b.** $g(x) = 2\cos x^3$ ∎

**Example 3** **Differentiating a Trigonometric Function**

Differentiate $f(x) = \tan^4 3x$.

**SOLUTION** By the Power Rule, you can write

$$\frac{d}{dx}[(\tan 3x)^4] = 4(\tan 3x)^3 \frac{d}{dx}[\tan 3x]$$

$$= 4(\tan^3 3x)(3)(\sec^2 3x)$$

$$= 12\tan^3 3x \sec^2 3x.$$

**TECHNOLOGY**

ⓣ When you use a symbolic differentiation utility to differentiate trigonometric functions, you can easily obtain results that appear to be different from those you would obtain by hand. Try using a symbolic differentiation utility to differentiate the function in Example 3. How does your result compare with the given solution?

✓**CHECKPOINT 3**

Differentiate each function.

**a.** $y = \sin^3 x$

**b.** $y = \cos^4 2x$ ∎

*Example 4* **Differentiating a Trigonometric Function**

Differentiate $y = \csc \dfrac{x}{2}$.

**SOLUTION**

$$y = \csc \frac{x}{2}$$ Write original function.

$$\frac{dy}{dx} = -\csc \frac{x}{2} \cot \frac{x}{2} \frac{d}{dx}\left[\frac{x}{2}\right]$$ Apply Cosecant Differentiation Rule.

$$= -\frac{1}{2} \csc \frac{x}{2} \cot \frac{x}{2}$$ Simplify.

✓ **CHECKPOINT 4**

Differentiate each function.

**a.** $y = \sec 4x$    **b.** $y = \cot x^2$ ▪

*Example 5* **Differentiating a Trigonometric Function**

Differentiate $f(t) = \sqrt{\sin 4t}$.

**SOLUTION** Begin by rewriting the function in rational exponent form. Then apply the General Power Rule to find the derivative.

$$f(t) = (\sin 4t)^{1/2}$$ Rewrite with rational exponent.

$$f'(t) = \left(\frac{1}{2}\right)(\sin 4t)^{-1/2} \frac{d}{dt}[\sin 4t]$$ Apply General Power Rule.

$$= \left(\frac{1}{2}\right)(\sin 4t)^{-1/2}(4 \cos 4t)$$

$$= \frac{2 \cos 4t}{\sqrt{\sin 4t}}$$ Simplify.

✓ **CHECKPOINT 5**

Differentiate each function.

**a.** $f(x) = \sqrt{\cos 2x}$

**b.** $f(x) = \sqrt[3]{\tan 3x}$ ▪

*Example 6* **Differentiating a Trigonometric Function**

Differentiate $y = x \sin x$.

**SOLUTION** Using the Product Rule, you can write

$$y = x \sin x$$ Write original function.

$$\frac{dy}{dx} = x \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[x]$$ Apply Product Rule.

$$= x \cos x + \sin x.$$ Simplify.
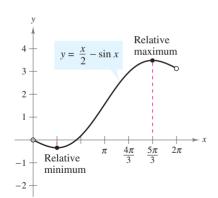
✓ **CHECKPOINT 6**

Differentiate each function.

**a.** $y = x^2 \cos x$    **b.** $y = t \sin 2t$ ▪

$y = \dfrac{x}{2} - \sin x$

Relative maximum

Relative minimum

**FIGURE 8.30**

✓ **CHECKPOINT 7**

Find the relative extrema of

$$y = \frac{x}{2} - \cos x$$

on the interval $(0, 2\pi)$. ■

## Relative Extrema of Trigonometric Functions

**Example 7**    **Finding Relative Extrema**

Find the relative extrema of

$$y = \frac{x}{2} - \sin x$$

on the interval $(0, 2\pi)$.

**SOLUTION**    To find the relative extrema of the function, begin by finding its critical numbers. The derivative of $y$ is

$$\frac{dy}{dx} = \frac{1}{2} - \cos x.$$

By setting the derivative equal to zero, you obtain $\cos x = \frac{1}{2}$. So, in the interval $(0, 2\pi)$, the critical numbers are $x = \pi/3$ and $x = 5\pi/3$. Using the First-Derivative Test, you can conclude that $\pi/3$ yields a relative minimum and $5\pi/3$ yields a relative maximum, as shown in Figure 8.30.

---

**STUDY TIP**

Recall that the critical numbers of a function $y = f(x)$ are the $x$-values for which $f'(x) = 0$ or $f'(x)$ is undefined.

---

**Example 8**    **Finding Relative Extrema**

Find the relative extrema of $f(x) = 2 \sin x - \cos 2x$ on the interval $(0, 2\pi)$.

**SOLUTION**

| | |
|---|---|
| $f(x) = 2 \sin x - \cos 2x$ | Write original function. |
| $f'(x) = 2 \cos x + 2 \sin 2x$ | Differentiate. |
| $0 = 2 \cos x + 2 \sin 2x$ | Set derivative equal to 0. |
| $0 = 2 \cos x + 4 \cos x \sin x$ | Identity: $\sin 2x = 2 \cos x \sin x$ |
| $0 = 2(\cos x)(1 + 2 \sin x)$ | Factor. |

From this, you can see that the critical numbers occur when $\cos x = 0$ and when $\sin x = -\frac{1}{2}$. So, in the interval $(0, 2\pi)$, the critical numbers are

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$

Using the First-Derivative Test, you can determine that $(\pi/2, 3)$ and $(3\pi/2, -1)$ are relative maxima, and $\left(7\pi/6, -\frac{3}{2}\right)$ and $\left(11\pi/6, -\frac{3}{2}\right)$ are relative minima, as shown in Figure 8.31.



$f(x) = 2 \sin x - \cos 2x$

$\left(\frac{\pi}{2}, 3\right)$ Relative maxima

$\left(\frac{3\pi}{2}, -1\right)$

$(0, -1)$

$(2\pi, -1)$

$\left(\frac{7\pi}{6}, -\frac{3}{2}\right)$ $\left(\frac{11\pi}{6}, -\frac{3}{2}\right)$

Relative minima

**FIGURE 8.31**

✓ **CHECKPOINT 8**

Find the relative extrema of $y = \frac{1}{2} \sin 2x + \cos x$ on the interval $(0, 2\pi)$. ■

## Applications

*Example 9*  **Modeling Seasonal Sales**  (R)

A fertilizer manufacturer finds that the sales of one of its fertilizer brands follows a seasonal pattern that can be modeled by

$$F = 100{,}000\left[1 + \sin\frac{2\pi(t - 60)}{365}\right], \quad t \geq 0$$

where $F$ is the amount sold (in pounds) and $t$ is the time (in days), with $t = 1$ corresponding to January 1. On which day of the year is the maximum amount of fertilizer sold?

**SOLUTION**   The derivative of the model is

$$\frac{dF}{dt} = 100{,}000\left(\frac{2\pi}{365}\right)\cos\frac{2\pi(t - 60)}{365}.$$

Setting this derivative equal to zero produces

$$\cos\frac{2\pi(t - 60)}{365} = 0.$$

Because cosine is zero at $\pi/2$ and $3\pi/2$, you can find the critical numbers as shown.

$$\frac{2\pi(t - 60)}{365} = \frac{\pi}{2} \qquad\qquad \frac{2\pi(t - 60)}{365} = \frac{3\pi}{2}$$

$$t - 60 = \frac{365}{4} \qquad\qquad t - 60 = \frac{3(365)}{4}$$
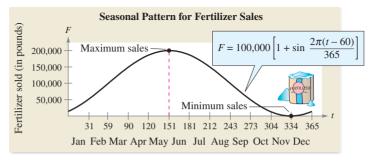
$$t = \frac{365}{4} + 60 \approx 151 \qquad\qquad t = \frac{3(365)}{4} + 60 \approx 334$$

The 151st day of the year is May 31 and the 334th day of the year is November 30. From the graph in Figure 8.32, you can see that, according to the model, the maximum sales occur on May 31.

**Seasonal Pattern for Fertilizer Sales**

Fertilizer sold (in pounds)

Maximum sales → $F = 100{,}000\left[1 + \sin\frac{2\pi(t-60)}{365}\right]$

FERTILIZER

Minimum sales →

31  59  90  120  151  181  212  243  273  304  334  365
Jan  Feb  Mar  Apr  May  Jun  Jul  Aug  Sep  Oct  Nov  Dec

**FIGURE 8.32**

✓**CHECKPOINT 9**
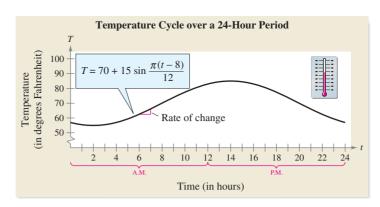
Using the model from Example 9, find the rate at which sales are changing when $t = 59$. ∎

*Example 10*
**MAKE A DECISION**    **Modeling Temperature Change**    Ⓡ

The temperature $T$ (in degrees Fahrenheit) during a given 24-hour period can be modeled by

$$T = 70 + 15 \sin \frac{\pi(t-8)}{12}, \quad t \geq 0$$

where $t$ is the time (in hours), with $t = 0$ corresponding to midnight, as shown in Figure 8.33. Find the rate at which the temperature is changing at 6 A.M.



**FIGURE 8.33**

**SOLUTION**
The rate of change of the temperature is given by the derivative

$$\frac{dT}{dt} = \frac{15\pi}{12} \cos \frac{\pi(t-8)}{12}.$$

Because 6 A.M. corresponds to $t = 6$, the rate of change at 6 A.M. is

$$\frac{15\pi}{12} \cos \left( \frac{-2\pi}{12} \right) = \frac{5\pi}{4} \cos \left( -\frac{\pi}{6} \right)$$

$$= \frac{5\pi}{4} \left( \frac{\sqrt{3}}{2} \right)$$

$$\approx 3.4° \text{ per hour.}$$

✓**CHECKPOINT 10**

In Example 10, find the rate at which the temperature is changing at 8 P.M. ■

**CONCEPT CHECK**

1. Given $\frac{d}{dx}[\sin u] = \cos u \frac{du}{dx}$, you know that the slope of the sine curve determines the value of what curve?

2. In the differentiation rules for all six trigonometric functions, identify each trigonometric function that has a negative sign in its derivative. What do these functions have in common?

3. Can the General Power Rule and the Product Rule be applied to the differentiation of trigonometric functions?

4. Identify the trigonometric function whose derivative is $-\sin u \frac{du}{dx}$.

**Skills Review 8.4**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.2, 2.4, 2.5, 3.2, and 8.2.

In Exercises 1–4, find the derivative of the function.

**1.** $f(x) = 3x^3 - 2x^2 + 4x - 7$

**2.** $g(x) = (x^3 + 4)^4$

**3.** $f(x) = (x - 1)(x^2 + 2x + 3)$

**4.** $g(x) = \dfrac{2x}{x^2 + 5}$

In Exercises 5 and 6, find the relative extrema of the function.

**5.** $f(x) = x^2 + 4x + 1$

**6.** $f(x) = \frac{1}{3}x^3 - 4x + 2$

In Exercises 7–10, solve the trigonometric equation for $x$ where $0 \le x \le 2\pi$.

**7.** $\sin x = \dfrac{\sqrt{3}}{2}$

**8.** $\cos x = -\dfrac{1}{2}$

**9.** $\cos \dfrac{x}{2} = 0$

**10.** $\sin \dfrac{x}{2} = -\dfrac{\sqrt{2}}{2}$

## Exercises 8.4

In Exercises 1–26, find the derivative of the function.

**1.** $y = \frac{1}{2} - 3\sin x$

**2.** $y = 5 + \sin x$

**3.** $y = x^2 - \cos x$

**4.** $g(t) = \pi \cos t - \dfrac{1}{t^2}$

**5.** $f(x) = 4\sqrt{x} + 3\cos x$

**6.** $f(x) = \sin x + \cos x$

**7.** $f(t) = t^2 \cos t$

**8.** $f(x) = (x + 1)\cos x$

**9.** $g(t) = \dfrac{\cos t}{t}$

**10.** $f(x) = \dfrac{\sin x}{x}$

**11.** $y = \tan x + x^2$

**12.** $y = x^3 - \cot x$

**13.** $y = e^{x^2}\sec x$

**14.** $y = e^{-x}\sin x$

**15.** $y = \cos 3x + \sin^2 x$

**16.** $y = \csc^2 x - \cos 2x$

**17.** $y = \sec \pi x$

**18.** $y = \frac{1}{2}\csc 2x$

**19.** $y = x\sin \dfrac{1}{x}$

**20.** $y = x^2 \cos \dfrac{1}{x}$

**21.** $y = 3\tan 4x$

**22.** $y = \tan e^x$

**23.** $y = 2\tan^2 4x$

**24.** $y = -\sin^4 2x$

**25.** $y = e^{2x}\sin 2x$

**26.** $y = e^{-x}\cos \dfrac{x}{2}$

In Exercises 27–38, find the derivative of the function and simplify your answer by using the trigonometric identities listed in Section 8.2.

**27.** $y = \cos^2 x$

**28.** $y = \frac{1}{4}\sin^2 2x$

**29.** $y = \cos^2 x - \sin^2 x$

**30.** $y = \dfrac{x}{2} + \dfrac{\sin 2x}{4}$

**31.** $y = \sin^2 x - \cos 2x$

**32.** $y = 3\sin x - 2\sin^3 x$

**33.** $y = \tan x - x$

**34.** $y = \cot x + x$

**35.** $y = \dfrac{\sin^3 x}{3} - \dfrac{\sin^5 x}{5}$

**36.** $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

**37.** $y = \ln(\sin^2 x)$

**38.** $y = \frac{1}{2}\ln(\cos^2 x)$

In Exercises 39–46, find an equation of the tangent line to the graph of the function at the given point.

| Function | Point |
|---|---|
| **39.** $y = \tan x$ | $\left(-\dfrac{\pi}{4}, -1\right)$ |
| **40.** $y = \sec x$ | $\left(\dfrac{\pi}{3}, 2\right)$ |
| **41.** $y = \sin 4x$ | $(\pi, 0)$ |
| **42.** $y = \csc^2 x$ | $\left(\dfrac{\pi}{2}, 1\right)$ |
| **43.** $y = \cot x$ | $\left(\dfrac{3\pi}{4}, -1\right)$ |
| **44.** $y = \sin x \cos x$ | $\left(\dfrac{3\pi}{2}, 0\right)$ |
| **45.** $y = \ln(\sin x + 2)$ | $\left(\dfrac{3\pi}{2}, 0\right)$ |
| **46.** $y = \sqrt{\sin x}$ | $\left(\dfrac{\pi}{6}, \dfrac{\sqrt{2}}{2}\right)$ |

In Exercises 47 and 48, use implicit differentiation to find $dy/dx$ and evaluate the derivative at the given point.

| Function | Point |
|---|---|
| **47.** $\sin x + \cos 2y = 1$ | $\left(\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$ |
| **48.** $\tan(x + y) = x$ | $(0, 0)$ |

In Exercises 49–52, show that the function satisfies the differential equation.

**49.** $y = 2 \sin x + 3 \cos x$

$y'' + y = 0$

**50.** $y = \dfrac{10 - \cos x}{x}$

$xy' + y = \sin x$

**51.** $y = \cos 2x + \sin 2x$
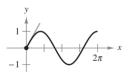
$y'' + 4y = 0$

**52.** $y = e^x \sin x$
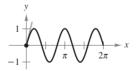
$y'' - 2y' + 2y = 0$

In Exercises 53–58, find the slope of the tangent line to the given sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$.
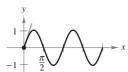
**53.** $y = \sin \dfrac{5x}{4}$

**54.** $y = \sin \dfrac{5x}{2}$

**55.** $y = \sin 2x$

**56.** $y = \sin \dfrac{3x}{2}$
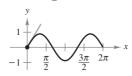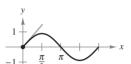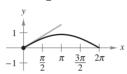
**57.** $y = \sin x$

**58.** $y = \sin \dfrac{x}{2}$

In Exercises 59–64, determine the relative extrema of the function on the interval $(0, 2\pi)$. Use a graphing utility to confirm your result.

**59.** $y = 2 \sin x + \sin 2x$

**60.** $y = 2 \cos x + \cos 2x$

**61.** $y = x - 2 \sin x$

**62.** $y = e^{-x} \sin x$

**63.** $y = e^x \cos x$

**64.** $y = \sec \dfrac{x}{2}$
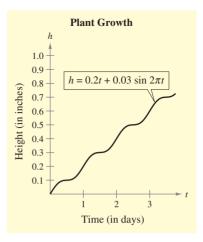
**65. Biology**  Plants do not grow at constant rates during a normal 24-hour period because their growth is affected by sunlight. Suppose that the growth of a certain plant species in a controlled environment is given by the model

$$h = 0.2t + 0.03 \sin 2\pi t$$

where $h$ is the height of the plant in inches and $t$ is the time in days, with $t = 0$ corresponding to midnight of day 1 (see figure). During what time of day is the rate of growth of this plant

(a) a maximum?        (b) a minimum?



**Plant Growth**

$$h = 0.2t + 0.03 \sin 2\pi t$$

Time (in days)

**66. Meteorology**  The normal average daily temperature in degrees Fahrenheit for a city is given by

$$T = 55 - 21 \cos \frac{2\pi(t - 32)}{365}$$

where $t$ is the time in days, with $t = 1$ corresponding to January 1. Find the expected date of

(a) the warmest day.        (b) the coldest day.

**67. Construction Workers**  The numbers $W$ (in thousands) of construction workers employed in the United States during 2006 can be modeled by

$$W = 7594 + 455.2 \sin(0.4t - 1.713)$$

where $t$ is the time in months, with $t = 1$ corresponding to January 1. Approximate the month $t$ in which the number of construction workers employed was a maximum. What was the maximum number of construction workers employed?  *(Source: U.S. Bureau of Labor Statistics)*

**68. Amusement Park Workers**  The numbers $W$ (in thousands) of amusement park workers employed in the United States during 2006 can be modeled by

$$W = 139.8 + 37.33 \sin(0.612t - 2.66)$$

where $t$ is the time in months, with $t = 1$ corresponding to January 1. Approximate the month $t$ in which the number of amusement park workers employed was a maximum. What was the maximum number of amusement park workers employed?  *(Source: U.S. Bureau of Labor Statistics)*

**69. Meteorology**   The number of hours of daylight $D$ in New Orleans can be modeled by

$$D = 12.13 - 1.87 \cos \frac{\pi(t - 0.07)}{6}, \quad 0 \le t \le 12$$

where $t$ represents the month, with $t = 0$ corresponding to January 1. Find the month $t$ in which New Orleans has the maximum number of daylight hours. What is this maximum number of daylight hours?   *(Source: U.S. Naval Observatory)*

**70. Tides**   Throughout the day, the depth of water $D$ in meters at the end of a dock varies with the tides. The depth for one particular day can be modeled by

$$D = 3.5 + 1.5 \cos \frac{\pi t}{6}, \quad 0 \le t \le 24$$

where $t = 0$ represents midnight.

(a) Determine $dD/dt$.

(b) Evaluate $dD/dt$ for $t = 4$ and $t = 20$ and interpret your results.

(c) Find the time(s) when the water depth is the greatest and the time(s) when the water depth is the least.

(d) What is the greatest depth? What is the least depth? Did you have to use calculus to determine these depths? Explain your reasoning.

(T) In Exercises 71–76, use a graphing utility (a) to graph $f$ and $f'$ on the same coordinate axes over the specified interval, (b) to find the critical numbers of $f$, and (c) to find the interval(s) on which $f'$ is positive and the interval(s) on which it is negative. Note the behavior of $f$ in relation to the sign of $f'$.

| Function | Interval |
|---|---|
| **71.** $f(t) = t^2 \sin t$ | $(0, 2\pi)$ |
| **72.** $f(x) = \dfrac{x}{2} + \cos \dfrac{x}{2}$ | $(0, 4\pi)$ |
| **73.** $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$ | $(0, \pi)$ |
| **74.** $f(x) = x \sin x$ | $(0, \pi)$ |
| **75.** $f(x) = \sqrt{2x} \sin x$ | $(0, 2\pi)$ |
| **76.** $f(x) = 4e^{-0.5x} \sin \pi x$ | $(0, 4)$ |

(T) In Exercises 77–82, use a graphing utility to find the relative extrema of the trigonometric function. Let $0 < x < 2\pi$.

**77.** $f(x) = \dfrac{x}{\sin x}$

**78.** $f(x) = \dfrac{x^2 - 2}{\sin x} - 5x$

**79.** $f(x) = \ln x \cos x$

**80.** $f(x) = \ln x \sin x$

**81.** $f(x) = \sin(0.1x^2)$

**82.** $f(x) = \sin \sqrt{x}$

**True or False?**   In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**83.** If $y = (1 - \cos x)^{1/2}$, then $y' = \frac{1}{2}(1 - \cos x)^{-1/2}$.

**84.** If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

**85.** If $y = x \sin^2 x$, then $y' = 2x \sin x$.

**86.** The minimum value of $y = 3 \sin x + 2$ is $-1$.

**87. Extended Application**   To work an extended application analyzing the mean monthly temperature and precipitation in Honolulu, Hawaii, visit this text's website at *college.hmco.com*.   *(Source: National Oceanic and Atmospheric Administration)*

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**Business Capsule**



Photo courtesy of Grandee Ann Ray/www.grandideas.net

After a successful career as a critical care nurse, Grandee Ann Ray started Grand Ideas, a corporate gift, specialty, and promotional products firm in Charleston, South Carolina. The company offers a wide variety of items, including office accessories, apparel, and glassware, that bear the logo of the client company. Ray started Grand Ideas from her home in 2001 with little more than a cell phone, a fax machine, and minimal inventory. Today, the company has sales approaching $1.5 million per year, and she has a team of 12 women working as full-time and part-time employees and independent contractors.

**88. Research Project**   Use your school's library, the Internet, or some other reference source to gather information on a company that offers unique products or services to its customers. Collect data about the revenue that the company has generated, and find a mathematical model of the data. Write a short paper that summarizes your findings.

# Integrals of Trigonometric Functions

- Find the six basic trigonometric integrals.
- Solve trigonometric integrals.
- Use trigonometric integrals to solve real-life problems.

## The Six Basic Trigonometric Integrals

For each trigonometric differentiation rule, there is a corresponding integration rule. For instance, corresponding to the differentiation rule

$$\frac{d}{dx}[\cos u] = -\sin u \frac{du}{dx}$$

is the integration rule

$$\int \sin u \, du = -\cos u + C.$$

The list below contains the integration formulas that correspond to the six basic trigonometric differentiation rules.

### Integrals Involving Trigonometric Functions

| *Differentiation Rule* | *Integration Rule* |
|---|---|
| $\dfrac{d}{dx}[\sin u] = \cos u \dfrac{du}{dx}$ | $\displaystyle\int \cos u \, du = \sin u + C$ |
| $\dfrac{d}{dx}[\cos u] = -\sin u \dfrac{du}{dx}$ | $\displaystyle\int \sin u \, du = -\cos u + C$ |
| $\dfrac{d}{dx}[\tan u] = \sec^2 u \dfrac{du}{dx}$ | $\displaystyle\int \sec^2 u \, du = \tan u + C$ |
| $\dfrac{d}{dx}[\sec u] = \sec u \tan u \dfrac{du}{dx}$ | $\displaystyle\int \sec u \tan u \, du = \sec u + C$ |
| $\dfrac{d}{dx}[\cot u] = -\csc^2 u \dfrac{du}{dx}$ | $\displaystyle\int \csc^2 u \, du = -\cot u + C$ |
| $\dfrac{d}{dx}[\csc u] = -\csc u \cot u \dfrac{du}{dx}$ | $\displaystyle\int \csc u \cot u \, du = -\csc u + C$ |

**STUDY TIP**

Note that this list gives you formulas for integrating only two of the six trigonometric functions: the sine function and the cosine function. The list does not show you how to integrate the other four trigonometric functions. Rules for integrating those functions are discussed later in this section.

*Example 1* **Integrating a Trigonometric Function**

Find $\int 2 \cos x \, dx$.

**SOLUTION** Let $u = x$. Then $du = dx$.

$$\int 2 \cos x \, dx = 2 \int \cos x \, dx \qquad \text{Apply Constant Multiple Rule.}$$

$$= 2 \int \cos u \, du \qquad \text{Substitute for } x \text{ and } dx.$$

$$= 2 \sin u + C \qquad \text{Integrate.}$$

$$= 2 \sin x + C \qquad \text{Substitute for } u.$$

✓**CHECKPOINT 1**

Find $\int 5 \sin x \, dx$. ■

*Example 2* **Integrating a Trigonometric Function**

Find $\int 3x^2 \sin x^3 \, dx$.

**SOLUTION** Let $u = x^3$. Then $du = 3x^2 \, dx$.

$$\int 3x^2 \sin x^3 \, dx = \int (\sin x^3) 3x^2 \, dx \qquad \text{Rewrite integrand.}$$

$$= \int \sin u \, du \qquad \text{Substitute for } x^3 \text{ and } 3x^2 \, dx.$$

$$= -\cos u + C \qquad \text{Integrate.}$$

$$= -\cos x^3 + C \qquad \text{Substitute for } u.$$

✓**CHECKPOINT 2**

Find $\int 4x^3 \cos x^4 \, dx$. ■

*Example 3* **Integrating a Trigonometric Function**

Find $\int \sec 3x \tan 3x \, dx$.

**SOLUTION** Let $u = 3x$. Then $du = 3 \, dx$.

$$\int \sec 3x \tan 3x \, dx = \frac{1}{3} \int (\sec 3x \tan 3x) 3 \, dx \qquad \text{Multiply and divide by 3.}$$

$$= \frac{1}{3} \int \sec u \tan u \, du \qquad \text{Substitute for } 3x \text{ and } 3 \, dx.$$

$$= \frac{1}{3} \sec u + C \qquad \text{Integrate.}$$

$$= \frac{1}{3} \sec 3x + C \qquad \text{Substitute for } u.$$

**TECHNOLOGY**

Ⓣ If you have access to a symbolic integration utility, try using it to integrate the functions in Examples 1, 2, and 3. Does your utility give the same results that are given in the examples?

✓**CHECKPOINT 3**

Find $\int \sec^2 5x \, dx$. ■

*Example 4*    **Integrating a Trigonometric Function**

Find $\int e^x \sec^2 e^x \, dx$.

**SOLUTION**    Let $u = e^x$. Then $du = e^x \, dx$.

$$\int e^x \sec^2 e^x \, dx = \int (\sec^2 e^x) e^x \, dx \qquad \text{Rewrite integrand.}$$

$$= \int \sec^2 u \, du \qquad \text{Substitute for } e^x \text{ and } e^x \, dx.$$

$$= \tan u + C \qquad \text{Integrate.}$$

$$= \tan e^x + C \qquad \text{Substitute for } u.$$

✓ **CHECKPOINT 4**

Find $\int 2 \csc 2x \cot 2x \, dx$.  ∎

The next two examples use the General Power Rule for integration and the General Log Rule for integration. Recall from Chapter 5 that these rules are

$$\int u^n \frac{du}{dx} \, dx = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \text{General Power Rule}$$

and

$$\int \frac{du/dx}{u} \, dx = \ln|u| + C. \qquad \text{General Log Rule}$$

The key to using these two rules is identifying the proper substitution for $u$. For instance, in the next example, the proper choice for $u$ is $\sin 4x$.

**STUDY TIP**

It is a good idea to check your answers to integration problems by differentiating. In Example 5, for instance, try differentiating the answer

$$y = \frac{1}{12} \sin^3 4x + C.$$

You should obtain the original integrand, as shown.

$$y' = \frac{1}{12} 3(\sin 4x)^2(\cos 4x)4$$

$$= \sin^2 4x \cos 4x$$

*Example 5*    **Using the General Power Rule**

Find $\int \sin^2 4x \cos 4x \, dx$.

**SOLUTION**    Let $u = \sin 4x$. Then $du/dx = 4 \cos 4x$.

$$\int \sin^2 4x \cos 4x \, dx = \frac{1}{4} \int \overbrace{(\sin 4x)^2}^{u^2} \overbrace{(4 \cos 4x)}^{du/dx} \, dx \qquad \text{Rewrite integrand.}$$

$$= \frac{1}{4} \int u^2 \, du \qquad \text{Substitute for } \sin 4x \text{ and } 4 \cos 4x \, dx.$$

$$= \frac{1}{4} \frac{u^3}{3} + C \qquad \text{Integrate.}$$

$$= \frac{1}{4} \frac{(\sin 4x)^3}{3} + C \qquad \text{Substitute for } u.$$

$$= \frac{1}{12} \sin^3 4x + C \qquad \text{Simplify.}$$

✓ **CHECKPOINT 5**

Find $\int \cos^3 2x \sin 2x \, dx$.  ∎

**Example 6**    **Using the Log Rule**

Find $\displaystyle\int \frac{\sin x}{\cos x}\, dx$.

**SOLUTION**    Let $u = \cos x$. Then $du/dx = -\sin x$.

$$\int \frac{\sin x}{\cos x}\, dx = -\int \frac{-\sin x}{\cos x}\, dx \qquad \text{Rewrite integrand.}$$

$$= -\int \frac{du/dx}{u}\, dx \qquad \text{Substitute for } \cos x \text{ and } -\sin x.$$

$$= -\ln|u| + C \qquad \text{Apply Log Rule.}$$

$$= -\ln|\cos x| + C \qquad \text{Substitute for } u.$$

✓**CHECKPOINT 6**

Find $\displaystyle\int \frac{\cos x}{\sin x}\, dx$.    ∎

**Example 7**    **Evaluating a Definite Integral**

Evaluate $\displaystyle\int_0^{\pi/4} \cos 2x\, dx$.

**SOLUTION**

$$\int_0^{\pi/4} \cos 2x\, dx = \left[\frac{1}{2}\sin 2x\right]_0^{\pi/4}$$

$$= \frac{1}{2} - 0 = \frac{1}{2}$$

✓**CHECKPOINT 7**

Find $\displaystyle\int_0^{\pi/2} \sin 2x\, dx$.    ∎

**Example 8**    **Finding Area by Integration**

Find the area of the region bounded by the $x$-axis and one arc of the graph of $y = \sin x$.

**SOLUTION**    As indicated in Figure 8.34, this area is given by

$$\text{Area} = \int_0^{\pi} \sin x\, dx$$

$$= \left[-\cos x\right]_0^{\pi}$$

$$= -(-1) - (-1)$$

$$= 2.$$

So, the region has an area of 2 square units.



**FIGURE 8.34**

✓**CHECKPOINT 8**

Find the area of the region bounded by the graphs of $y = \cos x$ and $y = 0$ for

$$0 \le x \le \frac{\pi}{2}. \quad ∎$$

## Other Trigonometric Integrals

At the beginning of this section, the integration rules for the sine and cosine functions were listed. Now, using the result of Example 6, you have an integration rule for the tangent function. That rule is

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln|\cos x| + C.$$

Integration formulas for the other three trigonometric functions can be developed in a similar way. For instance, to obtain the integration formula for the secant function, you can integrate as shown.

$$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \qquad \text{Use substitution with } u = \sec x + \tan x.$$

$$= \ln|\sec x + \tan x| + C$$

These formulas, and integration formulas for the other two trigonometric functions, are summarized below.

---

**Integrals of Trigonometric Functions**

$$\int \tan u \, du = -\ln|\cos u| + C \qquad \int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C \qquad \int \csc u \, du = \ln|\csc u - \cot u| + C$$

---

*Example 9*  **Integrating a Trigonometric Function**

Find $\int \tan 4x \, dx$.

**SOLUTION**  Let $u = 4x$. Then $du = 4 \, dx$.

$$\int \tan 4x \, dx = \frac{1}{4} \int (\tan 4x) 4 \, dx \qquad \text{Rewrite integrand.}$$

$$= \frac{1}{4} \int \tan u \, du \qquad \text{Substitute for } 4x \text{ and } 4 \, dx.$$

$$= -\frac{1}{4} \ln|\cos u| + C \qquad \text{Integrate.}$$

$$= -\frac{1}{4} \ln|\cos 4x| + C \qquad \text{Substitute for } u.$$

✓ **CHECKPOINT 9**

Find $\int \sec 2x \, dx$.

∎

## Application

In the next example, recall from Section 5.4 that the average value of a function $f$ over an interval $[a, b]$ is given by

$$\frac{1}{b-a} \int_a^b f(x)\, dx.$$

**Example 10**
**MAKE A DECISION** **Finding an Average Temperature** ®

The temperature $T$ (in degrees Fahrenheit) during a 24-hour period can be modeled by

$$T = 72 + 18 \sin \frac{\pi(t-8)}{12}$$

where $t$ is the time (in hours), with $t = 0$ corresponding to midnight. Will the average temperature during the four-hour period from noon to 4 P.M. be greater than 90°?

**SOLUTION** To find the average temperature $A$, use the formula for the average value of a function over an interval.

$$A = \frac{1}{4} \int_{12}^{16} \left[ 72 + 18 \sin \frac{\pi(t-8)}{12} \right] dt$$

$$= \frac{1}{4} \left[ 72t + 18\left(\frac{12}{\pi}\right)\left(-\cos \frac{\pi(t-8)}{12}\right) \right]_{12}^{16}$$

$$= \frac{1}{4} \left[ 72(16) + 18\left(\frac{12}{\pi}\right)\left(\frac{1}{2}\right) - 72(12) + 18\left(\frac{12}{\pi}\right)\left(\frac{1}{2}\right) \right]$$

$$= \frac{1}{4} \left( 288 + \frac{216}{\pi} \right)$$

$$= 72 + \frac{54}{\pi} \approx 89.2°$$

So, the average temperature is 89.2°, as indicated in Figure 8.35. No, the average temperature from noon to 4 P.M. will not be greater than 90°.

**Average Temperature**

$T = 72 + 18 \sin \dfrac{\pi(t-8)}{12}$

Average ≈ 89.2°

Temperature (in degrees Fahrenheit)

Time (in hours)
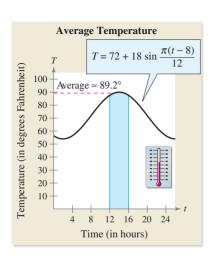
**FIGURE 8.35**

✓**CHECKPOINT 10**

Use the function in Example 10 to find the average temperature from 9 A.M. to noon. ■

---

**CONCEPT CHECK**

1. For each trigonometric differentiation rule, is there a corresponding integration rule?

2. For the differentiation rule $\frac{d}{dx}[\sin u] = \cos u \frac{du}{dx}$, what is the corresponding integration rule?

3. For the differentiation rule $\frac{d}{dx}[\cos u] = -\sin u \frac{du}{dx}$, what is the corresponding integration rule?

4. For the integration rule $\int \sec^2 u\, du = \tan u + C$, what is the corresponding differentiation rule?

| **Skills Review 8.5** | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 5.4 and 8.2. |

In Exercises 1–8, evaluate the trigonometric function.

**1.** $\cos \dfrac{5\pi}{4}$     **2.** $\sin \dfrac{7\pi}{6}$     **3.** $\sin\left(-\dfrac{\pi}{3}\right)$     **4.** $\cos\left(-\dfrac{\pi}{6}\right)$

**5.** $\tan \dfrac{5\pi}{6}$     **6.** $\cot \dfrac{5\pi}{3}$     **7.** $\sec \pi$     **8.** $\cos \dfrac{\pi}{2}$

In Exercises 9–16, simplify the expression using the trigonometric identities.

**9.** $\sin x \sec x$     **10.** $\csc x \cos x$

**11.** $\cos^2 x(\sec^2 x - 1)$     **12.** $\sin^2 x(\csc^2 x - 1)$

**13.** $\sec x \sin\left(\dfrac{\pi}{2} - x\right)$     **14.** $\cot x \cos\left(\dfrac{\pi}{2} - x\right)$

**15.** $\cot x \sec x$     **16.** $\cot x(\sin^2 x)$

In Exercises 17–20, evaluate the definite integral.

**17.** $\displaystyle\int_0^4 (x^2 + 3x - 4)\, dx$     **18.** $\displaystyle\int_{-1}^1 (1 - x^2)\, dx$

**19.** $\displaystyle\int_0^2 x(4 - x^2)\, dx$     **20.** $\displaystyle\int_0^1 x(9 - x^2)\, dx$

## Exercises 8.5

In Exercises 1–34, find the indefinite integral.

**1.** $\displaystyle\int (2\sin x + 3\cos x)\, dx$     **2.** $\displaystyle\int (t^2 - \sin t)\, dt$

**3.** $\displaystyle\int (1 - \csc t \cot t)\, dt$     **4.** $\displaystyle\int (\theta^2 + \sec^2 \theta)\, d\theta$

**5.** $\displaystyle\int (\csc^2 \theta - \cos \theta)\, d\theta$     **6.** $\displaystyle\int (\sec y \tan y - \sec^2 y)\, dy$

**7.** $\displaystyle\int \sin 2x\, dx$     **8.** $\displaystyle\int \cos 6x\, dx$

**9.** $\displaystyle\int 2x \cos x^2\, dx$     **10.** $\displaystyle\int 2x \sin x^2\, dx$

**11.** $\displaystyle\int \sec^2 \dfrac{x}{2}\, dx$     **12.** $\displaystyle\int \csc^2 4x\, dx$

**13.** $\displaystyle\int \tan 3x\, dx$     **14.** $\displaystyle\int \csc \dfrac{x}{3} \cot \dfrac{x}{3}\, dx$

**15.** $\displaystyle\int \tan^3 x \sec^2 x\, dx$     **16.** $\displaystyle\int \sqrt{\cot x}\, \csc^2 x\, dx$

**17.** $\displaystyle\int \cot \pi x\, dx$     **18.** $\displaystyle\int \tan 5x\, dx$

**19.** $\displaystyle\int \csc 2x\, dx$     **20.** $\displaystyle\int \sec \dfrac{x}{2}\, dx$

**21.** $\displaystyle\int \dfrac{\sec^2 x}{\tan x}\, dx$     **22.** $\displaystyle\int \dfrac{\sin x}{\cos^2 x}\, dx$

**23.** $\displaystyle\int \dfrac{\sec x \tan x}{\sec x - 1}\, dx$     **24.** $\displaystyle\int \dfrac{\cos t}{1 + \sin t}\, dt$

**25.** $\displaystyle\int \dfrac{\sin x}{1 + \cos x}\, dx$     **26.** $\displaystyle\int \dfrac{\sin \sqrt{x}}{\sqrt{x}}\, dx$

**27.** $\displaystyle\int \dfrac{\csc^2 x}{\cot^3 x}\, dx$     **28.** $\displaystyle\int \dfrac{1 - \cos \theta}{\theta - \sin \theta}\, d\theta$

**29.** $\displaystyle\int e^x \sin e^x\, dx$     **30.** $\displaystyle\int e^{-x} \tan e^{-x}\, dx$

**31.** $\displaystyle\int e^{\sin x} \cos x\, dx$     **32.** $\displaystyle\int e^{\sec x} \sec x \tan x\, dx$

**33.** $\displaystyle\int (\sin x + \cos x)^2\, dx$     **34.** $\displaystyle\int (1 + \tan \theta)^2\, d\theta$

In Exercises 35–38, use integration by parts to find the indefinite integral.

**35.** $\displaystyle\int x \cos x \, dx$

**36.** $\displaystyle\int x \sin x \, dx$

**37.** $\displaystyle\int x \sec^2 x \, dx$

**38.** $\displaystyle\int \theta \sec \theta \tan \theta \, d\theta$

In Exercises 39–46, evaluate the definite integral. Use a symbolic integration utility to verify your results.

**39.** $\displaystyle\int_0^{\pi/4} \cos \frac{4x}{3} \, dx$

**40.** $\displaystyle\int_0^{\pi/6} \sin 6x \, dx$

**41.** $\displaystyle\int_{\pi/2}^{2\pi/3} \sec^2 \frac{x}{2} \, dx$

**42.** $\displaystyle\int_0^{\pi/2} (x + \cos x) \, dx$

**43.** $\displaystyle\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x \, dx$

**44.** $\displaystyle\int_0^{\pi/8} \sin 2x \cos 2x \, dx$
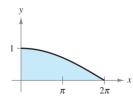
**45.** $\displaystyle\int_0^1 \tan(1 - x) \, dx$
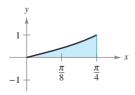
**46.** $\displaystyle\int_0^{\pi/4} \sec x \tan x \, dx$

In Exercises 47–52, determine the area of the region.

**47.** $y = \cos \dfrac{x}{4}$



**48.** $y = \tan x$
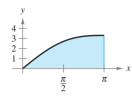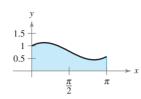


**49.** $y = x + \sin x$
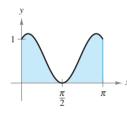


**50.** $y = \dfrac{x}{2} + \cos x$
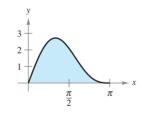


**51.** $y = \sin x + \cos 2x$



**52.** $y = 2 \sin x + \sin 2x$



**53. Consumer Trends** Energy consumption in the United States is seasonal. For instance, primary residential energy consumption can be approximated by the model

$$Q = 588 + 390 \cos(0.46t - 0.25), \quad 0 \le t \le 12$$

where $Q$ is the monthly consumption (in trillion Btu) and $t$ is the time in months, with $t = 1$ corresponding to January. Find the average consumption rate of domestic energy during a year. *(Source: Energy Information Administration)*

**54. Seasonal Sales** The monthly sales (in millions of units) of snow blowers can be modeled by

$$S = 15 + 6 \sin\frac{\pi(t - 8)}{6}, \quad 0 \le t \le 12$$

where $t$ is the time in months, with $t = 1$ corresponding to January. Find the average monthly sales

(a) during a year.

(b) from July through December.

**55. Inventory** The stockpile level of liquefied petroleum gases in the United States in 2006 can be approximated by the model

$$Q = 109 + 32 \cos \frac{\pi(t + 3)}{6}$$

where $Q$ is measured in millions of barrels and $t$ is the time in months, with $t = 1$ corresponding to January. Find the average levels given by this model during

(a) the first quarter $(0 \le t \le 3)$.

(b) the second quarter $(3 \le t \le 6)$.

(c) the entire year $(0 \le t \le 12)$.

*(Source: Energy Information Administration)*

(T) **56. Construction Workers** The number $W$ (in thousands) of construction workers employed in the United States during 2006 can be modeled by

$$W = 7594 + 455.2 \sin(0.41t - 1.713)$$

where $t$ is the time in months, with $t = 1$ corresponding to January. Use a graphing utility to estimate the average number of construction workers during

(a) the first quarter $(0 \le t \le 3)$.

(b) the second quarter $(3 \le t \le 6)$.

(c) the entire year $(0 \le t \le 12)$.

*(Source: U.S. Bureau of Labor Statistics)*

**57. Meteorology** The average monthly precipitation $P$ in inches, including rain, snow, and ice, for Sacramento, California can be modeled by

$$P = 2.47 \sin(0.40t + 1.80) + 2.08, \quad 0 \le t \le 12$$

where $t$ is the time in months, with $t = 1$ corresponding to January. Find the total annual precipitation for Sacramento. *(Source: National Oceanic and Atmospheric Administration)*

**58. Meteorology**    The average monthly precipitation $P$ in inches, including rain, snow, and ice, for Bismarck, North Dakota can be modeled by

$$P = 1.07 \sin(0.59t + 3.94) + 1.52, \quad 0 \le t \le 12$$

where $t$ is the time in months, with $t = 1$ corresponding to January. *(Source: National Oceanic and Atmospheric Administration)*
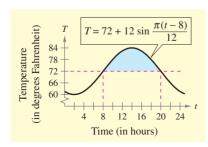
(a) Find the maximum and minimum precipitation and the month in which each occurs.

(b) Determine the average monthly precipitation for the year.

(c) Find the total annual precipitation for Bismarck.

Ⓣ **59. Cost**    Suppose that the temperature in degrees Fahrenheit is given by

$$T = 72 + 12 \sin \frac{\pi(t - 8)}{12}$$

where $t$ is the time in hours, with $t = 0$ corresponding to midnight. Furthermore, suppose that it costs $0.30 to cool a particular house 1° for 1 hour.
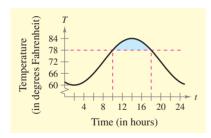
(a) Use the integration capabilities of a graphing utility to find the cost $C$ of cooling this house between 8 A.M. and 8 P.M., if the thermostat is set at 72° (see figure) and the cost is given by

$$C = 0.3 \int_8^{20} \left[ 72 + 12 \sin \frac{\pi(t - 8)}{12} - 72 \right] dt.$$



(b) Use the integration capabilities of a graphing utility to find the savings realized by resetting the thermostat to 78° (see figure) by evaluating the integral

$$C = 0.3 \int_{10}^{18} \left[ 72 + 12 \sin \frac{\pi(t - 8)}{12} - 78 \right] dt.$$



**60. Health**    For a person at rest, the velocity $v$ (in liters per second) of air flow into and out of the lungs during a respiratory cycle is approximated by

$$v = 0.9 \sin \frac{\pi t}{3}$$

where $t$ is the time in seconds. Find the volume in liters of air inhaled during one cycle by integrating this function over the interval $[0, 3]$.

**61. Health**    After exercising for a few minutes, a person has a respiratory cycle for which the velocity of air flow is approximated by

$$v = 1.75 \sin \frac{\pi t}{2}.$$

How much does the lung capacity of a person increase as a result of exercising? Use the results of Exercise 60 to determine how much more air is inhaled during a cycle after exercising than is inhaled during a cycle at rest. (Note that the cycle is shorter and you must integrate over the interval $[0, 2]$.)

**62. Sales**    In Example 9 in Section 8.4, the sales of a seasonal product were approximated by the model

$$F = 100{,}000 \left[ 1 + \sin \frac{2\pi(t - 60)}{365} \right], \quad t \ge 0$$

where $F$ was measured in pounds and $t$ was the time in days, with $t = 1$ corresponding to January 1. The manufacturer of this product wants to set up a manufacturing schedule to produce a uniform amount each day. What should this amount be? (Assume that there are 200 production days during the year.)

Ⓣ In Exercises 63–66, use a graphing utility and Simpson's Rule to approximate the integral.

| Integral | $n$ |
|---|---|
| **63.** $\displaystyle\int_0^{\pi/2} \sqrt{x} \sin x \, dx$ | 8 |
| **64.** $\displaystyle\int_0^{\pi/2} \cos \sqrt{x} \, dx$ | 8 |
| **65.** $\displaystyle\int_0^{\pi} \sqrt{1 + \cos^2 x} \, dx$ | 20 |
| **66.** $\displaystyle\int_0^2 (4 + x + \sin \pi x) \, dx$ | 20 |

**True or False?**    In Exercises 67 and 68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**67.** $\displaystyle\int_a^b \sin x \, dx = \int_a^{b + 2\pi} \sin x \, dx$

**68.** $4 \displaystyle\int \sin x \cos x \, dx = 0$

# Algebra Review

## Solving Trigonometric Equations

Solving a trigonometric equation requires the use of trigonometry, but it also requires the use of algebra. Some examples of solving trigonometric equations were presented on pages 585 and 586. Here are several others.

### Example 1    Solving a Trigonometric Equation

Solve each trigonometric equation.

**a.** $\sin x + \sqrt{2} = -\sin x$

**b.** $3 \tan^2 x = 1$

**c.** $\cot x \cos^2 x = 2 \cot x$

**SOLUTION**

**a.**

| | |
|---|---|
| $\sin x + \sqrt{2} = -\sin x$ | Write original equation. |
| $\sin x + \sin x = -\sqrt{2}$ | Add $\sin x$ to, and subtract $\sqrt{2}$ from, each side. |
| $2 \sin x = -\sqrt{2}$ | Combine like terms. |
| $\sin x = -\dfrac{\sqrt{2}}{2}$ | Divide each side by 2. |

$$x = \frac{5\pi}{4}, \frac{7\pi}{4}, \quad 0 \le x \le 2\pi$$

**b.**

| | |
|---|---|
| $3 \tan^2 x = 1$ | Write original equation. |
| $\tan^2 x = \dfrac{1}{3}$ | Divide each side by 3. |
| $\tan x = \pm\dfrac{\sqrt{3}}{3}$ | Extract square roots. |

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \quad 0 \le x \le 2\pi$$

**c.**

| | |
|---|---|
| $\cot x \cos^2 x = 2 \cot x$ | Write original equation. |
| $\cot x \cos^2 x - 2 \cot x = 0$ | Subtract $2 \cot x$ from each side. |
| $\cot x(\cos^2 x - 2) = 0$ | Factor. |

Setting each factor equal to zero, you obtain the solutions in the interval $0 \le x \le 2\pi$ as shown.

$$\cot x = 0 \quad \text{and} \quad \cos^2 x - 2 = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \cos^2 x = 2$$

$$\cos x = \pm\sqrt{2}$$

No solution is obtained from $\cos x = \pm\sqrt{2}$ because $\pm\sqrt{2}$ are outside the range of the cosine function.

*Example 2*  **Solving a Trigonometric Equation**

Solve each trigonometric equation.

**a.** $2 \sin^2 x - \sin x - 1 = 0$

**b.** $2 \sin^2 x + 3 \cos x - 3 = 0$

**c.** $2 \cos 3t - 1 = 0$

**SOLUTION**

**a.**     $2 \sin^2 x - \sin x - 1 = 0$              Write original equation.

$(2 \sin x + 1)(\sin x - 1) = 0$              Factor.

Setting each factor equal to zero, you obtain the solutions in the interval $[0, 2\pi]$ as shown.

$$2 \sin x + 1 = 0 \qquad \text{and} \quad \sin x - 1 = 0$$

$$\sin x = -\frac{1}{2} \qquad\qquad\qquad \sin x = 1$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6} \qquad\qquad\qquad x = \frac{\pi}{2}$$

**b.**     $2 \sin^2 x + 3 \cos x - 3 = 0$              Write original equation.

$2(1 - \cos^2 x) + 3 \cos x - 3 = 0$              Pythagorean Identity

$-2 \cos^2 x + 3 \cos x - 1 = 0$              Combine like terms.

$2 \cos^2 x - 3 \cos x + 1 = 0$              Multiply each side by $-1$.

$(2 \cos x - 1)(\cos x - 1) = 0$              Factor.

Setting each factor equal to zero, you obtain the solutions in the interval $[0, 2\pi]$ as shown.

$$2 \cos x - 1 = 0 \quad \text{and} \quad \cos x - 1 = 0$$

$$2 \cos x = 1 \qquad\qquad \cos x = 1$$

$$\cos x = \frac{1}{2} \qquad\qquad x = 0, 2\pi$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

**c.** $2 \cos 3t - 1 = 0$              Write original equation.

$2 \cos 3t = 1$              Add 1 to each side.

$\cos 3t = \frac{1}{2}$              Divide each side by 2.

$3t = \frac{\pi}{3}, \frac{5\pi}{3}, \quad 0 \le 3t \le 2\pi$

$t = \frac{\pi}{9}, \frac{5\pi}{9}, \quad 0 \le t \le \frac{2}{3}\pi$

In the interval $0 \le t \le 2\pi$, there are four other solutions.

# Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 623.
Answers to odd-numbered Review Exercises are given in the back of the text.*

| Section 8.1 | Review Exercises |
|---|---|
| ■ Find coterminal angles. | *1–8* |
| ■ Convert from degree to radian measure and from radian to degree measure. | *9–20* |
|    $\pi$ radians $= 180°$ | |
| ■ Use formulas relating to triangles. | *21–24* |
| ■ Use formulas relating to triangles to solve real-life problems. | *25, 26* |

## Section 8.2

| | Review Exercises |
|---|---|
| ■ Find the reference angles for given angles. | *27–34* |
| ■ Evaluate trigonometric functions exactly. | *35–46* |

*Right Triangle Definition:* $0 < \theta < \dfrac{\pi}{2}$.

$$\sin \theta = \frac{\text{opp.}}{\text{hyp.}} \qquad \cos \theta = \frac{\text{adj.}}{\text{hyp.}} \qquad \tan \theta = \frac{\text{opp.}}{\text{adj.}}$$

$$\csc \theta = \frac{\text{hyp.}}{\text{opp.}} \qquad \sec \theta = \frac{\text{hyp.}}{\text{adj.}} \qquad \cot \theta = \frac{\text{adj.}}{\text{opp.}}$$

*Circular Function Definition:* $\theta$ is any angle in standard position and $(x, y)$ is a point on the terminal ray of the angle.

$$\sin \theta = \frac{y}{r} \qquad \cos \theta = \frac{x}{r} \qquad \tan \theta = \frac{y}{x}$$

$$\csc \theta = \frac{r}{y} \qquad \sec \theta = \frac{r}{x} \qquad \cot \theta = \frac{x}{y}$$

| | Review Exercises |
|---|---|
| ■ Use a calculator to approximate values of trigonometric functions. | *47–54* |
| ■ Solve right triangles. | *55–58* |
| ■ Solve trigonometric equations. | *59–64* |
| ■ Use right triangles to solve real-life problems. | *65, 66* |

## Section 8.3

| | Review Exercises |
|---|---|
| ■ Sketch graphs of trigonometric functions. | *67–74* |
| ■ Use trigonometric functions to model real-life situations. | *75, 76* |

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\* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

## Section 8.4

■ Find derivatives of trigonometric functions.

$$\frac{d}{dx}[\sin u] = \cos u \, \frac{du}{dx} \qquad\qquad \frac{d}{dx}[\cos u] = -\sin u \, \frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \sec^2 u \, \frac{du}{dx} \qquad\qquad \frac{d}{dx}[\cot u] = -\csc^2 u \, \frac{du}{dx}$$

$$\frac{d}{dx}[\sec u] = \sec u \tan u \, \frac{du}{dx} \qquad\qquad \frac{d}{dx}[\csc u] = -\csc u \cot u \, \frac{du}{dx}$$

■ Find the equations of tangent lines to graphs of trigonometric functions.

■ Analyze the graphs of trigonometric functions.

■ Use relative extrema to solve real-life problems.

## Section 8.5

■ Solve trigonometric integrals.

$$\int \cos u \, du = \sin u + C \qquad\qquad \int \sin u \, du = -\cos u + C$$

$$\int \sec^2 u \, du = \tan u + C \qquad\qquad \int \sec u \tan u \, du = \sec u + C$$

$$\int \csc^2 u \, du = -\cot u + C \qquad\qquad \int \csc u \cot u \, du = -\csc u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C \qquad\qquad \int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C \qquad\qquad \int \csc u \, du = \ln|\csc u - \cot u| + C$$

■ Find the areas of regions in the plane.

■ Use trigonometric integrals to solve real-life problems.
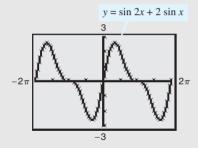
## Study Strategies

■ **Degree and Radian Modes**  When using a computer or calculator to evaluate or graph a trigonometric function, be sure that you use the proper mode—*radian* mode or *degree* mode.

■ **Checking the Form of an Answer**  Because of the abundance of trigonometric identities, solutions of problems in this chapter can take a variety of forms. For instance, the expressions $-\ln|\cot x| + C$ and $\ln|\tan x| + C$ are equivalent. So, when you are checking your solutions with those given in the back of the text, remember that your solution might be correct, even if its form doesn't agree precisely with that given in the text.

■ **Using Technology**  Throughout this chapter, remember that technology can help you *graph* trigonometric functions, *evaluate* trigonometric functions, *differentiate* trigonometric functions, and *integrate* trigonometric functions. Consider, for instance, the difficulty of sketching the graph of the function below *without* using a graphing utility.

$y = \sin 2x + 2 \sin x$

## Review Exercises

In Exercises 1–8, determine two coterminal angles (one positive and one negative) for the angle.

**1.** $\dfrac{7\pi}{4}$

**2.** $\dfrac{9\pi}{5}$

**3.** $\dfrac{3\pi}{2}$

**4.** $\dfrac{\pi}{2}$

**5.** $135°$

**6.** $210°$

**7.** $-405°$

**8.** $-315°$

In Exercises 9–16, convert the degree measure to radian measure. Use a calculator to verify your results.

**9.** $210°$

**10.** $300°$

**11.** $-60°$

**12.** $-30°$

**13.** $-480°$

**14.** $-540°$

**15.** $110°$

**16.** $320°$

In Exercises 17–20, convert the radian measure to degree measure. Use a calculator to verify your results.

**17.** $\dfrac{4\pi}{3}$
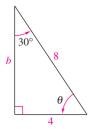
**18.** $\dfrac{5\pi}{6}$

**19.** $-\dfrac{2\pi}{3}$
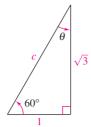
**20.** $-\dfrac{11\pi}{6}$

In Exercises 21–24, solve the triangle for the indicated side and/or angle.
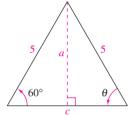
**21.**
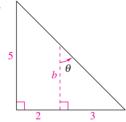


**22.**
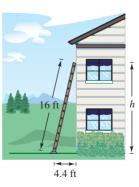


**23.**



**24.**



**25. Height**  A ladder of length 16 feet leans against the side of a house. The bottom of the ladder is 4.4 feet from the house (see figure). Find the height $h$ of the top of the ladder.
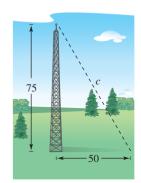


Figure for 25      Figure for 26

**26. Length**  To stabilize a 75-foot tower for a radio antenna, a guy wire must be attached from the top of the tower to an anchor 50 feet from the base. How long is the wire?

In Exercises 27–34, find the reference angle for the given angle.

**27.** $\dfrac{2\pi}{3}$

**28.** $\dfrac{9\pi}{4}$

**29.** $-\dfrac{5\pi}{6}$

**30.** $-\dfrac{5\pi}{3}$

**31.** $240°$

**32.** $300°$

**33.** $420°$

**34.** $480°$

In Exercises 35–46, evaluate the trigonometric function without using a calculator.

**35.** $\cos(-45°)$

**36.** $\sin 240°$

**37.** $\tan\dfrac{2\pi}{3}$

**38.** $\sec\dfrac{\pi}{4}$

**39.** $\sin\dfrac{5\pi}{3}$

**40.** $\cos\dfrac{5\pi}{2}$

**41.** $\cot\left(-\dfrac{5\pi}{6}\right)$

**42.** $\tan\left(-\dfrac{5\pi}{3}\right)$

**43.** $\sec(-180°)$

**44.** $\csc(-270°)$

**45.** $\cos\left(-\dfrac{4\pi}{3}\right)$

**46.** $\cot\left(-\dfrac{11\pi}{6}\right)$

In Exercises 47–54, use a calculator to evaluate the trigonometric function. Round to four decimal places.

**47.** $\tan 33°$
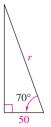
**48.** $\cot 216°$

**49.** $\sec \dfrac{12\pi}{5}$

**50.** $\csc \dfrac{2\pi}{9}$

**51.** $\sin\left(-\dfrac{\pi}{9}\right)$

**52.** $\cos\left(-\dfrac{3\pi}{7}\right)$

**53.** $\cos 105°$
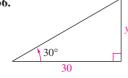
**54.** $\sin 224°$

In Exercises 55–58, solve for $x$, $y$, or $r$ as indicated.

**55.**
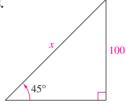
**56.**

**57.**
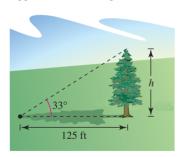
**58.**
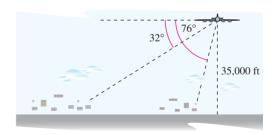
In Exercises 59–64, solve the trigonometric equation for $x$ $(0 \le x \le 2\pi)$.

**59.** $2 \cos x + 1 = 0$

**60.** $2 \cos^2 x = 1$

**61.** $2 \sin^2 x + 3 \sin x + 1 = 0$

**62.** $\cos^3 x = \cos x$

**63.** $\sec^2 x - \sec x - 2 = 0$

**64.** $2 \sec^2 x + \tan^2 x - 3 = 0$

**65. Height**    The length of a shadow of a tree is 125 feet when the angle of elevation of the sun is 33° (see figure). Approximate the height $h$ of the tree.

**66. Distance**    A passenger in an airplane flying at 35,000 feet sees two towns directly to the left of the airplane. The angles of depression to the towns are 32° and 76° (see figure). How far apart are the towns?

In Exercises 67–74, sketch a graph of the trigonometric function.

**67.** $y = 2 \cos 6x$

**68.** $y = \sin 2\pi x$

**69.** $y = \dfrac{1}{3} \tan x$

**70.** $y = \cot \dfrac{x}{2}$

**71.** $y = 3 \sin \dfrac{2x}{5}$

**72.** $y = 8 \cos\left(-\dfrac{x}{4}\right)$

**73.** $y = \sec 2\pi x$

**74.** $y = 3 \csc 2x$

**Ⓣ 75. Seasonal Sales**    A company's daily sales $S$ (in thousands of dollars) of jet skis can be modeled by

$$S = 74 + \dfrac{3}{365}t - 40 \cos \dfrac{2\pi t}{365}$$

where $t$ is the time in days, with $t = 1$ corresponding to January 1. Use a graphing utility to graph this model over a one-year period.

**Ⓣ 76. Seasonal Sales**    A company's daily sales $S$ (in hundreds of dollars) of bathing suits can be modeled by

$$S = 25 + \dfrac{2}{365}t + 20 \sin \dfrac{2\pi t}{365}$$

where $t$ is the time in days, with $t = 1$ corresponding to January 1. Use a graphing utility to graph this model over a one-year period.

In Exercises 77–88, find the derivative of the function.

**77.** $y = \sin 5\pi x$

**78.** $y = \tan(4x - \pi)$

**79.** $y = -x \tan x$

**80.** $y = \csc 3x + \cot 3x$

**81.** $y = \dfrac{\cos x}{x^2}$

**82.** $y = \dfrac{\cos(x - 1)}{x - 1}$

**83.** $y = \sin^2 x + x$

**84.** $y = x \cos x - \sin x$

**85.** $y = \csc^4 x$

**86.** $y = \sec^2 2x$

**87.** $y = e^x \cot x$

**88.** $y = e^{\sin x}$

In Exercises 89–94, find an equation of the tangent line to the graph of the function at the given point.

**89.** $y = \cos 2x,$ $\left(\dfrac{\pi}{4}, 0\right)$    **90.** $y = -x \cos x,$ $(0, 0)$

**91.** $y = \dfrac{1}{2} \sin^2 x,$ $\left(\dfrac{\pi}{2}, \dfrac{1}{2}\right)$    **92.** $y = \dfrac{x}{\cos x},$ $(0, 0)$

**93.** $y = x \tan 2x,$ $(0, 0)$    **94.** $y = \tan \pi e^x,$ $(0, 0)$
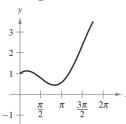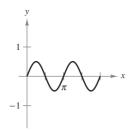
In Exercises 95–98, find the relative extrema of the function on the interval $(0, 2\pi)$.

**95.** $f(x) = \dfrac{x}{2} + \cos x$    **96.** $f(x) = \sin x \cos x$
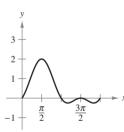


**97.** $f(x) = \sin^2 x + \sin x$    **98.** $f(x) = \dfrac{1}{2 + \sin x}$
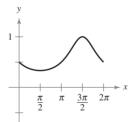


**(T)** **99. Seasonal Sales** Refer to the model given in Exercise 75.

(a) Use a graphing utility to find the maximum daily sales of jet skis. On what day of the year does the maximum daily revenue occur?

(b) Use a graphing utility to find the minimum daily sales of jet skis. On what day of the year does the minimum daily revenue occur?

**(T)** **100. Seasonal Sales** Refer to the model given in Exercise 76.

(a) Use a graphing utility to find the maximum daily sales of bathing suits. On what day of the year do the maximum daily sales occur?

(b) Use a graphing utility to find the minimum daily sales of bathing suits. On what day of the year do the minimum daily sales occur?

In Exercises 101–112, find or evaluate the integral.

**101.** $\displaystyle\int (3 \sin x - 2 \cos x)\, dx$    **102.** $\displaystyle\int \csc 5x \cot 5x\, dx$

**103.** $\displaystyle\int \sin^3 x \cos x\, dx$    **104.** $\displaystyle\int 2x \sec^2 x^2\, dx$

**105.** $\displaystyle\int_0^\pi (1 + \sin x)\, dx$    **106.** $\displaystyle\int_{-\pi}^\pi \dfrac{1}{2}(1 + \cos 2x)\, dx$

**107.** $\displaystyle\int_{-\pi/6}^{\pi/6} \sec^2 x\, dx$    **108.** $\displaystyle\int_{\pi/6}^{\pi/2} \csc^2 x\, dx$

**109.** $\displaystyle\int_{-\pi/3}^{\pi/3} 4 \sec x \tan x\, dx$    **110.** $\displaystyle\int_{\pi/6}^{\pi/3} \csc x \cot x\, dx$
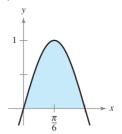
**111.** $\displaystyle\int_{-\pi/2}^{\pi/2} (2x + \cos x)\, dx$    **112.** $\displaystyle\int_0^\pi 2x \sin x^2\, dx$
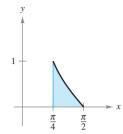
In Exercises 113–116, find the area of the region.

**113.** $y = \sin 3x$    **114.** $y = \cot x$
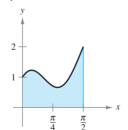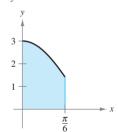


**115.** $y = 2 \sin x + \cos 3x$    **116.** $y = 2 \cos x + \cos 2x$



**117. Meteorology** The average monthly precipitation $P$ in inches, including rain, snow, and ice, for San Francisco, California can be modeled by

$$P = 2.91 \sin(0.4t + 1.81) + 2.38, \quad 0 \le t \le 12$$

where $t$ is the time in months, with $t = 1$ corresponding to January. Find the total annual precipitation for San Francisco. *(Source: National Oceanic and Atmospheric Administration)*

**118. Sales** The sales $S$ (in billions of dollars per year) for Safeway for the years 1996 through 2005 can be modeled by $S = 98.9 \sin(0.07t + 0.53) - 61.8$, $6 \le t \le 15$, where $t$ is the year, with $t = 6$ corresponding to 1996. *(Source: Safeway, Inc.)*

**(T)** (a) Use a graphing utility to graph the model.

(b) Find the rates at which the sales were changing in 2000, 2002, and 2005. Explain your results.

(c) Determine Safeway's total sales from 1996 through 2005.

## Chapter Test

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–6, copy and complete the table. Use a calculator if necessary.

| | Function | $\theta$ (deg) | $\theta$ (rad) | Function value |
|---|---|---|---|---|
| **1.** | sin | 67.5° | | |
| **2.** | cos | | $\dfrac{\pi}{5}$ | |
| **3.** | tan | 15° | | |
| **4.** | cot | | $-\dfrac{\pi}{6}$ | |
| **5.** | sec | $-40°$ | | |
| **6.** | csc | | $-\dfrac{5\pi}{4}$ | |

**7.** A digital camera tripod has a height of 25 inches and an angle of 24° is formed between the height and the leg of length $\ell$ (see figure). What is $\ell$?

In Exercises 8–10, solve the equation for $\theta$ $(0 \le \theta \le 2\pi)$.

**8.** $2 \sin \theta - \sqrt{2} = 0$    **9.** $\cos^2 \theta - \sin^2 \theta = 0$    **10.** $\csc \theta = \sqrt{3} \sec \theta$

In Exercises 11–13, sketch the graph of the function.

**11.** $y = 3 \sin 2x$    **12.** $y = 4 \cos 3\pi x$    **13.** $y = \cot \dfrac{\pi x}{5}$

In Exercises 14–16, (a) find the derivative of the function and (b) find the relative extrema of the function on the interval $(0, 2\pi)$.

**14.** $y = \cos x - \cos^2 x$    **15.** $y = \sec\left(x - \dfrac{\pi}{4}\right)$    **16.** $y = \dfrac{1}{3 - \sin(x + \pi)}$

In Exercises 17–22, find or evaluate the integral.

**17.** $\displaystyle\int \sin 5x \, dx$    **18.** $\displaystyle\int_{1/4}^{1/2} \cos \pi x \, dx$    **19.** $\displaystyle\int x \csc x^2 \, dx$

**20.** $\displaystyle\int_0^{\pi} \sec^2 \dfrac{x}{3} \tan \dfrac{x}{3} \, dx$    **21.** $\displaystyle\int \dfrac{\cos \sqrt{x}}{2\sqrt{x}} \, dx$    **22.** $\displaystyle\int_{\pi/4}^{\pi/2} \dfrac{e^{\cot x}}{\sin^2 x} \, dx$

**23.** The monthly sales (in thousands of dollars) of a company that produces insect repellent can be modeled by

$$S = 20.3 - 17.2 \cos \dfrac{\pi t}{6}$$

where $t$ is the time in months, with $t = 1$ corresponding to January.

(a) Find the total sales during the year $(0 \le t \le 12)$.

(b) Find the average monthly sales from April through October $(3 \le t \le 10)$.

Figure for 7

# Probability and Calculus

# 9



Calculus and probability theory can be used to determine the expected time for recovery after a certain medical procedure. (See Section 9.3, Exercise 40.)

**9.1** Discrete Probability
**9.2** Continuous Random Variables
**9.3** Expected Value and Variance

## Applications

Probability has many real-life applications. The applications listed below represent a sample of the applications in this chapter.

- Health, Exercise 39, page 637
- Waiting Time, Exercise 30, page 644
- Demand, Exercise 31, page 644
- Education, Exercise 55, page 654
- Fuel Mileage, Exercise 56, page 654

## Section 9.1

# Discrete Probability

■ Describe sample spaces for experiments.
■ Assign values to, and form frequency distributions for, discrete random variables.
■ Find the probabilities of events for discrete random variables.
■ Find the expected values or means of discrete random variables.
■ Find the variances and standard deviations of discrete random variables.

## Sample Spaces

When assigning measurements to the uncertainties of everyday life, people often use ambiguous terminology such as "fairly certain," "probable," and "highly unlikely." Probability theory allows you to remove this ambiguity by assigning a number to the likelihood of the occurrence of an event. This number is called the **probability** that the event will occur. For example, if you toss a fair coin, the probability that it will land heads up is one-half or 0.5.

In probability theory, any happening whose result is uncertain is called an **experiment.** The possible results of the experiment are **outcomes,** the set of all possible outcomes of the experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event.**

For instance, consider an experiment in which a coin is tossed. The sample space of this experiment consists of two outcomes: either the coin will land heads up (denoted by $H$) or it will land tails up (denoted by $T$). So, the sample space $S$ is

$$S = \{H, T\}. \qquad \text{Sample space}$$

In this text, all outcomes of a sample space are assumed to be equally likely. For instance, when a coin is tossed, $H$ and $T$ are assumed to be equally likely.

Thomas Wiewandt/Getty Images

When a weather forecaster states that there is a 50% chance of thunderstorms, it means that thunderstorms have occurred on half of all days that have had similar weather conditions.

*Example 1*  **Finding a Sample Space** Ⓡ

An experiment consists of tossing a six-sided die.

**a.** What is the sample space?

**b.** Describe the event corresponding to a number greater than 2 turning up.

**SOLUTION**

**a.** The sample space $S$ consists of six outcomes, which can be represented by the numbers 1 through 6. That is

$$S = \{1, 2, 3, 4, 5, 6\}. \qquad \text{Sample space}$$

Note that each of the outcomes in the sample space is equally likely.

**b.** The event $E$ corresponding to a number greater than 2 turning up is a subset of $S$. That is

$$E = \{3, 4, 5, 6\}. \qquad \text{Event}$$

✓ **CHECKPOINT 1**

An experiment consists of tossing two six-sided dice.

**a.** What is the sample space?

**b.** Describe the event corresponding to a sum greater than or equal to seven points when the dice are tossed. ■
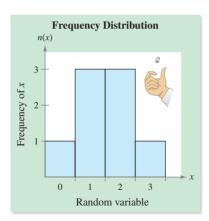
## Discrete Random Variables

A function that assigns a numerical value to each of the outcomes in a sample space is called a **random variable.** For instance, in the sample space $S = \{HH, HT, TH, TT\}$, the outcomes could be assigned the numbers 2, 1, and 0, depending on the number of heads in the outcome.

---

**Definition of Discrete Random Variable**

Let $S$ be a sample space. A **random variable** is a function $x$ that assigns a numerical value to each outcome in $S$. If the set of values taken on by the random variable is finite, then the random variable is **discrete**. The number of times a specific value of $x$ occurs is the **frequency** of $x$ and is denoted by $n(x)$.

---

**Example 2** Finding Frequencies ®

Three coins are tossed. A random variable assigns the number 0, 1, 2, or 3 to each possible outcome, depending on the number of heads that turn up.

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$\qquad\quad 3 \quad\;\; 2 \quad\;\; 2 \quad\;\; 1 \quad\;\; 2 \quad\;\; 1 \quad\;\; 1 \quad\;\; 0$$

Find the frequencies of 0, 1, 2, and 3. Then use a bar graph to represent the result.

**SOLUTION** To find the frequencies, simply count the number of occurrences of each value of the random variable, as shown in the table.


**Frequency Distribution**
$n(x)$
Frequency of $x$
Random variable

**FIGURE 9.1**

| Random variable, $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Frequency of $x$, $n(x)$ | 1 | 3 | 3 | 1 |

This table is called a **frequency distribution** of the random variable. The result is shown graphically by the bar graph in Figure 9.1.

✓**CHECKPOINT 2**

Use a graphing utility to create a bar graph similar to the one shown in Figure 9.1, representing the frequency for tossing two six-sided dice. Let the random variable be the sum of the points when the dice are tossed. ■

---

**STUDY TIP**

In Example 2, note that the sample space consists of *eight* outcomes, each of which is *equally likely*. The sample space does *not* consist of the outcomes "zero heads," "one head," "two heads," and "three heads." You cannot consider these events to be outcomes because they are not equally likely.

## Discrete Probability

The probability of a random variable $x$ is

$$P(x) = \frac{\text{Frequency of } x}{\text{Number of outcomes in } S}$$

$$= \frac{n(x)}{n(S)}$$    Probability

where $n(S)$ is the number of equally likely outcomes in the sample space. By this definition, it follows that the probability of an event must be a number between 0 and 1. That is, $0 \le P(x) \le 1$.

The collection of probabilities corresponding to the values of the random variable is called the **probability distribution** of the random variable. If the range of a discrete random variable consists of $m$ different values $\{x_1, x_2, x_3, \ldots, x_m\}$, then the sum of the probabilities of $x_i$ is 1. This can be written as

$$P(x_1) + P(x_2) + P(x_3) + \cdots + P(x_m) = 1.$$

### Example 3    Finding a Probability Distribution    ®

Five coins are tossed. Graph the probability distribution for the random variable giving the number of heads that turn up.

**SOLUTION**

| $x$ | Event | $n(x)$ |
|---|---|---|
| 0 | *TTTTT* | 1 |
| 1 | *HTTTT, THTTT, TTHTT, TTTHT, TTTTH* | 5 |
| 2 | *HHTTT, HTHTT, HTTHT, HTTTH, THHTT* | 10 |
|   | *THTHT, THTTH, TTHHT, TTHTH, TTTHH* | |
| 3 | *HHHTT, HHTHT, HHTTH, HTHHT, HTHTH* | 10 |
|   | *HTTHH, THHHT, THHTH, THTHH, TTHHH* | |
| 4 | *HHHHT, HHHTH, HHTHH, HTHHH, THHHH* | 5 |
| 5 | *HHHHH* | 1 |

The number of outcomes in the sample space is $n(S) = 32$. The probability of each value of the random variable is shown in the table.

| Random variable, $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Probability, $P(x)$ | $\frac{1}{32}$ | $\frac{5}{32}$ | $\frac{10}{32}$ | $\frac{10}{32}$ | $\frac{5}{32}$ | $\frac{1}{32}$ |

A graph of this probability distribution is shown in Figure 9.2. Note that values of the random variable are represented by intervals on the $x$-axis. Observe that the sum of the probabilities is 1.

✓**CHECKPOINT 3**

Two six-sided dice are tossed. Graph the probability distribution for the random variable giving the sum of the points when the dice are tossed. ∎
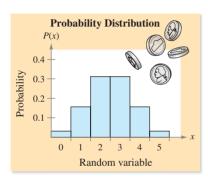


**Probability Distribution**

**FIGURE 9.2**

## Expected Value

Suppose you repeated the coin-tossing experiment in Example 3 several times. On the average, how many heads would you expect to turn up? From Figure 9.2, it seems reasonable that the average number of heads would be $2\frac{1}{2}$. This "average" is the **expected value** of the random variable.

---

**Definition of Expected Value**

If the range of a discrete random variable consists of $m$ different values $\{x_1, x_2, x_3, \ldots, x_m\}$, then the **expected value** of the random variable is

$$E(x) = x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3) + \cdots + x_m P(x_m).$$

The expected value is also called the **mean** of the random variable.

---

✓**CHECKPOINT 4**

Two six-sided dice are tossed. Find the expected value for a sum of seven points. ■



**FIGURE 9.3**

*Example 4*  **Finding an Expected Value** Ⓡ

Five coins are tossed. Find the expected value of the number of heads that will turn up.

**SOLUTION**  Using the results of Example 3, you obtain the expected value as shown.

| 0 Heads | 1 Head | 2 Heads | 3 Heads | 4 Heads | 5 Heads |
|---|---|---|---|---|---|

$$E(x) = (0)\left(\tfrac{1}{32}\right) + (1)\left(\tfrac{5}{32}\right) + (2)\left(\tfrac{10}{32}\right) + (3)\left(\tfrac{10}{32}\right) + (4)\left(\tfrac{5}{32}\right) + (5)\left(\tfrac{1}{32}\right)$$

$$= \tfrac{80}{32} = 2.5$$

*Example 5*  **Finding an Expected Value** Ⓡ

Over a period of 1 year (225 selling days), a sales representative sold from zero to eight units per day, as shown in Figure 9.3. From these data, what is the average number of units per day the sales representative should expect to sell?

**SOLUTION**  One way to answer this question is to calculate the expected value of the number of units.

$$E(x) = (0)\left(\tfrac{33}{225}\right) + (1)\left(\tfrac{45}{225}\right) + (2)\left(\tfrac{52}{225}\right) + (3)\left(\tfrac{46}{225}\right) + (4)\left(\tfrac{24}{225}\right) +$$
$$\quad (5)\left(\tfrac{11}{225}\right) + (6)\left(\tfrac{8}{225}\right) + (7)\left(\tfrac{5}{225}\right) + (8)\left(\tfrac{1}{225}\right)$$

$$= \tfrac{529}{225} \approx 2.35 \text{ units per day}$$

✓**CHECKPOINT 5**

Over a period of 1 year, a salesperson worked 6 days a week (312 selling days) and sold from zero to six units per day. Using the data in the table shown below, what is the average number of units per day the sales representative should expect to sell?

| Number of units per day | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| Number of days | 39 | 60 | 75 | 62 | 48 | 18 | 10 |

## Variance and Standard Deviation

The expected value or mean gives a measure of the average value assigned by a random variable. But the mean does not tell the whole story. For instance, all three of the distributions shown below have a mean of 2.

Distribution 1

| Random variable, $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Frequency of $x$, $n(x)$ | 2 | 2 | 2 | 2 | 2 |

Distribution 2

| Random variable, $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Frequency of $x$, $n(x)$ | 0 | 3 | 4 | 3 | 0 |

Distribution 3

| Random variable, $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Frequency of $x$, $n(x)$ | 5 | 0 | 0 | 0 | 5 |

Even though each distribution has the same mean, the patterns of the distributions are quite different. In the first distribution, each value has the same frequency. In the second, the values are clustered about the mean. In the third distribution, the values are far from the mean. To measure how much the distribution varies from the mean, you can use the concepts of variance and standard deviation.

---

### Definitions of Variance and Standard Deviation

Consider a random variable whose range is $\{x_1, x_2, x_3, \ldots, x_m\}$ with a mean of $\mu$. The **variance** of the random variable is

$$V(x) = (x_1 - \mu)^2 P(x_1) + (x_2 - \mu)^2 P(x_2) + \cdots + (x_m - \mu)^2 P(x_m).$$

The **standard deviation** of the random variable is
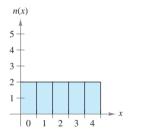
$$\sigma = \sqrt{V(x)}$$

($\sigma$ is the lowercase Greek letter sigma).

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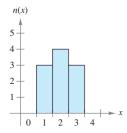
### *DISCOVERY*

The average grade on the calculus final in a class of 20 students was 80 out of 100 possible points. Describe a distribution of grades for which 10 students scored above 95 points. Describe another distribution of grades for which only one student scored above 85. In general, how does the standard deviation influence the grade distribution in a course?

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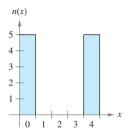
When the standard deviation is small, most of the values of the random variable are clustered near the mean. As the standard deviation becomes larger, the distribution becomes more and more spread out. For instance, in the three distributions above, you would expect the second to have the smallest standard deviation and the third to have the largest. This is confirmed in Example 6.

(a) Mean = 2; standard deviation ≈ 1.41



(b) Mean = 2; standard deviation ≈ 0.77



(c) Mean = 2; standard deviation = 2

**FIGURE 9.4**

*Example 6*   **Finding Variance and Standard Deviation**

Find the variance and standard deviation of each of the three distributions shown on page 632.

**SOLUTION**

**a.** For distribution 1, the mean is $\mu = 2$, the variance is

$$V(x) = (0 - 2)^2\left(\tfrac{2}{10}\right) + (1 - 2)^2\left(\tfrac{2}{10}\right) + (2 - 2)^2\left(\tfrac{2}{10}\right) +$$
$$(3 - 2)^2\left(\tfrac{2}{10}\right) + (4 - 2)^2\left(\tfrac{2}{10}\right)$$
$$= 2 \qquad\qquad\qquad \text{Variance}$$

and the standard deviation is $\sigma = \sqrt{2} \approx 1.41$.

**b.** For distribution 2, the mean is $\mu = 2$, the variance is

$$V(x) = (0 - 2)^2\left(\tfrac{0}{10}\right) + (1 - 2)^2\left(\tfrac{3}{10}\right) + (2 - 2)^2\left(\tfrac{4}{10}\right) +$$
$$(3 - 2)^2\left(\tfrac{3}{10}\right) + (4 - 2)^2\left(\tfrac{0}{10}\right)$$
$$= 0.6 \qquad\qquad\qquad \text{Variance}$$

and the standard deviation is $\sigma = \sqrt{0.6} \approx 0.77$.

**c.** For distribution 3, the mean is $\mu = 2$, the variance is

$$V(x) = (0 - 2)^2\left(\tfrac{5}{10}\right) + (1 - 2)^2\left(\tfrac{0}{10}\right) + (2 - 2)^2\left(\tfrac{0}{10}\right) +$$
$$(3 - 2)^2\left(\tfrac{0}{10}\right) + (4 - 2)^2\left(\tfrac{5}{10}\right)$$
$$= 4 \qquad\qquad\qquad \text{Variance}$$

and the standard deviation is $\sigma = \sqrt{4} = 2$.

As you can see in Figure 9.4, the second distribution has the smallest standard deviation and the third distribution has the largest.

✓**CHECKPOINT 6**

Find the variance and standard deviation of the distribution shown in the table. Then graph the distribution.

| Random variable, $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Frequency of $x$, $n(x)$ | 1 | 2 | 4 | 2 | 1 |

**CONCEPT CHECK**

1. What is an experiment?
2. What is a sample space?
3. Complete the following: The expected value of a random variable is also called the _____ of the random variable.
4. What is a probability distribution?

**Skills Review 9.1**    The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3 and 0.5.

In Exercises 1 and 2, solve for *x*.

**1.** $\dfrac{1}{9} + \dfrac{2}{3} + \dfrac{2}{9} = x$

**2.** $\dfrac{1}{3} + \dfrac{5}{12} + \dfrac{1}{8} + \dfrac{1}{12} + \dfrac{x}{24} = 1$

In Exercises 3–6, evaluate the expression.

**3.** $0\left(\dfrac{1}{16}\right) + 1\left(\dfrac{3}{16}\right) + 2\left(\dfrac{8}{16}\right) + 3\left(\dfrac{3}{16}\right) + 4\left(\dfrac{1}{16}\right)$

**4.** $0\left(\dfrac{1}{12}\right) + 1\left(\dfrac{2}{12}\right) + 2\left(\dfrac{6}{12}\right) + 3\left(\dfrac{2}{12}\right) + 4\left(\dfrac{1}{12}\right)$

**5.** $(0-1)^2\left(\dfrac{1}{4}\right) + (1-1)^2\left(\dfrac{1}{2}\right) + (2-1)^2\left(\dfrac{1}{4}\right)$

**6.** $(0-2)^2\left(\dfrac{1}{12}\right) + (1-2)^2\left(\dfrac{2}{12}\right) + (2-2)^2\left(\dfrac{6}{12}\right) + (3-2)^2\left(\dfrac{2}{12}\right) + (4-2)^2\left(\dfrac{1}{12}\right)$

In Exercises 7–10, write the fraction as a percent. Round your answers to 2 decimal places, if necessary.

**7.** $\dfrac{3}{8}$          **8.** $\dfrac{9}{11}$          **9.** $\dfrac{13}{24}$          **10.** $\dfrac{112}{256}$

## Exercises 9.1

In Exercises 1–4, list or describe the elements in the specified set.

**1. Coin Toss** A coin is tossed three times.

(a) The sample space *S*

(b) The event *A* that at least two heads occur

(c) The event *B* that no more than one head occurs

**2. Coin Toss** A coin is tossed. If a head occurs, the coin is tossed again; otherwise, a die is tossed.

(a) The sample space *S*

(b) The event *A* that 4, 5, or 6 occurs on the die

(c) The event *B* that two heads occur

**3. Poll** Three people are asked their opinions on a political issue. They can answer "In favor" (I), "Opposed" (O), or "Undecided" (U).

(a) The sample space *S*

(b) The event *A* that at least two people are in favor

(c) The event *B* that no more than one person is opposed

**4. Credit Card Fraud** Four cases of credit card fraud are examined. The method of fraud is "stolen card" (S), "counterfeit card" (C), "mail order" (M), or "other" (O).

(a) The sample space *S*

(b) The event *A* that at least three cases are mail order fraud

(c) The event *B* that no more than one case is counterfeit card fraud

**5. Coin Toss** Two coins are tossed. A random variable assigns the number 0, 1, or 2 to each possible outcome, depending on the number of heads that turn up. Find the frequencies of 0, 1, and 2.

**6. Coin Toss** Four coins are tossed. A random variable assigns the number 0, 1, 2, 3, or 4 to each possible outcome, depending on the number of heads that turn up. Find the frequencies of 0, 1, 2, 3, and 4.

**7. Exam** Three students answer a true-false question on an examination. A random variable assigns the number 0, 1, 2, or 3 to each outcome, depending on the number of answers of *true* among the three students. Find the frequencies of 0, 1, 2, and 3.

**8. Exam** Four students answer a true-false question on an examination. A random variable assigns the number 0, 1, 2, 3, or 4 to each outcome, depending on the number of answers of *true* among the four students. Find the frequencies of 0, 1, 2, 3, and 4.

**9. Poll**   Three people have been nominated for president of a college class. From a small poll it is estimated that Jane has a probability of 0.29 of winning and Larry has a probability of 0.47. What is the probability of the third candidate winning the election?

**10. Random Selection**   In a class of 72 students, 44 are girls and, of these, 12 are going to college. Of the 28 boys in the class, 9 are going to college. If a student is selected at random from the class, what is the probability that the person chosen is (a) going to college, (b) not going to college, and (c) a girl who is not going to college?

**11. Quality Control**   A component of a spacecraft has both a main system and a backup system. The probability of at least one of the systems performing satisfactorily throughout the duration of the flight is 0.9855. What is the probability of both of them failing?

**12. Random Selection**   A card is chosen at random from a standard 52-card deck of playing cards. What is the probability that the card will be black and a face card?

In Exercises 13 and 14, find the missing value of the probability distribution.

**13.**

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(x)$ | 0.20 | 0.35 | 0.15 | ? | 0.05 |

**14.**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(x)$ | 0.05 | ? | 0.25 | 0.30 | 0.15 | 0.10 |

In Exercises 15–18, determine whether the table represents a probability distribution. If it is a probability distribution, sketch its graph. If it is not a probability distribution, state any properties that are not satisfied.

**15.**

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(x)$ | 0.10 | 0.45 | 0.30 | 0.15 |

**16.**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(x)$ | 0.05 | 0.30 | 0.10 | 0.40 | 0.15 | 0.20 |

**17.**

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(x)$ | $\frac{12}{50}$ | $\frac{20}{50}$ | $\frac{8}{50}$ | $\frac{10}{50}$ | $-\frac{5}{50}$ |

**18.**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(x)$ | $\frac{8}{30}$ | $\frac{2}{30}$ | $\frac{6}{30}$ | $\frac{3}{30}$ | $\frac{4}{30}$ | $\frac{7}{30}$ |

In Exercises 19–22, sketch a graph of the probability distribution and find the required probabilities.

**19.**

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(x)$ | $\frac{1}{20}$ | $\frac{3}{20}$ | $\frac{6}{20}$ | $\frac{6}{20}$ | $\frac{4}{20}$ |

(a) $P(1 \le x \le 3)$

(b) $P(x \ge 2)$

**20.**

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(x)$ | $\frac{8}{20}$ | $\frac{6}{20}$ | $\frac{3}{20}$ | $\frac{2}{20}$ | $\frac{1}{20}$ |

(a) $P(x \le 2)$

(b) $P(x > 2)$

**21.**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(x)$ | 0.041 | 0.189 | 0.247 | 0.326 | 0.159 | 0.038 |

(a) $P(x \le 3)$

(b) $P(x > 3)$

**22.**

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(x)$ | 0.027 | 0.189 | 0.441 | 0.343 |

(a) $P(1 \le x \le 2)$

(b) $P(x < 2)$

**23. Biology**   Consider a couple who have four children. Assume that it is equally likely that each child is a girl or a boy.

(a) Complete the set to form the sample space consisting of 16 elements.

$S = \{gggg, gggb, ggbg, \ldots\}$

(b) Complete the table, in which the random variable $x$ is the number of girls in the family.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(x)$ | | | | | |

(c) Use the table in part (b) to sketch the graph of the probability distribution.

(d) Use the table in part (b) to find the probability that at least one of the children is a boy.

**24. Die Toss**   Consider the experiment of tossing a 12-sided die twice.

(a) Complete the set to form the sample space of 144 elements. Note that each element is an ordered pair in which the entries are the numbers of points on the first and second tosses, respectively.

$S = \{(1, 1), (1, 2), \ldots, (2, 1), (2, 2), \ldots\}$

(b) Complete the table, in which the random variable $x$ is the sum of the number of points.

| $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|---|---|---|---|---|---|---|
| $P(x)$ | | | | | | | | |

| $x$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|------|----|----|----|----|----|----|----|----|
| $P(x)$ | | | | | | | | |

| $x$ | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|------|----|----|----|----|----|----|----|
| $P(x)$ | | | | | | | |

(c) Use the table in part (b) to sketch the graph of the probability distribution.

(d) Use the table in part (b) to find $P(15 \leq x \leq 19)$.

In Exercises 25–28, find $E(x)$, $V(x)$, and $\sigma$ for the given probability distribution.

**25.**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|------|---|---|---|---|---|
| $P(x)$ | $\frac{1}{16}$ | $\frac{3}{16}$ | $\frac{8}{16}$ | $\frac{3}{16}$ | $\frac{1}{16}$ |

**26.**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|------|---|---|---|---|---|
| $P(x)$ | $\frac{4}{10}$ | $\frac{2}{10}$ | $\frac{2}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |

**27.**

| $x$ | $-3$ | $-1$ | 0 | 3 | 5 |
|------|------|------|---|---|---|
| $P(x)$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |

**28.**

| $x$ | $-5000$ | $-2500$ | 300 |
|------|---------|---------|-----|
| $P(x)$ | 0.008 | 0.052 | 0.940 |

In Exercises 29 and 30, find the mean and variance of the discrete random variable $x$.

**29. Die Toss**   $x$ is (a) the number of points when a four-sided die is tossed once and (b) the sum of the points when the four-sided die is tossed twice.

**30. Coin Toss**   $x$ is the number of heads when a coin is tossed four times.

**31. Revenue**   A publishing company introduces a new weekly magazine that sells for \$4.95 on the newsstand. The marketing group of the company estimates that sales $x$ (in thousands) will be approximated by the following probability function.

| $x$ | 10 | 15 | 20 | 30 | 40 |
|------|----|----|----|----|----|
| $P(x)$ | 0.25 | 0.30 | 0.25 | 0.15 | 0.05 |

(a) Find $E(x)$ and $\sigma$.

(b) Find the expected revenue.

**32. Personal Income**   The probability distribution of the random variable $x$, the annual income of a family (in thousands of dollars) in a certain section of a large city, is shown in the table.

| $x$ | 30 | 40 | 50 | 60 | 80 |
|------|----|----|----|----|----|
| $P(x)$ | 0.10 | 0.20 | 0.50 | 0.15 | 0.05 |

Find $E(x)$ and $\sigma$.

**33. Insurance**   An insurance company needs to determine the annual premium required to break even on fire protection policies with a face value of \$90,000. If $x$ is the claim size on these policies and the analysis is restricted to the losses \$30,000, \$60,000, and \$90,000, then the probability distribution of $x$ is as shown in the table.

| $x$ | 0 | 30,000 | 60,000 | 90,000 |
|------|---|--------|--------|--------|
| $P(x)$ | 0.995 | 0.0036 | 0.0011 | 0.0003 |

What premium should customers be charged for the company to break even?

**34. Insurance**   An insurance company needs to determine the annual premium required to break even for collision protection for cars with a value of \$10,000. If $x$ is the claim size on these policies and the analysis is restricted to the losses \$1000, \$5000, and \$10,000, then the probability distribution of $x$ is as shown in the table.

| $x$ | 0 | 1000 | 5000 | 10,000 |
|------|---|------|------|--------|
| $P(x)$ | 0.936 | 0.040 | 0.020 | 0.004 |

What premium should customers be charged for the company to break even?

**Games of Chance** If $x$ is the net gain to a player in a game of chance, then $E(x)$ is usually negative. This value gives the average amount per game the player can expect to lose over the long run. In Exercises 35 and 36, find the expected net gain to the player for one play of the specified game.

**35.** In roulette, the wheel has the 38 numbers 00, 0, 1, 2, . . . , 34, 35, and 36, marked on equally spaced slots. If a player bets $1 on a number and wins, then the player keeps the dollar and receives an additional $35. Otherwise, the dollar is lost.

**36.** A service organization is selling $2 raffle tickets as part of a fundraising program. The first prize is a boat valued at $2950, and the second prize is a camping tent valued at $400. In addition to the first and second prizes, there are 25 $20 gift certificates to be awarded. The number of tickets sold is 3000.

**37. Market Analysis** After considerable market study, a sporting goods company has decided on two possible cities in which to open a new store. Management estimates that city 1 will yield $20 million in revenues if successful and will lose $4 million if not, whereas city 2 will yield $50 million in revenues if successful and lose $9 million if not. City 1 has a 0.3 probability of being successful and city 2 has a 0.2 probability of being successful. In which city should the sporting goods company open the new store with respect to the expected return from each store?

**38.** Repeat Exercise 37 for the case in which the probabilities of city 1 and city 2 being successful are 0.4 and 0.25, respectively.

**39. Health** The table shows the probability distribution of the numbers of AIDS cases diagnosed in the United States in 2005 by age group. *(Source: Centers for Disease Control and Prevention)*

| Age, $a$ | 14 and under | 15–24 | 25–34 | 35–44 |
|---|---|---|---|---|
| $P(a)$ | 0.003 | 0.056 | 0.212 | 0.380 |

| Age, $a$ | 45–54 | 55-64 | 65 and over |
|---|---|---|---|
| $P(a)$ | 0.254 | 0.075 | 0.020 |

(a) Sketch the probability distribution.

(b) Find the probability that an individual diagnosed with AIDS was from 15 to 44 years of age.

(c) Find the probability that an individual diagnosed with AIDS was at least 35 years of age.

(d) Find the probability that an individual diagnosed with AIDS was at most 24 years of age.

**40. Education** The table gives the probability distribution of the educational attainments of people in the United States in 2005, ages 25 years old and over, where $x = 0$ represents no high school diploma, $x = 1$ represents a high school diploma, $x = 2$ represents some college, $x = 3$ represents an associate's degree, $x = 4$ represents a bachelor's degree, and $x = 5$ represents an advanced degree. *(Source: U.S. Census Bureau)*

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(x)$ | 0.148 | 0.322 | 0.168 | 0.086 | 0.181 | 0.095 |

(a) Sketch the probability distribution.

(b) Determine $E(x)$, $V(x)$, and $\sigma$. Explain the meanings of these values.

**41. Athletics** A baseball fan examined the record of a favorite baseball player's performance during his last 50 games. The numbers of games in which the player had zero, one, two, three, and four hits are recorded in the table shown below.

| Number of hits | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Frequency | 14 | 26 | 7 | 2 | 1 |

(a) Complete the table below, where $x$ is the number of hits.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(x)$ | | | | | |

(b) Use the table in part (a) to sketch the graph of the probability distribution.

(c) Use the table in part (a) to find $P(1 \le x \le 3)$.

(d) Determine $E(x)$, $V(x)$, and $\sigma$. Explain your results.

**Ⓑ 42. Economics: Investment** Suppose you are trying to make a decision about how to invest $10,000 over the next year. One option is a low-risk bank deposit paying 5% interest per year. The other is a high-risk corporate stock with a 5% dividend, plus a 50% chance of a 30% price decline and a 50% chance of a 30% price increase. Determine the expected value of each option and choose one of the options. Explain your choice. How would your decision change if the corporate stock offered a 20% dividend instead of a 5% dividend? *(Source: Adapted from Taylor, Economics, Fifth Edition)*

**43. Extended Application** To work an extended application analyzing the health insurance coverage status of people in the United States by age, visit this text's website at *college.hmco.com.* *(Data Source: U.S. Census Bureau)*

**Section 9.2**

## Continuous Random Variables

■ Verify continuous probability density functions and use continuous probability density functions to find probabilities.
■ Use continuous probability density functions to answer questions about real-life situations.

### Continuous Random Variables

In many applications of probability, it is useful to consider a random variable whose range is an interval on the real number line. Such a random variable is called **continuous.** For instance, the random variable that measures the height of a person in a population is continuous.

To define the probability of an event involving a continuous random variable, you cannot simply count the number of ways the event can occur (as you can with a discrete random variable). Rather, you need to define a function $f$ called a **probability density function.**

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**Definition of Probability Density Function**
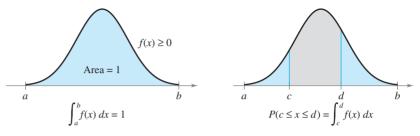
Consider a function $f$ of a continuous random variable $x$ whose range is the interval $[a, b]$. The function is a **probability density function** if it is nonnegative and continuous on the interval $[a, b]$ and if

$$\int_a^b f(x)\, dx = 1.$$

The probability that $x$ lies in the interval $[c, d]$ is

$$P(c \leq x \leq d) = \int_c^d f(x)\, dx$$

as shown in Figure 9.5. If the range of the continuous random variable is an infinite interval, then the integrals are improper integrals.

---



$f(x) \geq 0$

Area = 1

$\int_a^b f(x)\, dx = 1$

$P(c \leq x \leq d) = \int_c^d f(x)\, dx$

**FIGURE 9.5**     Probability Density Function

$f(x) = 12x(1 - x)^2$

Area = 1
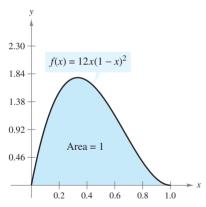
**FIGURE 9.6**

✓**CHECKPOINT 1**

Show that $f(x) = \frac{1}{2}x$ is a probability density function over the interval $[0, 2]$. ■
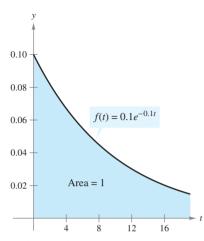
*Example 1*  **Verifying a Probability Density Function**

Show that

$$f(x) = 12x(1 - x)^2$$

is a probability density function over the interval $[0, 1]$.

**SOLUTION**    Begin by observing that $f$ is continuous and nonnegative on the interval $[0, 1]$.

$$f(x) = 12x(1 - x)^2 \geq 0, \quad 0 \leq x \leq 1 \qquad \textcolor{red}{f(x)\text{ is nonnegative on }[0, 1].}$$

Next, evaluate the integral below.

$$\int_0^1 12x(1 - x)^2 \, dx = 12 \int_0^1 (x^3 - 2x^2 + x) \, dx \qquad \textcolor{red}{\text{Expand polynomial.}}$$

$$= 12\left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1 \qquad \textcolor{red}{\text{Integrate.}}$$

$$= 12\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) \qquad \textcolor{red}{\text{Apply Fundamental Theorem of Calculus.}}$$

$$= 1 \qquad \textcolor{red}{\text{Simplify.}}$$

Because this value is 1, you can conclude that $f$ is a probability density function over the interval $[0, 1]$. The graph of $f$ is shown in Figure 9.6. _____

The next example deals with an infinite interval and its corresponding improper integral.

*Example 2*  **Verifying a Probability Density Function**

Show that

$$f(t) = 0.1e^{-0.1t}$$

is a probability density function over the infinite interval $[0, \infty)$.

**SOLUTION**    Begin by observing that $f$ is continuous and nonnegative on the interval $[0, \infty)$.

$$f(t) = 0.1e^{-0.1t} \geq 0, \quad 0 \leq t \qquad \textcolor{red}{f(t)\text{ is nonnegative on }[0, \infty).}$$

Next, evaluate the integral below.

$$\int_0^\infty 0.1e^{-0.1t} \, dt = \lim_{b \to \infty}\left[-e^{-0.1t}\right]_0^b \qquad \textcolor{red}{\text{Improper integral}}$$

$$= \lim_{b \to \infty}(-e^{-0.1b} + 1) \qquad \textcolor{red}{\text{Evaluate limit.}}$$

$$= 1$$

Because this value is 1, you can conclude that $f$ is a probability density function over the interval $[0, \infty)$. The graph of $f$ is shown in Figure 9.7. _____



$f(t) = 0.1e^{-0.1t}$

Area = 1

**FIGURE 9.7**

✓**CHECKPOINT 2**

Show that $f(x) = 2e^{-2x}$ is a probability density function over the interval $[0, \infty)$. ■

**FIGURE 9.8**

**Example 3** Finding a Probability

For the probability density function in Example 1

$$f(x) = 12x(1 - x)^2$$

find the probability that $x$ lies in the interval $\frac{1}{2} \le x \le \frac{3}{4}$.

**SOLUTION**

$$P\left(\frac{1}{2} \le x \le \frac{3}{4}\right) = 12\int_{1/2}^{3/4} x(1 - x)^2 \, dx \qquad \text{Integrate } f(x) \text{ over } \left[\frac{1}{2}, \frac{3}{4}\right].$$

$$= 12\int_{1/2}^{3/4} (x^3 - 2x^2 + x) \, dx \qquad \text{Expand polynomial.}$$

$$= 12\left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right]_{1/2}^{3/4} \qquad \text{Integrate.}$$

$$= 12\left[\frac{\left(\frac{3}{4}\right)^4}{4} - \frac{2\left(\frac{3}{4}\right)^3}{3} + \frac{\left(\frac{3}{4}\right)^2}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{2\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^2}{2}\right]$$

$$\approx 0.262 \qquad \text{Simplify.}$$

So, the probability that $x$ lies in the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$ is approximately 0.262 or 26.2%, as indicated in Figure 9.8.
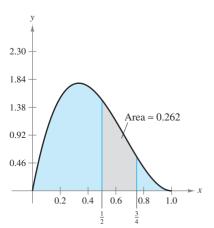
✓ **CHECKPOINT 3**

Find the probability that $x$ lies in the interval $\frac{1}{2} \le x \le 1$ for the probability density function in Checkpoint 1. ∎

In Example 3, note that if you had been asked to find the probability that $x$ lies in any of the intervals $\frac{1}{2} < x < \frac{3}{4}$, $\frac{1}{2} \le x < \frac{3}{4}$, or $\frac{1}{2} < x \le \frac{3}{4}$, you would have obtained the same solution. In other words, the inclusion of either endpoint adds nothing to the probability. This demonstrates an important difference between discrete and continuous random variables. For a continuous random variable, the probability that $x$ will be precisely one value (such as 0.5) is considered to be zero, because

$$P(0.5 \le x \le 0.5) = \int_{0.5}^{0.5} f(x) \, dx = 0.$$

You should not interpret this result to mean that it is impossible for the continuous random variable $x$ to have the value 0.5. It simply means that the probability that $x$ will have this *exact* value is insignificant.

**Example 4** Finding a Probability

✓ **CHECKPOINT 4**

A probability density function is defined over the interval $[0, 4]$. The probability that $x$ lies in $[0, 1]$ is 0.6. What is the probability that $x$ lies in $[1, 4]$? ∎

Consider a probability density function defined over the interval $[0, 5]$. If the probability that $x$ lies in the interval $[0, 2]$ is 0.7, what is the probability that $x$ lies in the interval $[2, 5]$?

**SOLUTION**  Because the probability that $x$ lies in the interval $[0, 5]$ is 1, you can conclude that the probability that $x$ lies in the interval $[2, 5]$ is $1 - 0.7 = 0.3$.

## Applications



(a) $P(0 \leq t \leq 2) \approx 0.181$



(b) $P(2 < t \leq 4) \approx 0.148$



(c) $P(4 < t < \infty) \approx 0.670$

**FIGURE 9.9**

> *Example 5*  **Modeling the Lifetime of a Product**  Ⓡ
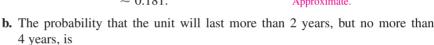
The useful lifetime (in years) of a product is modeled by the probability density function $f(t) = 0.1e^{-0.1t}$ for $0 \leq t < \infty$. Find the probability that a randomly selected unit will have a lifetime falling in each interval.

**a.** No more than 2 years

**b.** More than 2 years, but no more than 4 years

**c.** More than 4 years

**SOLUTION**

**a.** The probability that the unit will last no more than 2 years is

$$P(0 \leq t \leq 2) = 0.1 \int_0^2 e^{-0.1t}\, dt \qquad \text{Integrate } f(t) \text{ over } [0, 2].$$

$$= \left[ -e^{-0.1t} \right]_0^2 \qquad \text{Find antiderivative.}$$

$$= -e^{-0.2} + 1 \qquad \text{Apply Fundamental Theorem of Calculus.}$$

$$\approx 0.181. \qquad \text{Approximate.}$$

**b.** The probability that the unit will last more than 2 years, but no more than 4 years, is

$$P(2 < t \leq 4) = 0.1 \int_2^4 e^{-0.1t}\, dt \qquad \text{Integrate } f(t) \text{ over } [2, 4].$$

$$= \left[ -e^{-0.1t} \right]_2^4 \qquad \text{Find antiderivative.}$$

$$= -e^{-0.4} + e^{-0.2} \qquad \text{Apply Fundamental Theorem of Calculus.}$$

$$\approx 0.148. \qquad \text{Approximate.}$$

**c.** The probability that the unit will last more than 4 years is
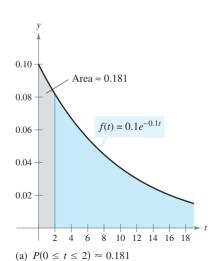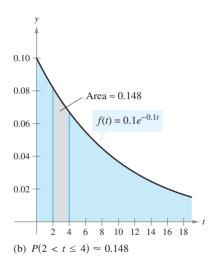
$$P(4 < t < \infty) = 0.1 \int_4^\infty e^{-0.1t}\, dt \qquad \text{Integrate } f(t) \text{ over } [4, \infty).$$

$$= \lim_{b \to \infty} \left[ -e^{-0.1t} \right]_4^b \qquad \text{Improper integral}$$

$$= \lim_{b \to \infty} \left( -e^{-0.1b} + e^{-0.4} \right) \qquad \text{Evaluate limit.}$$

$$= e^{-0.4}$$

$$\approx 0.670. \qquad \text{Approximate.}$$

These three probabilities are illustrated graphically in Figure 9.9. Note that the sum of the three probabilities is 1.
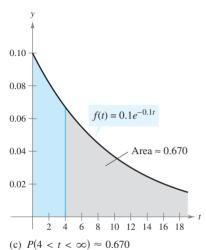
**✓ CHECKPOINT 5**

For the product in Example 5, find the probability that a randomly selected unit will have a lifetime of more than 10 years. ■

*Example 6*
**MAKE A DECISION**    **Modeling Weekly Demand**    (R)

The weekly demand for a product is modeled by the probability density function
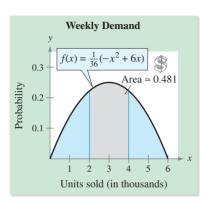
$$f(x) = \frac{1}{36}(-x^2 + 6x), \quad 0 \le x \le 6$$

where $x$ is the number of units sold (in thousands). What are the minimum and maximum weekly sales? Find the probability that the sales for a randomly chosen week will be between 2000 and 4000 units. Will this probability be at least 50%?

**SOLUTION**    Because $0 \le x \le 6$, the weekly sales vary from a minimum of 0 to a maximum of 6000 units. The probability is given by the integral

$$\begin{aligned}
P(2 \le x \le 4) &= \frac{1}{36}\int_2^4 (-x^2 + 6x)\, dx & \text{Integrate } f(x) \text{ over } [2, 4]. \\
&= \frac{1}{36}\left[-\frac{x^3}{3} + 3x^2\right]_2^4 & \text{Find antiderivative.} \\
&= \frac{1}{36}\left(-\frac{64}{3} + 48 + \frac{8}{3} - 12\right) & \text{Apply Fundamental Theorem of Calculus.} \\
&= \frac{13}{27} & \text{Simplify.} \\
&\approx 0.481. & \text{Approximate.}
\end{aligned}$$

So, the probability that the weekly sales will be between 2000 and 4000 units is about 0.481 or 48.1%, as indicated in Figure 9.10. No, the probability will not be at least 50%.



**Weekly Demand**

$f(x) = \frac{1}{36}(-x^2 + 6x)$

Area ≈ 0.481

Units sold (in thousands)

**FIGURE 9.10**

✓**CHECKPOINT 6**

Find the probabilities that the sales of the product in Example 6 for a randomly chosen week will be (a) less than 2000 units and (b) more than 4000 units. Explain how you can find these probabilities without integration.  ■

---

**CONCEPT CHECK**

1. Which random variable's range is an interval on the real number line?

2. For an event involving a continuous random variable, can you count the number of ways that the event can occur?

3. In Example 6, is $P(2 \le x \le 4) = P(2 < x < 4)$? (Do not calculate.)

4. List the conditions that determine if a function is a probability density function.

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.6, 5.4, and 6.5.

In Exercises 1–4, determine whether $f$ is continuous and nonnegative on the given interval.

**1.** $f(x) = \dfrac{1}{x}$, $[1, 4]$

**2.** $f(x) = x^2 - 1$, $[0, 1]$

**3.** $f(x) = 3 - x$, $[1, 5]$

**4.** $f(x) = e^{-x}$, $[0, 1]$

In Exercises 5–10, evaluate the definite integral.

**5.** $\displaystyle\int_0^4 \dfrac{1}{4}\, dx$

**6.** $\displaystyle\int_1^3 \dfrac{1}{4}\, dx$

**7.** $\displaystyle\int_0^2 \dfrac{2 - x}{2}\, dx$

**8.** $\displaystyle\int_1^2 \dfrac{2 - x}{2}\, dx$

**9.** $\displaystyle\int_0^\infty 0.4e^{-0.4t}\, dt$

**10.** $\displaystyle\int_0^\infty 3e^{-3t}\, dt$

## Exercises 9.2

(T) In Exercises 1–14, use a graphing utility to graph the function. Then determine whether the function $f$ represents a probability density function over the given interval. If $f$ is not a probability density function, identify the condition(s) that is (are) not satisfied.

**1.** $f(x) = \frac{1}{8}$, $[0, 8]$

**2.** $f(x) = \frac{1}{5}$, $[0, 4]$

**3.** $f(x) = \dfrac{4 - x}{8}$, $[0, 4]$

**4.** $f(x) = \dfrac{x}{18}$, $[0, 6]$

**5.** $f(x) = 6x(1 - 2x)$, $[0, 1]$

**6.** $f(x) = \dfrac{x(6 - x)}{36}$, $[0, 6]$

**7.** $f(x) = \frac{1}{5}e^{-x/5}$, $[0, 5]$

**8.** $f(x) = \frac{1}{6}e^{-x/6}$, $[0, \infty)$

**9.** $f(x) = 2\sqrt{4 - x}$, $[0, 2]$

**10.** $f(x) = 12x^2(1 - x)$, $[0, 2]$

**11.** $f(x) = \frac{4}{27}x^2(3 - x)$, $[0, 3]$

**12.** $f(x) = \frac{2}{9}x(3 - x)$, $[0, 3]$

**13.** $f(x) = \frac{1}{3}e^{-x/3}$, $[0, \infty)$

**14.** $f(x) = \frac{1}{4}$, $[8, 12]$

In Exercises 15–20, find the constant $k$ such that the function $f$ is a probability density function over the given interval.

**15.** $f(x) = kx$, $[1, 4]$

**16.** $f(x) = kx^3$, $[0, 4]$

**17.** $f(x) = k(4 - x^2)$, $[-2, 2]$

**18.** $f(x) = k\sqrt{x}(1 - x)$, $[0, 1]$

**19.** $f(x) = ke^{-x/2}$, $[0, \infty)$

**20.** $f(x) = \dfrac{k}{b - a}$, $[a, b]$

In Exercises 21–28, sketch the graph of the probability density function over the indicated interval and find the indicated probabilities.

**21.** $f(x) = \frac{1}{5}$, $[0, 5]$

(a) $P(0 < x < 3)$  (b) $P(1 < x < 3)$

(c) $P(3 < x < 5)$  (d) $P(x \geq 1)$

**22.** $f(x) = \frac{1}{10}$, $[0, 10]$

(a) $P(0 < x < 6)$  (b) $P(4 < x < 6)$

(c) $P(8 < x < 10)$  (d) $P(x \geq 2)$

**23.** $f(x) = \dfrac{x}{50}$, $[0, 10]$

(a) $P(0 < x < 6)$  (b) $P(4 < x < 6)$

(c) $P(8 < x < 10)$  (d) $P(x \geq 2)$

**24.** $f(x) = \dfrac{2x}{25}$, $[0, 5]$

(a) $P(0 < x < 3)$  (b) $P(1 < x < 3)$

(c) $P(3 < x < 5)$  (d) $P(x \geq 1)$

**25.** $f(x) = \frac{3}{16}\sqrt{x}$, $[0, 4]$

(a) $P(0 < x < 2)$  (b) $P(2 < x < 4)$

(c) $P(1 < x < 3)$  (d) $P(x \leq 3)$

**26.** $f(x) = \dfrac{5}{4(x+1)^2}$, $[0, 4]$

(a) $P(0 < x < 2)$      (b) $P(2 < x < 4)$

(c) $P(1 < x < 3)$      (d) $P(x \le 3)$

**27.** $f(t) = \frac{1}{3}e^{-t/3}$, $[0, \infty)$

(a) $P(t < 2)$      (b) $P(t \ge 2)$

(c) $P(1 < t < 4)$      (d) $P(t = 3)$

**28.** $f(t) = \frac{3}{256}(16 - t^2)$, $[-4, 4]$

(a) $P(t < -2)$      (b) $P(t > 2)$

(c) $P(-1 < t < 1)$      (d) $P(t > -2)$

**29. Waiting Time**    Buses arrive and depart from a college every 30 minutes. The probability density function for the waiting time $t$ (in minutes) for a person arriving at the bus stop is

$f(t) = \frac{1}{30}$, $[0, 30]$.

Find the probabilities that the person will wait (a) no more than 5 minutes and (b) at least 18 minutes.

**30. Waiting Time**    Commuter trains arrive and depart from a station every 15 minutes during rush hour. The probability density function for the waiting time $t$ (in minutes) for a person arriving at the station is

$f(t) = \frac{1}{15}$, $[0, 15]$.

Find the probabilities that the person will wait (a) no more than 5 minutes and (b) at least 10 minutes.

**31. Demand**    The daily demand for gasoline $x$ (in millions of gallons) in a city is described by the probability density function

$f(x) = 0.41 - 0.08x$, $[0, 4]$.

Find the probabilities that the daily demand for gasoline will be (a) no more than 3 million gallons and (b) at least 2 million gallons.

**32. Learning Theory**    The time $t$ (in hours) required for a new employee to successfully learn to operate a machine in a manufacturing process is described by the probability density function

$f(t) = \frac{5}{324}t\sqrt{9 - t}$, $[0, 9]$.

Find the probabilities that a new employee will learn to operate the machine (a) in less than 3 hours and (b) in more than 4 hours but less than 8 hours.

Ⓣ In Exercises 33–36, use a symbolic integration utility to find the required probabilities using the *exponential density function*

$f(t) = \dfrac{1}{\lambda}e^{-t/\lambda}$, $[0, \infty)$.

**33. Waiting Time**    The waiting time (in minutes) for service at the checkout at a grocery store is exponentially

distributed with $\lambda = 3$. Find the probabilities of waiting (a) less than 2 minutes, (b) more than 2 minutes but less than 4 minutes, and (c) at least 2 minutes.

**34. Waiting Time**    The length of time (in hours) required to unload trucks at a depot is exponentially distributed with $\lambda = \frac{3}{4}$. What proportion of the trucks can be unloaded in less than 1 hour?

**35. Useful Life**    The lifetime (in years) of a battery is exponentially distributed with $\lambda = 5$. Find the probabilities that the lifetime of a given battery will be (a) less than 6 years, (b) more than 2 years but less than 6 years, and (c) more than 8 years.

**36. Useful Life**    The time (in years) until failure of a component in a machine is exponentially distributed with $\lambda = 3.5$. A manufacturer has a large number of these machines and plans to replace the components in all the machines during regularly scheduled maintenance periods. How much time should elapse between maintenance periods if at least 90% of the components are to remain working throughout the period?

**37. Demand**    The weekly demand $x$ (in tons) for a certain product is a continuous random variable with the density function

$f(x) = \frac{1}{36}xe^{-x/6}$, $[0, \infty)$.

Find the probabilities.

(a) $P(x < 6)$

(b) $P(6 < x < 12)$

(c) $P(x > 12) = 1 - P(x \le 12)$

**38. Demand**    Given the conditions of Exercise 37, determine the number of tons that should be ordered each week so that the demand can be met for 90% of the weeks.

**39. Meteorology**    A meteorologist predicts that the amount of rainfall (in inches) expected for a certain coastal community during a hurricane has the probability density function

$f(x) = \dfrac{\pi}{30}\sin\dfrac{\pi x}{15}$, $0 \le x \le 15$.

Find and interpret the probabilities.

(a) $P(0 \le x \le 10)$      (b) $P(10 \le x \le 15)$

(c) $P(0 \le x < 5)$      (d) $P(12 \le x \le 15)$

Ⓣ **40. Coin Toss**    The probability of obtaining 49, 50, or 51 heads when a fair coin is tossed 100 times is

$P(49 \le x \le 51) \approx \displaystyle\int_{48.5}^{51.5} \dfrac{1}{5\sqrt{2\pi}}\, e^{-(x-50)^2/50}\, dx.$

Use a computer or graphing utility and Simpson's Rule (with $n = 12$) to approximate this integral.

## Section 9.3

# Expected Value and Variance

■ Find the expected values or means of continuous probability density functions.
■ Find the variances and standard deviations of continuous probability density functions.
■ Find the medians of continuous probability density functions.
■ Use special probability density functions to answer questions about real-life situations.

## Expected Value

In Section 9.1, you studied the concepts of expected value (or mean), variance, and standard deviation of *discrete* random variables. In this section, you will extend these concepts to *continuous* random variables.

### Definition of Expected Value

If $f$ is a probability density function of a continuous random variable $x$ over the interval $[a, b]$, then the **expected value** or **mean** of $x$ is

$$\mu = E(x) = \int_a^b xf(x)\, dx.$$

$f(x) = \frac{1}{36}(-x^2 + 6x)$

Expected value = 3

**FIGURE 9.11**

✓ **CHECKPOINT 1**

Find the expected value of the probability density function $f(x) = \frac{1}{32}3x(4 - x)$ on the interval $[0, 4]$. ■

*Example 1*  **Finding Average Weekly Demand**  ®

In Example 6 in Section 9.2, the weekly demand for a product was modeled by the probability density function

$$f(x) = \frac{1}{36}(-x^2 + 6x), \quad 0 \le x \le 6.$$

Find the expected weekly demand for this product.

**SOLUTION**

$$\begin{aligned}
\mu = E(x) &= \frac{1}{36}\int_0^6 x(-x^2 + 6x)\, dx \\
&= \frac{1}{36}\int_0^6 (-x^3 + 6x^2)\, dx \\
&= \frac{1}{36}\left[\frac{-x^4}{4} + 2x^3\right]_0^6 \\
&= 3
\end{aligned}$$

In Figure 9.11, you can see that an expected value of 3 seems reasonable because the region is symmetric about the line $x = 3$.

Standard deviation = 1.0



Standard deviation = 1.5



Standard deviation = 2.0

**FIGURE 9.12**

## Variance and Standard Deviation

**Definitions of Variance and Standard Deviation**

If $f$ is a probability density function of a continuous random variable $x$ over the interval $[a, b]$, then the **variance** of $x$ is

$$V(x) = \int_a^b (x - \mu)^2 f(x) \, dx$$

where $\mu$ is the mean of $x$. The **standard deviation** of $x$ is

$$\sigma = \sqrt{V(x)}.$$

Recall from Section 9.1 that distributions that are clustered about the mean tend to have smaller standard deviations than distributions that are more dispersed. For instance, all three of the probability density distributions shown in Figure 9.12 have a mean of $\mu = 0$, but they have different standard deviations. Because the first distribution is clustered more toward the mean, its standard deviation is the smallest of the three.

### Example 2   Finding Variance and Standard Deviation

Find the variance and standard deviation of the probability density function

$$f(x) = 2 - 2x, \quad 0 \le x \le 1.$$

**SOLUTION**   Begin by finding the mean.

$$\mu = \int_0^1 x(2 - 2x) \, dx = \frac{1}{3} \qquad \text{Mean}$$

Next, apply the formula for variance.

$$
\begin{aligned}
V(x) &= \int_0^1 \left( x - \frac{1}{3} \right)^2 (2 - 2x) \, dx \\
&= \int_0^1 \left( -2x^3 + \frac{10x^2}{3} - \frac{14x}{9} + \frac{2}{9} \right) dx \\
&= \left[ -\frac{x^4}{2} + \frac{10x^3}{9} - \frac{7x^2}{9} + \frac{2x}{9} \right]_0^1 \\
&= \frac{1}{18} \qquad \text{Variance}
\end{aligned}
$$

Finally, you can conclude that the standard deviation is

$$\sigma = \sqrt{\frac{1}{18}} \approx 0.236. \qquad \text{Standard deviation}$$

### ✔CHECKPOINT 2

Find the variance and standard deviation of the probability density function in Checkpoint 1. ■

The integral for variance can be difficult to evaluate. The following alternative formula is often simpler.

---

**Alternative Formula for Variance**

If $f$ is a probability density function of a continuous random variable $x$ over the interval $[a, b]$, then the **variance** of $x$ is

$$V(x) = \int_a^b x^2 f(x)\, dx - \mu^2$$

where $\mu$ is the mean of $x$.

---

**Example 3**    **Using the Alternative Formula** Ⓣ

Find the standard deviation of the probability density function

$$f(x) = \frac{2}{\pi(x^2 - 2x + 2)}, \quad 0 \le x \le 2.$$

What percent of the distribution lies within one standard deviation of the mean?

**SOLUTION**    Begin by using a symbolic integration utility to find the mean.

$$\mu = \int_0^2 \left[ \frac{2}{\pi(x^2 - 2x + 2)} \right] (x)\, dx$$

$$= 1 \qquad\qquad\qquad\qquad\qquad\qquad \text{Mean}$$

Next, use a symbolic integration utility to find the variance.

$$V(x) = \int_0^2 \left[ \frac{2}{\pi(x^2 - 2x + 2)} \right] (x^2)\, dx - 1^2$$

$$\approx 0.273 \qquad\qquad\qquad\qquad\qquad \text{Variance}$$

This implies that the standard deviation is

$$\sigma \approx \sqrt{0.273}$$

$$\approx 0.522. \qquad\qquad\qquad\qquad \text{Standard deviation}$$

To find the percent of the distribution that lies within one standard deviation of the mean, integrate the probability density function between $\mu - \sigma = 0.478$ and $\mu + \sigma = 1.522$.

$$\int_{0.478}^{1.522} \frac{2}{\pi(x^2 - 2x + 2)}\, dx \approx 0.613$$

So, about 61.3% of the distribution lies within one standard deviation of the mean. This result is illustrated in Figure 9.13.



$$f(x) = \frac{2}{\pi(x^2 - 2x + 2)}$$

Mean = 1

**FIGURE 9.13**

✓ **CHECKPOINT 3**

Use a symbolic integration utility to find the percent of the distribution in Example 3 that lies within 1.5 standard deviations of the mean. ■

## Median

The mean of a probability density function is an example of a **measure of central tendency.** Another useful measure of central tendency is the **median.**

---

**Definition of Median**

If $f$ is a probability density function of a continuous random variable $x$ over the interval $[a, b]$, then the **median** of $x$ is the number $m$ such that

$$\int_a^m f(x)\, dx = 0.5.$$

---

*Example 4*  **Comparing Mean and Median**  Ⓡ

In Example 5 in Section 9.2, the probability density function

$$f(t) = 0.1e^{-0.1t}, \quad 0 \le t < \infty$$

was used to model the useful lifetime of a product. Find the mean and median useful lifetimes.

**SOLUTION**   Using integration by parts or a symbolic integration utility, you can find the mean to be

$$\mu = \int_0^\infty 0.1te^{-0.1t}\, dt$$

$$= 10 \text{ years.} \qquad\qquad \text{Mean}$$

The median is given by

$$\int_0^m 0.1e^{-0.1t}\, dt = 0.5$$

$$-e^{-0.1m} + 1 = 0.5$$

$$e^{-0.1m} = 0.5$$

$$-0.1m = \ln 0.5$$

$$m = -10 \ln 0.5$$

$$m \approx 6.93 \text{ years.} \qquad \text{Median}$$

From this, you can see that the mean and median of a probability distribution can be quite different. Using the mean, the "average" lifetime of a product is 10 years, but using the median, the "average" lifetime is 6.93 years. In Figure 9.14, note that half of the products have usable lifetimes of 6.93 years or less.



**Useful Lifetime of a Product**

$f(t) = 0.1e^{-0.1t}$

Area = $\frac{1}{2}$

Area = $\frac{1}{2}$

Median ≈ 6.93

Time (in years)
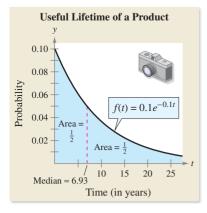
**FIGURE 9.14**

### ✓ CHECKPOINT 4

Find the mean and median of the probability density function $f(x) = 2e^{-2x}$, $0 \le x < \infty$. ■

## Special Probability Density Functions

The remainder of this section describes three common types of probability density functions: uniform, exponential, and normal. The **uniform probability density function** is defined as

$$f(x) = \frac{1}{b - a}, \quad a \le x \le b.$$    Uniform probability density function

This probability density function represents a continuous random variable for which each outcome is equally likely.

### Example 5    Analyzing a Probability Density Function

Find the expected value and standard deviation of the uniform probability density function

$$f(x) = \frac{1}{8}, \quad 0 \le x \le 8.$$

**SOLUTION**    The expected value (or mean) is

$$\mu = \int_0^8 \frac{1}{8} x \, dx$$

$$= \left[ \frac{x^2}{16} \right]_0^8$$

$$= 4.$$    Expected value

The variance is

$$V(x) = \int_0^8 \frac{1}{8} x^2 \, dx - 4^2$$

$$= \left[ \frac{x^3}{24} \right]_0^8 - 16$$

$$\approx 5.333.$$    Variance
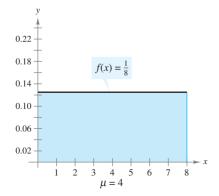
The standard deviation is

$$\sigma \approx \sqrt{5.333}$$

$$\approx 2.309.$$    Standard deviation

The graph of $f$ is shown in Figure 9.15.



$f(x) = \frac{1}{8}$

$\mu = 4$

**FIGURE 9.15**    Uniform Probability Density Function

### ✓ CHECKPOINT 5

Find the expected value and standard deviation of the uniform probability density function $f(x) = \frac{1}{2}, 0 \le x \le 2$. ◼

### STUDY TIP

Try showing that the mean and the variance of the general uniform probability density function $f(x) = 1/(b - a)$ are $\mu = \frac{1}{2}(a + b)$ and $V(x) = \frac{1}{12}(b - a)^2$.

The second special type of probability density function is the **exponential probability density function** and has the form

$$f(x) = ae^{-ax}, \quad 0 \le x < \infty.$$

<span style="color:magenta">Exponential probability density function, $a > 0$</span>

The probability density function in Example 4 is of this type. Try showing that this function has a mean of $1/a$ and a variance of $1/a^2$.
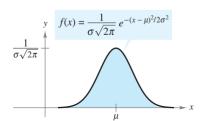
The third special type of probability density function (and the most widely used) is the **normal probability density function** given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty.$$

<span style="color:magenta">Normal probability density function</span>

The expected value of this function is $\mu$, and the standard deviation is $\sigma$. Figure 9.16 shows the graph of a typical normal probability density function.

A normal probability density function for which $\mu = 0$ and $\sigma = 1$ is called a **standard normal probability density function.**



$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$

**FIGURE 9.16**   Normal Probability Density Function

### Example 6   Finding a Probability   ®

In 2006, the scores for the *Graduate Management Admission Test* (GMAT) could be modeled by a normal probability density function with a mean of $\mu = 527$ and a standard deviation of $\sigma = 117$. If you select a person who took the GMAT in 2006, what is the probability that the person scored between 600 and 700? What is the probability that the person scored between 700 and 800?

**SOLUTION**   Using a calculator or computer, you can find the first probability to be

$$P(600 \le x \le 700) = \int_{600}^{700} \frac{1}{117\sqrt{2\pi}}e^{-(x-527)^2/2(117)^2}\,dx$$
$$\approx 0.197.$$

So, the probability of choosing a person who scored between 600 and 700 is about 19.7%. In a similar way, you can find the probability of choosing a person who scored between 700 and 800 to be

$$P(700 \le x \le 800) = \int_{700}^{800} \frac{1}{117\sqrt{2\pi}}e^{-(x-527)^2/2(117)^2}\,dx$$
$$\approx 0.060$$

or about 6.0%.

### ✓ CHECKPOINT 6

From Example 6, find the probability that a person selected at random scored between 400 and 600. ■

**TECHNOLOGY**

Ⓣ The normal probability density function does not have an antiderivative that is an elementary function. However, it does have an antiderivative. Finding its antiderivative analytically is beyond the scope of this text. You can use a graphing utility, a symbolic integration utility, or a spreadsheet to evaluate this function.* Use one of these tools to evaluate the function in Example 6.

**STUDY TIP**

In Example 6, note that probabilities connected with normal distributions must be evaluated with a table of values, with a symbolic integration utility, or with a spreadsheet.

*Specific calculator keystroke instructions for operations in this and other technology boxes can be found at *college.hmco.com/info/larsonapplied*.

*Example 7* Modeling Body Weight ®

Assume the weights of adult male rhesus monkeys are normally distributed with a mean of 15 pounds and a standard deviation of 3 pounds. In a typical population of adult male rhesus monkeys, what percent of the monkeys would have weights within one standard deviation of the mean?

**SOLUTION** For this population, the normal probability density function is

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-(x-15)^2/18}.$$

The probability that a randomly chosen adult male monkey will weigh between 12 and 18 pounds (that is, within 3 pounds of 15 pounds) is

$$P(12 \le x \le 18) = \int_{12}^{18} \frac{1}{3\sqrt{2\pi}} e^{-(x-15)^2/18} \, dx$$
$$\approx 0.683.$$

So, about 68% of the adult male rhesus monkeys have weights that lie within one standard deviation of the mean, as shown in Figure 9.17.

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In addition to being popular zoo animals, rhesus monkeys are commonly used in medical and behavioral research. Research on rhesus monkeys led to the discovery of the Rh factor in human red blood cells.

✓ **CHECKPOINT 7**

Use the results of Example 7 to find the probability that an adult male rhesus monkey chosen at random will weigh more than 18 pounds. ■



**FIGURE 9.17**

The result described in Example 7 can be generalized to all normal distributions. That is, in any normal distribution, the probability that $x$ lies within one standard deviation of the mean is about 68%. For normal distributions, 95.4% of the $x$-values lie within two standard deviations of the mean, and almost all (99.7%) of the $x$-values lie within three standard deviations of the mean.

**CONCEPT CHECK**

1. Complete the following: The mean and median are both measures of _____ _____.
2. Which probability density function represents a continuous random variable for which each outcome is equally likely?
3. What is the mean of the standard normal probability density function?
4. What is the standard deviation of the standard normal probability density function?

**Skills Review 9.3**    The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 5.4 and 9.2.

In Exercises 1–4, solve for $m$.

**1.** $\displaystyle\int_0^m \frac{1}{10}\,dx = 0.5$

**2.** $\displaystyle\int_0^m \frac{1}{16}\,dx = 0.5$

**3.** $\displaystyle\int_0^m \frac{1}{3}e^{-t/3}\,dt = 0.5$

**4.** $\displaystyle\int_0^m \frac{1}{9}e^{-t/9}\,dt = 0.5$

In Exercises 5–8, evaluate the definite integral.

**5.** $\displaystyle\int_0^2 \frac{x^2}{2}\,dx$

**6.** $\displaystyle\int_1^2 x(4 - 2x)\,dx$

**7.** $\displaystyle\int_2^5 x^2\left(\frac{1}{3}\right)dx - \left(\frac{7}{2}\right)^2$

**8.** $\displaystyle\int_2^4 x^2\left(\frac{4 - x}{2}\right)dx - \left(\frac{8}{3}\right)^2$

In Exercises 9 and 10, find the indicated probability using the given probability density function.

**9.** $f(x) = \frac{1}{8}$, $[0, 8]$

(a) $P(x \le 2)$

(b) $P(3 < x < 7)$

**10.** $f(x) = 6x - 6x^2$, $[0, 1]$

(a) $P\left(x \le \frac{1}{2}\right)$

(b) $P\left(\frac{1}{4} \le x \le \frac{3}{4}\right)$

# Exercises 9.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, use the given probability density function over the indicated interval to find the (a) mean, (b) variance, and (c) standard deviation of the random variable. Sketch the graph of the density function and locate the mean on the graph.

**1.** $f(x) = \frac{1}{3}$, $[0, 3]$

**2.** $f(x) = \frac{1}{4}$, $[0, 4]$

**3.** $f(t) = \frac{t}{18}$, $[0, 6]$

**4.** $f(x) = \frac{4}{3x^2}$, $[1, 4]$

**5.** $f(x) = \frac{5}{2}x^{3/2}$, $[0, 1]$

**6.** $f(x) = \frac{3}{16}\sqrt{4 - x}$, $[0, 4]$

(T) In Exercises 7–10, use a graphing utility to graph the function and approximate the mean. Then find the mean analytically. Compare your results.

**7.** $f(x) = 6x(1 - x)$, $[0, 1]$

**8.** $f(x) = \frac{3}{32}x(4 - x)$, $[0, 4]$

**9.** $f(x) = \frac{4}{3(x + 1)^2}$, $[0, 3]$

**10.** $f(x) = \frac{1}{18}\sqrt{9 - x}$, $[0, 9]$

In Exercises 11 and 12, find the median of the exponential probability density function.

**11.** $f(t) = \frac{1}{9}e^{-t/9}$, $[0, \infty)$

**12.** $f(t) = \frac{2}{5}e^{-2t/5}$, $[0, \infty)$

In Exercises 13–18, identify the probability density function. Then find the mean, variance, and standard deviation without integrating.

**13.** $f(x) = \frac{1}{10}$, $[0, 10]$

**14.** $f(x) = 0.2$, $[0, 5]$

**15.** $f(x) = \frac{1}{8}e^{-x/8}$, $[0, \infty)$

**16.** $f(x) = \frac{5}{3}e^{-5x/3}$, $[0, \infty)$

**17.** $f(x) = \frac{1}{11\sqrt{2\pi}}e^{-(x-100)^2/242}$, $(-\infty, \infty)$

**18.** $f(x) = \frac{1}{6\sqrt{2\pi}}e^{-(x-30)^2/72}$, $(-\infty, \infty)$

(T) In Exercises 19–24, use a symbolic integration utility to find the mean, standard deviation, and given probability.

| Function | Probability |
|----------|-------------|

**19.** $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $\qquad$ $P(0 \le x \le 0.85)$

**20.** $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $\qquad$ $P(-1.21 \le x \le 1.21)$

**21.** $f(x) = \frac{1}{6} e^{-x/6}$ $\qquad$ $P(x \ge 2.23)$

**22.** $f(x) = \frac{3}{4} e^{-3x/4}$ $\qquad$ $P(x \ge 0.27)$

**23.** $f(x) = \dfrac{1}{2\sqrt{2\pi}} e^{-(x-8)^2/8}$ $\qquad$ $P(3 \le x \le 13)$

**24.** $f(x) = \dfrac{1}{1.5\sqrt{2\pi}} e^{-(x-2)^2/4.5}$ $\qquad$ $P(-2.5 \le x \le 2.5)$

(T) In Exercises 25 and 26, let $x$ be a random variable that is normally distributed with the given mean and standard deviation. Find the indicated probabilities using a symbolic integration utility.

**25.** $\mu = 50, \ \sigma = 10$

(a) $P(x > 55)$ $\qquad$ (b) $P(x > 60)$

(c) $P(x < 60)$ $\qquad$ (d) $P(30 < x < 55)$

**26.** $\mu = 70, \ \sigma = 14$

(a) $P(x > 65)$ $\qquad$ (b) $P(x < 98)$

(c) $P(x < 49)$ $\qquad$ (d) $P(56 < x < 75)$

**27. Transportation** The arrival time $t$ of a bus at a bus stop is uniformly distributed between 10:00 A.M. and 10:10 A.M.

(a) Find the mean and standard deviation of the random variable $t$.

(b) What is the probability that you will miss the bus if you arrive at the bus stop at 10:03 A.M.?

**28. Transportation** Repeat Exercise 27 for a bus that arrives between 10:00 A.M. and 10:05 A.M.

**29. Useful Life** The time $t$ until failure of an appliance is exponentially distributed with a mean of 2 years.

(a) Find the probability density function of the random variable $t$.

(b) Find the probability that the appliance will fail in less than 1 year.

**30. Useful Life** The lifetime of a battery is normally distributed with a mean of 400 hours and a standard deviation of 24 hours. You purchased one of the batteries, and its useful life was 340 hours.

(a) How far, in standard deviations, did the useful life of your battery fall short of the expected life?

(b) What percent of all other batteries of this type have useful lives that exceed yours?

**31. Waiting Time** The waiting time $t$ for service in a store is exponentially distributed with a mean of 5 minutes.

(a) Find the probability density function of the random variable $t$.

(b) Find the probability that $t$ is within one standard deviation of the mean.

**32. License Renewal** The time $t$ spent at a driver's license renewal center is exponentially distributed with a mean of 15 minutes.

(a) Find the probability density function of the random variable $t$.

(b) Find the probability that $t$ is within one standard deviation of the mean.

**33. Education** The scores on a national exam are normally distributed with a mean of 150 and a standard deviation of 16. You scored 174 on the exam.

(a) How far, in standard deviations, did your score exceed the national mean?

(b) What percent of those who took the exam had scores lower than yours?

**34. Education** The scores on a qualifying exam for entrance into a post secondary school are normally distributed with a mean of 120 and a standard deviation of 10.5. To qualify for admittance, the candidates must score in the top 10%. Find the lowest possible qualifying score.

**35. Demand** The daily demand $x$ for a certain product (in hundreds of pounds) is a random variable with the probability density function $f(x) = \frac{1}{36} x (6 - x), [0, 6]$.

(a) Determine the expected value and the standard deviation of the demand.

(b) Determine the median of the random variable.

(c) Find the probability that $x$ is within one standard deviation of the mean.

**36. Demand** Repeat Exercise 35 for a probability density function of $f(x) = \frac{3}{256}(x - 2)(10 - x), [2, 10]$.

**37. Learning Theory** The percent recall $x$ in a learning experiment is a random variable with the probability density function $f(x) = \frac{15}{4} x \sqrt{1 - x}, [0, 1]$.

Determine the mean and variance of the random variable $x$.

**38. Metallurgy** The percent of iron $x$ in samples of ore is a random variable with the probability density function $f(x) = \frac{1155}{32} x^3 (1 - x)^{3/2}, [0, 1]$. Determine the expected percent of iron in each ore sample.

**39. Demand** The daily demand $x$ for a certain product (in thousands of units) is a random variable with the probability density function $f(x) = \frac{1}{25} x e^{-x/5}, [0, \infty)$.

(a) Determine the expected daily demand.

(b) Find $P(x \le 4)$.

**40. Medicine**  The time $t$ (in days) until recovery after a certain medical procedure is a random variable with the probability density function

$$f(t) = \frac{1}{2\sqrt{t-2}}, \quad [3, 6].$$

(a) Find the probability that a patient selected at random will take more than 4 days to recover.

(b) Determine the expected time for recovery.

In Exercises 41–46, find the mean and median.

**41.** $f(x) = \frac{1}{11}, \quad [0, 11]$      **42.** $f(x) = 0.05, \quad [0, 20]$

**43.** $f(x) = 4(1 - 2x), \quad [0, \frac{1}{2}]$      **44.** $f(x) = \frac{4}{3} - \frac{2}{3}x, \quad [0, 1]$

**45.** $f(x) = \frac{1}{5}e^{-x/5}, \quad [0, \infty)$      **46.** $f(x) = \frac{2}{3}e^{-2x/3}, \quad [0, \infty)$

**47. Cost**  The daily cost (in dollars) of electricity $x$ in a city is a random variable with the probability density function $f(x) = 0.28e^{-0.28x}, 0 \le x < \infty$. Find the median daily cost of electricity.

**48. Consumer Trends**  The number of coupons used by a customer in a grocery store is a random variable with the probability density function

$$f(x) = \frac{2x + 1}{12}, \quad 0 \le x \le 3.$$

Find the expected number of coupons a customer will use.

**49. Demand**  The daily demand $x$ for water (in millions of gallons) in a town is a random variable with the probability density function $f(x) = \frac{1}{9}xe^{-x/3}, [0, \infty)$.

(a) Determine the expected value and the standard deviation of the demand.

(b) Find the probability that the demand is greater than 4 million gallons on a given day.

**50. Useful Life**  The lifetime of a tire is normally distributed with a mean of 50,000 miles and a standard deviation of 3000 miles. How many miles should this tire be guaranteed if the manufacturer does not want to replace any more than 10% of the tires during the mileage covered by the guarantee?

**51. Manufacturing**  An automatic filling machine fills cans so that the weights are normally distributed with a mean of $\mu$ and a standard deviation of $\sigma$. The value of $\mu$ can be controlled by settings on the machine, but $\sigma$ depends on the precision and design of the machine. For a particular substance, $\sigma = 0.15$ ounce. If 12-ounce cans are being filled, determine the setting for $\mu$ such that no more than 5% of the cans weigh less than the stated weight.

**(T) 52. MAKE A DECISION: USEFUL LIFE**  A storage battery has an expected lifetime of 4.5 years with a standard deviation of 0.5 year. Assume that the useful lives of these batteries are normally distributed.

(a) Use a computer or graphing utility and Simpson's Rule (with $n = 12$) to approximate the probability that a given battery will last for 4 to 5 years.

(b) Will 10% of the batteries last less than 3 years?

**(T) 53. MAKE A DECISION: WAGES**  The employees of a large corporation are paid an average wage of $14.50 per hour with a standard deviation of $1.50. Assume that these wages are normally distributed.

(a) Use a computer or graphing utility and Simpson's Rule (with $n = 10$) to approximate the percent of employees that earn hourly wages of $11.00 to $14.00.

(b) Will 20% of the employees be paid more than $16.00 per hour?

**(T) 54. Medical Science**  A medical research team has determined that for a group of 500 females, the length of pregnancy from conception to birth varies according to an approximately normal distribution with a mean of 266 days and a standard deviation of 16 days.

(a) Use a graphing utility to graph the distribution.

(b) Use a symbolic integration utility to approximate the probability that a pregnancy will last from 240 days to 280 days.

(c) Use a symbolic integration utility to approximate the probability that a pregnancy will last more than 280 days.

**(T) 55. Education**  In 2006, the scores for the ACT Test could be modeled by a normal probability density function with a mean of 21.1 and a standard deviation of 4.8.  *(Source: ACT, Inc.)*

(a) Use a graphing utility to graph the distribution.

(b) Use a symbolic integration utility to approximate the probability that a person who took the ACT scored between 24 and 36.

(c) Use a symbolic integration utility to approximate the probability that a person who took the ACT scored more than 26.

**(T) 56. Fuel Mileage**  Assume the fuel mileage of all 2007 model vehicles weighing less than 8500 pounds are normally distributed with a mean of 20.6 miles per gallon and a standard deviation of 4.9 miles per gallon.  *(Source: U.S. Environmental Protection Agency)*

(a) Use a graphing utility to graph the distribution.

(b) Use a symbolic integration utility to approximate the probability that a vehicle's fuel mileage is between 25 and 30 miles per gallon.

(c) Use a symbolic integration utility to approximate the probability that a vehicle's fuel mileage is less than 18 miles per gallon.

# Algebra Review

## Using Counting Principles

In discrete probability, one of the basic skills is being able to count the number of ways an event can happen. To do this, the strategies below can be helpful.

1. *The Fundamental Counting Principle:* The number of ways that two or more events can occur is the product of the numbers of ways each event can occur by itself. These ways can be listed graphically using a tree diagram.

2. *Permutations:* The number of permutations of $n$ elements is $n!$.

3. *Combinations:* The number of combinations of $n$ elements taken $r$ at a time is

$$_nC_r = \frac{n!}{(n - r)!r!}.$$

### Example 1  Counting the Ways an Event Can Happen

a. How many ways can you form a five-letter password if no letter is used more than once?

b. Your class is divided into five work groups containing three, four, four, three, and five people. How many ways can you poll one person from each group?

c. In how many orders can seven runners finish a race if there are no ties?

d. You have 12 phone calls to return. In how many orders can you return them?

**SOLUTION**

a. For the first letter of the password, you have 26 choices. For the second letter, you have 25 choices. For the third letter, you have 24 choices, and so on.

$$\text{Number of ways} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \qquad \text{Counting Principle}$$
$$= 7{,}893{,}600 \qquad \text{Multiply.}$$

b. $\text{Number of ways} = 3 \cdot 4 \cdot 4 \cdot 3 \cdot 5 \qquad$ Counting Principle
$$= 720 \qquad \text{Multiply.}$$

c. The solution is given by the number of permutations of the seven runners.

$$\text{Number of ways} = 7! = 5040 \qquad \text{Multiply.}$$

d. The solution is given by the number of permutations of the 12 phone calls.

$$\text{Number of ways} = 12! \qquad \text{Use a calculator.}$$
$$= 479{,}001{,}600$$

**TECHNOLOGY**

(T) Most graphing utilities have a factorial key. Consult your user's manual for specific keystrokes for your graphing utility.

**Example 2**  **Counting the Ways an Event Can Happen**

How many different ways can you choose a three-person group from a class of 20 people? From a class of 40 people?

**SOLUTION**    The number of ways to choose a three-person group from a class of 20 is given by the number of combinations of 20 elements taken three at a time.

$$\text{Number of ways} = {}_{20}C_3 \qquad\qquad\qquad \text{Combination}$$

$$= \frac{20!}{17!\,3!} \qquad\qquad\qquad \text{Formula for combination}$$

$$= \frac{20 \cdot 19 \cdot 18 \cdot 17!}{17! \cdot 3!}$$

$$= \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} \qquad\qquad\qquad \text{Divide out like factors.}$$

$$= 20 \cdot 19 \cdot 3 \qquad\qquad\qquad \text{Divide out like factors.}$$

$$= 1140 \qquad\qquad\qquad\qquad \text{Multiply.}$$

The number of ways to choose a three-person group from a class of 40 is given by ${}_{40}C_3$, which is 9880.

**Example 3**  **Counting the Ways an Event Can Happen**

To test for defective units, you are choosing a sample of 10 from a manufacturing production of 2000 units. How many different samples of 10 are possible?

**SOLUTION**    The solution is given by the number of combinations of 2000 elements taken 10 at a time.

$$\text{Number of ways} = {}_{2000}C_{10} \qquad\qquad\qquad \text{Combination}$$

$$= \frac{2000!}{1990!\,10!} \qquad\qquad\qquad \text{Formula for combination}$$

$$\approx 2.76 \times 10^{26} \qquad\qquad\qquad \text{Use a graphing utility.}$$

$$= 276{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000$$

From these examples, you can see that combinations and permutations can be very large numbers.

**TECHNOLOGY**

Ⓣ Most graphing utilities have a combination key. Consult your user's manual for specific keystrokes for your graphing utility.

# Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 658.
Answers to odd-numbered Review Exercises are given in the back of the text.*

## Section 9.1                                                                    Review Exercises

- Describe sample spaces for experiments.                                             *1–4*
- Assign values to discrete random variables.                                         *5, 6*
- Form frequency distributions for discrete random variables.                         *7, 8*
- Find the probabilities of events for discrete random variables.                     *9–12*

$$P(x) = \frac{\text{Frequency of } x}{\text{Number of outcomes in } S} = \frac{n(x)}{n(S)}$$

- Find the expected values or means of discrete random variables.   (Section 9.1)    *13–16*

$$\mu = E(x) = x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3) + \cdots + x_m P(x_m)$$

- Find the variances and standard deviations of discrete random variables.            *17–20*

$$V(x) = (x_1 - \mu)^2 P(x_1) + \cdots + (x_m - \mu)^2 P(x_m), \quad \sigma = \sqrt{V(x)}$$

## Section 9.2

- Verify probability density functions.                                               *21–26*
- Use probability density functions to find probabilities.                            *27–30*

$$P(c \le x \le d) = \int_c^d f(x)\, dx$$

- Use probability density functions to answer questions about real-life situations.   *31, 32*

## Section 9.3

- Find the means of probability density functions.                                    *33–36*

$$\mu = E(x) = \int_a^b x f(x)\, dx$$

- Find the variances and standard deviations of continuous probability density functions.   *37–40*

$$V(x) = \int_a^b (x - \mu)^2 f(x)\, dx, \quad \sigma = \sqrt{V(x)}$$

- Find the medians of probability density functions.                                  *41–44*

$$\int_a^m f(x)\, dx = 0.5$$

- Use special probability density functions to answer questions about real-life situations.   *45–52*

## Study Strategies

- **Using Technology**   Integrals that arise with continuous probability density functions tend to be difficult to evaluate by hand. When evaluating such integrals, we suggest that you use a symbolic integration utility or that you use a numerical integration technique such as Simpson's Rule with a programmable calculator.

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\* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

## Review Exercises

In Exercises 1–4, describe the sample space of the experiment.

**1.** A month of the year is chosen for vacation.

**2.** A letter from the word *calculus* is selected.

**3.** A student must answer three questions from a selection of four essay questions.

**4.** A winner in a game show must choose two out of five prizes.

**5. Lottery**   Three numbers are drawn in a lottery. Each number is a digit from 0 to 9. Find the sample space giving the number of 7's drawn.

**6. Quality Control**   As cans of soft drink are filled on the production line, four are randomly selected and labeled with an "S" if the weight is satisfactory or with a "U" if the weight is unsatisfactory. Find the sample space giving the satisfactory/unsatisfactory classification of the four cans in the selected group.

In Exercises 7 and 8, complete the table to form the frequency distribution of the random variable *x*. Then construct a bar graph to represent the result.

**7.** A computer randomly selects a three-digit bar code. Each digit can be 0 or 1, and *x* is the number of 1's in the bar code.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $n(x)$ | | | | |

**8.** A cat has a litter of four kittens. Let *x* represent the number of male kittens.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $n(x)$ | | | | | |

In Exercises 9 and 10, sketch a graph of the given probability distribution and find the required probabilities.

**9.**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $P(x)$ | $\frac{1}{18}$ | $\frac{7}{18}$ | $\frac{5}{18}$ | $\frac{3}{18}$ | $\frac{2}{18}$ |

(a) $P(2 \leq x \leq 4)$

(b) $P(x \geq 3)$

**10.**

| $x$ | $-2$ | $-1$ | 1 | 3 | 5 |
|---|---|---|---|---|---|
| $P(x)$ | $\frac{1}{11}$ | $\frac{2}{11}$ | $\frac{4}{11}$ | $\frac{3}{11}$ | $\frac{1}{11}$ |

(a) $P(x < 0)$

(b) $P(x > 1)$

**11. Dice Toss**   Consider an experiment in which two six-sided dice are tossed. Find the indicated probabilities.

(a) The probability that the total is 8

(b) The probability that the total is greater than 4

(c) The probability that doubles are thrown

(d) The probability of getting double 6's

**12. Random Selection**   Consider an experiment in which one card is randomly selected from a standard deck of 52 playing cards. Find the probabilities of

(a) selecting a face card.

(b) selecting a card that is not a face card.

(c) selecting a black card that is not a face card.

(d) selecting a card whose value is 6 or less.

**13. Education**   An instructor gave a 25-point quiz to 52 students. Use the frequency distribution shown below to find the mean quiz score.

| Score | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|
| Frequency | 1 | 0 | 1 | 0 | 0 | 0 | 3 | 4 | 7 |

| Score | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|
| Frequency | 3 | 0 | 9 | 11 | 6 | 3 | 0 | 4 |

**14. Cost Increases**   A pharmaceutical company uses three different chemicals, A, B, and C, to create a nutritional supplement. The table shown below gives the cost and the percent increase of the cost of each of the three chemicals. Find the mean percent increase of the three chemicals.

| Chemical | Percent Increase | Cost of Materials |
|---|---|---|
| A | 8% | $650 |
| B | 23% | $375 |
| C | 16% | $800 |

**15. Revenue**  A publishing company introduces a new weekly magazine that sells for $3.95. The marketing group of the company estimates that sales $x$ (in thousands) will be approximated by the probability function shown in the table.

| $x$ | 10 | 15 | 20 | 30 | 40 |
|-----|-----|-----|-----|-----|-----|
| $P(x)$ | 0.10 | 0.20 | 0.50 | 0.15 | 0.05 |

(a) Find $E(x)$.

(b) Find the expected revenue.

**16. Games of Chance**  A service organization is selling $5 raffle tickets as part of a fundraising program. The first and second prizes are $3000 and $1000, respectively. In addition to the first and second prizes, there are 50 $20 gift certificates to be awarded. The number of tickets sold is 2000. Find the expected net gain to the player when one ticket is purchased.

**17. Sales**  A consumer electronics retailer sells five different models of personal computers. During one month the sales for the five models were as shown.

Model 1     24 sold at $450 each

Model 2     12 sold at $460 each

Model 3     35 sold at $360 each

Model 4     5 sold at $1000 each

Model 5     4 sold at $1099 each

Find the variance and standard deviation of the prices.

**18. Inventory**  A discount retailer stocks multiple brands of digital cameras. The quantities and prices per camera are shown below.

Brand 1     30 cameras at $50 each

Brand 2     25 cameras at $60 each

Brand 3     20 cameras at $70 each

Brand 4     18 cameras at $85 each

Brand 5     12 cameras at $100 each

Find the variance and standard deviation of the prices.

**19. Consumer Trends**  A random survey of households recorded the number of cars per household. The results of the survey are shown in the table.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|-----|-----|-----|-----|-----|-----|
| $P(x)$ | 0.10 | 0.28 | 0.39 | 0.17 | 0.04 | 0.02 |

Find the variance and standard deviation of $x$.

**20. Vital Statistics**  The probability distribution for the numbers of children in a sample of families is shown in the table.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|-----|-----|-----|-----|-----|-----|
| $P(x)$ | 0.12 | 0.31 | 0.43 | 0.12 | 0.02 |

Find the variance and standard deviation of $x$.

(T) In Exercises 21–26, use a graphing utility to graph the function. Then determine whether the function $f$ represents a probability density function over the given interval. If $f$ is not a probability density function, identify the condition(s) that is (are) not satisfied.

**21.** $f(x) = \frac{1}{12}$,  $[0, 12]$

**22.** $f(x) = \frac{1}{8}$,  $[1, 8]$

**23.** $f(x) = \frac{1}{4}(3 - x)$,  $[0, 4]$

**24.** $f(x) = \frac{3}{4}x^2(2 - x)$,  $[0, 2]$

**25.** $f(x) = \dfrac{1}{4\sqrt{x}}$,  $[1, 9]$

**26.** $f(x) = 8.75x^{3/2}(1 - x)$,  $[0, 2]$

In Exercises 27–30, find the indicated probability for the probability density function.

**27.** $f(x) = \frac{1}{50}(10 - x)$,  $[0, 10]$

$P(0 < x < 2)$

**28.** $f(x) = \frac{1}{36}(9 - x^2)$,  $[-3, 3]$

$P(-1 < x < 2)$

**29.** $f(x) = \dfrac{2}{(x + 1)^2}$,  $[0, 1]$     **30.** $f(x) = \dfrac{3}{128}\sqrt{x}$,  $[0, 16]$

$P\left(0 < x < \dfrac{1}{2}\right)$          $P(4 < x < 9)$

**31. Waiting Time**  Buses arrive and depart from a college every 20 minutes. The probability density function of the waiting time $t$ (in minutes) for a person arriving at the bus stop is

$f(t) = \frac{1}{20}$,  $[0, 20]$.

Find the probabilities that the person will wait (a) no more than 10 minutes and (b) at least 15 minutes.

**32. Medicine**  The time $t$ (in days) until recovery after a certain medical procedure is a random variable with the probability density function

$f(t) = \dfrac{1}{4\sqrt{t - 4}}$,  $[5, 13]$.

Find the probability that a patient selected at random will take more than 8 days to recover.

In Exercises 33–36, find the mean of the probability density function.

**33.** $f(x) = \frac{1}{7}$,   $[0, 7]$

**34.** $f(x) = \dfrac{8 - x}{32}$,   $[0, 8]$

**35.** $f(x) = \frac{1}{6}e^{-x/6}$,   $[0, \infty)$

**36.** $f(x) = 0.3e^{-0.3x}$,   $[0, \infty)$

In Exercises 37–40, find the variance and standard deviation of the probability density function.

**37.** $f(x) = \frac{2}{9}x(3 - x)$,   $[0, 3]$

**38.** $f(x) = \frac{3}{16}\sqrt{x}$,   $[0, 4]$

**39.** $f(x) = \frac{1}{2}e^{-x/2}$,   $[0, \infty)$

**40.** $f(x) = 0.8e^{-0.8x}$,   $[0, \infty)$

In Exercises 41–44, find the median of the probability density function.

**41.** $f(x) = 6x(1 - x)$,   $[0, 1]$

**42.** $f(x) = 12x^2(1 - x)$,   $[0, 1]$

**43.** $f(x) = 0.25e^{-x/4}$,   $[0, \infty)$

**44.** $f(x) = \frac{5}{6}e^{-5x/6}$,   $[0, \infty)$

**45. Waiting Time**   The waiting time $t$ (in minutes) for service at the checkout at a grocery store is exponentially distributed with the probability density function

$f(t) = \frac{1}{15}e^{-t/15}$,   $[0, \infty)$.
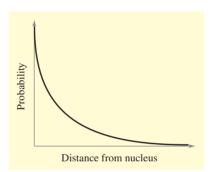
Find the probabilities of waiting (a) less than 10 minutes and (b) more than 10 minutes but less than 20 minutes.

**46. Useful Life**   The lifetime $t$ (in hours) of a mechanical unit is exponentially distributed with the density function

$f(t) = \frac{1}{350}e^{-t/350}$,   $[0, \infty)$.

Find the probability that a given unit chosen at random will perform satisfactorily for more than 400 hours.

**47. Botany**   In a botany experiment, plants are grown in a nutrient solution. The heights of the plants are found to be normally distributed with a mean of 42 centimeters and a standard deviation of 3 centimeters. Find the probability that a plant in the experiment is at least 50 centimeters tall.

**48. Wages**   The hourly wages for the workers at a certain company are normally distributed with a mean of $14.50 and a standard deviation of $1.40.

(a) What percent of the workers receive hourly wages from $13 to $15, inclusive?

(b) The highest 10% of the hourly wages are greater than what amount?

(T) **49. Heart Transplants**   Assume the waiting times for heart transplants are normally distributed with a mean of 130 days and a standard deviation of 25 days.   *(Source: Organ Procurement and Transplant Network)*

(a) Use a graphing utility to graph the distribution.

(b) Use a symbolic integration utility to approximate the probability that a waiting time is between 70 and 105 days.

(c) Use a symbolic integration utility to approximate the probability that a waiting time is more than 120 days.

(T) **50. Health**   Assume the heights of American men (from 20 to 29 years old) are normally distributed with a mean of 70 inches and a standard deviation of 3 inches.   *(Source: U.S. National Center for Health Statistics)*

(a) Use a graphing utility to graph the distribution.

(b) Use a symbolic integration utility to approximate the probability that a man's height is between 72 and 75 inches.

(c) Use a symbolic integration utility to approximate the probability that a man's height is less than 68 inches.

(T) **51. Meteorology**   The monthly rainfall $x$ in a certain state is normally distributed with a mean of 3.75 inches and a standard deviation of 0.5 inch. Use a computer or a graphing utility and Simpson's Rule (with $n = 12$) to approximate the probability that in a randomly selected month the rainfall is between 3.5 and 4 inches.

(B) **52. Chemistry: Hydrogen Orbitals**   In chemistry, the probability of finding an electron at a particular position is greatest close to the nucleus and drops off rapidly as the distance from the nucleus increases. The graph displays the probability of finding the electron at points along a line drawn from the nucleus outward in any direction for the hydrogen $1s$ orbital. Make a sketch of this graph, and add to your sketch an indication of where you think the median might be.   *(Source: Adapted from Zumdahl, Chemistry, Seventh Edition)*

## Chapter Test

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

1. A coin is tossed four times.

    (a) Write the sample space and frequency distribution for the possible outcomes.

    (b) What is the probability that at least two heads occur?

2. A card is chosen at random from a standard 52-card deck of playing cards. What is the probability that the card will be red and not a face card?

In Exercises 3 and 4, sketch a graph of the probability distribution and find the indicated probabilities.

3.

| $x$ | 1 | 2 | 3 | 4 |
|------|------|------|------|------|
| $P(x)$ | $\frac{3}{16}$ | $\frac{7}{16}$ | $\frac{1}{16}$ | $\frac{5}{16}$ |

4.

| $x$ | 7 | 8 | 9 | 10 | 11 |
|------|------|------|------|------|------|
| $P(x)$ | 0.21 | 0.13 | 0.19 | 0.42 | 0.05 |

(a) $P(x < 3)$   (b) $P(x \geq 3)$

(a) $P(7 \leq x \leq 10)$   (b) $P(x > 8)$

In Exercises 5 and 6, find $E(x)$, $V(x)$, and $\sigma$ for the given probability distribution.

5.

| $x$ | 0 | 1 | 2 | 3 |
|------|------|------|------|------|
| $P(x)$ | $\frac{2}{10}$ | $\frac{1}{10}$ | $\frac{4}{10}$ | $\frac{3}{10}$ |

6.

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|------|------|------|------|------|------|
| $P(x)$ | 0.141 | 0.305 | 0.257 | 0.063 | 0.234 |

(T) In Exercises 7–9, use a graphing utility to graph the function. Then determine whether the function represents a probability density function over the given interval. If $f$ is not a probability density function, identify the condition(s) that is (are) not satisfied.

7. $f(x) = \dfrac{\pi}{2}\sin \pi x, \quad [0, 1]$   8. $f(x) = \dfrac{3 - x}{6}, \quad [-1, 1]$   9. $f(x) = \dfrac{2x}{x^2 + 1}, \quad [0, \infty)$

In Exercises 10–12, find the indicated probabilities for the probability density function.

10. $f(x) = \dfrac{x}{32}, \quad [0, 8]$   (a) $P(1 \leq x \leq 4)$   (b) $P(3 \leq x \leq 6)$

11. $f(x) = 4(x - x^3), \quad [0, 1]$   (a) $P(0 < x < 0.5)$   (b) $P(0.25 \leq x < 1)$

12. $f(x) = 2xe^{-x^2}, \quad [0, \infty)$   (a) $P(x < 1)$   (b) $P(x \geq 1)$

In Exercises 13–15, find the mean, variance, and standard deviation of the probability density function.

13. $f(x) = \frac{1}{14}, \quad [0, 14]$   14. $f(x) = 3x - \frac{3}{2}x^2, \quad [0, 1]$

15. $f(x) = e^{-x}, \quad [0, \infty)$

(T) 16. An *intelligence quotient* or IQ is a number that is meant to measure intelligence. The IQs of students in a school are normally distributed with a mean of 110 and a standard deviation of 10. Use a symbolic integration utility to find the probability that a student selected at random will have an IQ within one standard deviation of the mean.

# 10

# Series and Taylor Polynomials



Newton's Method can be used to approximate the advertising expenditure that will yield a profit from sales of digital audio players. (See Section 10.6, Exercise 47.)

## Applications

Series and Taylor polynomials have many real-life applications. The applications listed below represent a sample of the applications in this chapter.

Justin Sullivan/Getty Images

## Section 10.1

# Sequences

■ Find the terms of sequences.
■ Determine the convergence or divergence of sequences and find the limits of convergent sequences.
■ Find patterns for sequences.
■ Use sequences to answer questions about real-life situations.

## Sequences

In mathematics, the word "sequence" is used in much the same way as in ordinary English. To say that a collection of objects or events is *in sequence* usually means that the collection is ordered so that it has an identified first member, second member, third member, and so on.

Mathematically, a sequence is defined as a function whose domain is the set of positive integers. Although a sequence is a function, it is common to represent sequences by subscript notation rather than by the standard function notation. For instance, the equation $a_n = 2^n$ defines the sequence below.

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \ldots, \quad a_n, \ldots$$
$$2, \quad 4, \quad 8, \quad 16, \quad \ldots, \quad 2^n, \ldots$$

### STUDY TIP

Occasionally it is convenient to begin subscripting a sequence with zero (or some other integer). In such cases, you can write

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

### Definition of Sequence

A **sequence** $\{a_n\}$ is a function whose domain is the set of positive integers. The function values $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the **terms** of the sequence. The number $a_n$ is the **$n$th term** of the sequence.

### *Example 1*   Finding Terms of a Sequence

Write the first four terms of each sequence.

**a.** $a_n = 2n + 1$      **b.** $b_n = \dfrac{3}{n+1}$

**SOLUTION**

**a.** The first four terms of the sequence whose $n$th term is $a_n = 2n + 1$ are

$$a_1, \qquad a_2, \qquad a_3, \qquad a_4$$
$$2(1) + 1 = 3, \quad 2(2) + 1 = 5, \quad 2(3) + 1 = 7, \quad 2(4) + 1 = 9.$$

**b.** The first four terms of the sequence whose $n$th term is $b_n = \dfrac{3}{n+1}$ are

$$b_1, \qquad b_2, \qquad b_3, \qquad b_4$$
$$\frac{3}{(1)+1} = \frac{3}{2}, \quad \frac{3}{(2)+1} = \frac{3}{3}, \quad \frac{3}{(3)+1} = \frac{3}{4}, \quad \frac{3}{(4)+1} = \frac{3}{5}.$$

### ✓ CHECKPOINT 1

Write the first four terms, starting with $n = 1$, of each sequence.

**a.** $a_n = 3n - 1$

**b.** $b_n = \dfrac{n}{n^2 + 1}$. ◾

## The Limit of a Sequence

The primary focus of this chapter is sequences whose terms approach limiting values. Such sequences are said to **converge.** If the limit of a sequence does not exist, then the sequence **diverges.** For instance, the terms of the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

approach 0 as $n$ increases. You can write this limit as $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{1}{2^n} = 0$.

Although there are technical differences, you can for the most part operate with limits of sequences just as you did with limits of continuous functions in Section 3.6. For instance, to evaluate the limit of the sequence whose $n$th term is

$$a_n = \frac{2n}{n + 1}$$

you can write

$$\lim\limits_{n \to \infty} \frac{2n}{n + 1} = \lim\limits_{n \to \infty} \frac{2}{1 + (1/n)} \qquad \text{Divide numerator and denominator by } n.$$

$$= \frac{2}{1 + 0} \qquad \text{Take limit as } n \to \infty.$$

$$= 2. \qquad \text{Limit of sequence}$$

### *Example 2* Finding the Limit of a Sequence

Find the limit of each sequence (if it exists) as $n$ approaches infinity.

**a.** $a_n = 3 + (-1)^n$ **b.** $a_n = \dfrac{n}{1 - 2n}$ **c.** $a_n = \dfrac{2^n}{2^n - 1}$

**SOLUTION**

**a.** The terms of the sequence whose $n$th term is $a_n = 3 + (-1)^n$ oscillate between 2 and 4.

$$a_1 = 2, \quad a_2 = 4, \quad a_3 = 2, \quad a_4 = 4, \ldots$$

So, the limit as $n \to \infty$ does not exist, and the sequence diverges.

**b.** The limit of the sequence whose $n$th term is $a_n = n/(1 - 2n)$ is

$$\lim\limits_{n \to \infty} \frac{n}{1 - 2n} = \lim\limits_{n \to \infty} \frac{1}{(1/n) - 2} \qquad \text{Divide numerator and denominator by } n.$$

$$= -\frac{1}{2}. \qquad \text{Take limit as } n \to \infty.$$

So, the sequence converges to $-\frac{1}{2}$.

**c.** The limit of the sequence whose $n$th term is $a_n = 2^n/(2^n - 1)$ is

$$\lim\limits_{n \to \infty} \frac{2^n}{2^n - 1} = \lim\limits_{n \to \infty} \frac{1}{1 - (1/2^n)} \qquad \text{Divide numerator and denominator by } 2^n.$$

$$= 1. \qquad \text{Take limit as } n \to \infty.$$

So, the sequence converges to 1.

**TECHNOLOGY**

(T) Symbolic algebra utilities are capable of evaluating the limit of a sequence. Use a symbolic algebra utility to evaluate the limits in Example 2.

✓ **CHECKPOINT 2**

Find the limit of each sequence (if it exists) as $n$ approaches infinity.

**a.** $a_n = \dfrac{n}{1 - n^2}$

**b.** $b_n = \dfrac{2^{n+1}}{2^n + 1}$ ∎

In this chapter you will learn that many important sequences in calculus involve factorials. If $n$ is a positive integer, then $n$ **factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n - 1) \cdot n.$$    $n$ factorial

As a special case, 0! is defined to be 1.

$$0! = 1$$
$$1! = 1$$
$$2! = 1 \cdot 2 = 2$$
$$3! = 1 \cdot 2 \cdot 3 = 6$$
$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

Factorials follow the same conventions for order of operations as exponents. That is, just as $2x^3$ and $(2x)^3$ imply different orders of operations, $2n!$ and $(2n)!$ imply different orders, as shown.

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)$$
$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \cdot (n + 1) \cdots (2n)$$

Try evaluating $n!$ for several values of $n$. You will find that $n$ does not have to be very large before $n!$ becomes huge. For instance, $10! = 3,628,800$.

**Example 3**    **Finding the Limit of a Sequence**

Find the limit of the sequence whose $n$th term is $a_n = \dfrac{(-1)^n}{n!}$.

**SOLUTION**    One way to determine the limit is to write several terms of the sequence and look for a pattern.

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \quad a_6$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$-\frac{1}{1}, \quad \frac{1}{2}, \quad -\frac{1}{6}, \quad \frac{1}{24}, \quad -\frac{1}{120}, \quad \frac{1}{720}$$

From these terms, it is clear that the denominator is increasing without bound while the numerator is bounded. So, you can write

$$\lim_{n \to \infty} \frac{(-1)^n}{n!} = 0.$$

This result is shown graphically in Figure 10.1. Note that the terms of the sequence oscillate between positive and negative values.



**FIGURE 10.1**

$a_n = \dfrac{(-1)^n}{n!}$

**✓CHECKPOINT 3**

Find the limit of the sequence whose $n$th term is
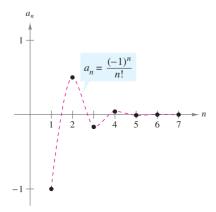
$$a_n = \frac{(-1)^{n+1}}{(n + 1)!}. \quad \blacksquare$$

**STUDY TIP**

Try using a symbolic algebra utility to evaluate the limit in Example 3.

## Pattern Recognition for Sequences

Sometimes the terms of a sequence are generated by a rule that does not explicitly identify the $n$th term of the sequence. In such cases, you need to discover a *pattern* in the sequence and find a formula for the $n$th term.

### Example 4 Finding a Pattern for a Sequence

Consider the function given by $f(x) = e^{x/3}$. Determine the convergence or divergence of the sequence whose $n$th term is

$$a_n = f^{(n-1)}(0)$$

where $f^{(0)}(x) = f(x)$ and $f^{(n)}$ is the $n$th derivative of $f$.

**SOLUTION** Begin by calculating several derivatives of $f$ and evaluating the results for $x = 0$, as shown in the table.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | $\cdots$ | $n$ |
|---|---|---|---|---|---|---|---|---|
| $f^{(n-1)}(x)$ | $e^{x/3}$ | $\dfrac{e^{x/3}}{3}$ | $\dfrac{e^{x/3}}{3^2}$ | $\dfrac{e^{x/3}}{3^3}$ | $\dfrac{e^{x/3}}{3^4}$ | $\dfrac{e^{x/3}}{3^5}$ | $\cdots$ | $\dfrac{e^{x/3}}{3^{n-1}}$ |
| $f^{(n-1)}(0)$ | 1 | $\dfrac{1}{3}$ | $\dfrac{1}{3^2}$ | $\dfrac{1}{3^3}$ | $\dfrac{1}{3^4}$ | $\dfrac{1}{3^5}$ | $\cdots$ | $\dfrac{1}{3^{n-1}}$ |

From this table, you can see that the pattern is

$$a_n = \frac{1}{3^{n-1}}.$$

So, you can write

$$\lim_{n\to\infty} \frac{1}{3^{n-1}} = 0$$

which means that the sequence converges to zero.

### ✓ CHECKPOINT 4

Complete a table similar to the one shown in Example 4 to find the first four terms of the sequence whose $n$th term is $a_n = f^{(n-1)}(0)$ where $f(x) = e^{2x}$, $f^{(0)} = f(x)$, and $f^{(n)}$ is the $n$th derivative of $f$. Then determine the convergence or divergence of the sequence. ■

Searching for a pattern for the $n$th term of a sequence can be difficult. It helps to consider the patterns below.

| *nth Term* | *Terms* | *Type of Sequence* |
|---|---|---|
| $(-1)^n$ | $-1, 1, -1, 1, -1, 1, -1, 1, \ldots$ | Changes in sign |
| $(-1)^{n+1}$ | $1, -1, 1, -1, 1, -1, 1, -1, \ldots$ | Changes in sign |
| $an + b$ | $a + b, 2a + b, 3a + b, 4a + b, \ldots$ | Arithmetic |
| $ar^{n-1}$ | $a, ar, ar^2, ar^3, ar^4, \ldots$ | Geometric |
| $n!$ | $1, 2, 6, 24, 120, 720, \ldots$ | Factorial |
| $n^p$ | $1, 2^p, 3^p, 4^p, 5^p, 6^p, \ldots$ | Power |

Determine the first eight terms of the sequence whose $n$th term is

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

What is the limit of this sequence? How does this sequence relate to the definition of the number $e$ given on page 299?

Without a specific rule for the $n$th term of a sequence, it is not possible to determine the convergence or divergence of the sequence; knowing the first several terms is not enough. For instance, the first three terms of the four sequences below are identical. Yet, from their $n$th terms, you can determine that two of the sequences converge to zero, one converges to $\frac{1}{9}$, and one diverges.

$$\{a_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots\right\}$$

$$\{b_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2 - n + 6)}, \ldots\right\}$$

$$\{c_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{7}{62}, \ldots, \frac{n^2 - 3n + 3}{9n^2 - 25n + 18}, \ldots\right\}$$

$$\{d_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0, \ldots, \frac{-n(n+1)(n-4)}{6(n^2 + 3n - 2)}, \ldots\right\}$$

So, if only the first several terms of a sequence are given, there are many possible patterns that can be used to write a formula for the $n$th term. In such a situation, remember that your decision as to whether the sequence converges or diverges depends on your description of the $n$th term.

---

**Example 5**   **Finding a Pattern for a Sequence**

Determine an $n$th term for the sequence

$$-\frac{1}{1}, \frac{3}{2}, -\frac{7}{6}, \frac{15}{24}, -\frac{31}{120}, \ldots$$

**SOLUTION**   Begin by observing that the numerators are 1 less than $2^n$. So, you can generate the numerators by the rule

$$2^n - 1, \quad n = 1, \quad 2, \quad 3, \quad 4, \quad 5, \ldots$$

Factoring the denominators produces

$$1 = 1!$$
$$2 = 1 \cdot 2 = 2!$$
$$6 = 1 \cdot 2 \cdot 3 = 3!$$
$$24 = 1 \cdot 2 \cdot 3 \cdot 4 = 4!$$
$$120 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5!$$

So, the denominators can be represented by $n!$. Finally, because the signs alternate, you can write

$$a_n = (-1)^n\left(\frac{2^n - 1}{n!}\right)$$

as one possible formula for the $n$th term of this sequence.

**✓ CHECKPOINT 5**

Determine an $n$th term for the sequence $\dfrac{1}{2}, -\dfrac{4}{6}, \dfrac{9}{24}, -\dfrac{16}{120}, \ldots$   ■

## Application

There are many applications of sequences in business and economics. The next example involves the balance in an account for which the interest is compounded monthly. The terms of the sequence are the balances at the end of the first month, the end of the second month, and so on.

### Example 6  Finding Balances  (R)

Interest rates on savings accounts are related to inflation rates. Throughout most of the twentieth century, the United States experienced very little inflation. In fact, from 1914 to 1939, the average annual inflation rate was only 1.6%. From 1940 to 1969, the average annual inflation rate was about 3.4%, from 1970 to 1982, the average annual inflation rate was 7.8%, and from 1983 to 2007, the average annual inflation rate was 3.1%.

A deposit of $1000 is made in an account that earns 6% interest, compounded monthly. Find a sequence that represents the monthly balances.

**SOLUTION**   Because an annual interest rate of 6% compounded monthly corresponds to a monthly rate of 0.5%, the balance after 1 month is

$$A_1 = 1000 + 1000(0.005) = 1000(1.005).$$

After 2 months, the balance is

$$A_2 = 1000(1.005)(1.005) = 1000(1.005)^2.$$

Continuing this pattern, you can determine that the balance after $n$ months is

$$A_n = 1000(1.005)^n.$$

This implies that the first several terms of the sequence are

$$1000(1.005), 1000(1.005)^2, 1000(1.005)^3, 1000(1.005)^4, \ldots$$

or

$$\$1005.00, \$1010.03, \$1015.08, \$1020.15, \ldots$$

Note that this sequence is of the form $P[1 + (r/12)]^n$, which agrees with the formula for the balance in an account.

### ✓ CHECKPOINT 6

Find the sequence for the balance in Example 6 if the interest is compounded quarterly.  ■

---

### CONCEPT CHECK

1. What is the domain of a sequence?
2. Complete the following: The function values $a_1, a_2, a_3, a_4, \ldots a_n, \ldots$ are called the _____ of a sequence.
3. The terms of a sequence approach a limiting value. Does the sequence converge or diverge?
4. The terms of a sequence have no limit. Does the sequence converge or diverge?

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.4, 0.5, 1.5, and 3.6.

In Exercises 1–6, find the limit.

**1.** $\displaystyle \lim_{x \to \infty} \frac{1}{x^3}$

**2.** $\displaystyle \lim_{x \to \infty} \frac{1}{x^2 - 4}$

**3.** $\displaystyle \lim_{x \to \infty} \frac{2x^2}{x^2 + 1}$

**4.** $\displaystyle \lim_{x \to \infty} \frac{x^3 - 1}{x^2 + 2}$

**5.** $\displaystyle \lim_{x \to \infty} e^{-2x}$

**6.** $\displaystyle \lim_{x \to \infty} \frac{1}{2^{x-1}}$

In Exercises 7–10, simplify the expression.

**7.** $\dfrac{n^2 - 4}{n^2 + 2n}$

**8.** $\dfrac{n^2 + n - 12}{n^2 - 16}$

**9.** $\dfrac{3}{n} + \dfrac{1}{n^3}$

**10.** $\dfrac{1}{n - 1} + \dfrac{1}{n + 2}, \quad n \geq 2$

## Exercises 10.1

In Exercises 1–10, write the first five terms of the sequence.

**1.** $a_n = 2n - 1$

**2.** $a_n = 5n + 2$

**3.** $a_n = 3^n$

**4.** $a_n = \left(-\frac{1}{2}\right)^n$

**5.** $a_n = \dfrac{n}{n + 1}$

**6.** $a_n = \dfrac{n - 1}{n^2 + 2}$

**7.** $a_n = \dfrac{3^n}{n!}$

**8.** $a_n = \dfrac{3n!}{(n - 1)!}$

**9.** $a_n = \dfrac{(-1)^n}{n^2}$

**10.** $a_n = 5 - \dfrac{1}{n} + \dfrac{1}{n^2}$

In Exercises 11–22, determine the convergence or divergence of the sequence. If the sequence converges, find its limit.

**11.** $a_n = \dfrac{5}{n}$

**12.** $a_n = \dfrac{n}{2}$

**13.** $a_n = \dfrac{n + 1}{n}$

**14.** $a_n = \dfrac{1}{n^{3/2}}$

**15.** $a_n = \dfrac{n^2 + 3n - 4}{2n^2 + n - 3}$

**16.** $a_n = \dfrac{\sqrt{n}}{\sqrt{n} + 1}$

**17.** $a_n = \dfrac{n^2 - 25}{n + 5}$

**18.** $a_n = \dfrac{n + 2}{n^2 + 1}$

**19.** $a_n = \dfrac{1 + (-1)^n}{n}$

**20.** $a_n = 1 + (-1)^n$

**21.** $a_n = \dfrac{n!}{n}$

**22.** $a_n = \dfrac{n!}{(n + 1)!}$

(T) In Exercises 23–32, determine the convergence or divergence of the sequence. If the sequence converges, use a symbolic algebra utility to find its limit.

**23.** $a_n = 3 - \dfrac{1}{2^n}$

**24.** $a_n = 5 - \dfrac{1}{4^n}$

**25.** $a_n = \dfrac{n}{n^2 + 1}$

**26.** $a_n = \dfrac{n + 1}{n^2 - 3}$

**27.** $a_n = \dfrac{3^n}{4^n}$

**28.** $a_n = (0.5)^n$
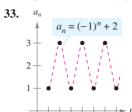
**29.** $a_n = \dfrac{(n + 1)!}{n!}$

**30.** $a_n = \dfrac{(n - 2)!}{n!}$
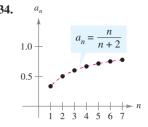
**31.** $a_n = (-1)^n \left(\dfrac{n}{n + 1}\right)$

**32.** $a_n = (-1)^n \dfrac{n}{n^2 + 1}$

In Exercises 33 and 34, use the graph of the sequence to decide whether the sequence converges or diverges. Then verify your result analytically.

**33.**



$a_n = (-1)^n + 2$

**34.**



$a_n = \dfrac{n}{n + 2}$

In Exercises 35–48, write an expression for the $n$th term of the sequence. (There is more than one correct answer.)

**35.** $1, 4, 7, 10, \ldots$

**36.** $3, 7, 11, 15, \ldots$

**37.** $-1, 4, 9, 14, \ldots$

**38.** $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$

**39.** $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

**40.** $2, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

**41.** $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$

**42.** $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \ldots$

**43.** $2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, \ldots$

**44.** $1 + \frac{1}{2}, 1 + \frac{1}{4}, 1 + \frac{1}{8}, 1 + \frac{1}{16}, \ldots$

**45.** $-2, 2, -2, 2, \ldots$    **46.** $2, -4, 6, -8, 10, \ldots$

**47.** $-x, \frac{x^2}{2}, -\frac{x^3}{3}, \frac{x^4}{4}, \ldots$    **48.** $1, x, \frac{x^2}{2}, \frac{x^3}{6}, \frac{x^4}{24}, \frac{x^5}{120}, \ldots$

In Exercises 49–52, write the next two terms of the arithmetic sequence. Describe the pattern you used to find these terms.

**49.** $2, 5, 8, 11, \ldots$    **50.** $\frac{7}{2}, 4, \frac{9}{2}, 5, \ldots$

**51.** $1, \frac{5}{3}, \frac{7}{3}, 3, \ldots$    **52.** $\frac{1}{2}, \frac{5}{4}, 2, \frac{11}{4}, \ldots$

In Exercises 53–56, write the next two terms of the geometric sequence. Describe the pattern you used to find these terms.

**53.** $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$    **54.** $5, 10, 20, 40, \ldots$

**55.** $2, 6, 18, 54, \ldots$    **56.** $9, 6, 4, \frac{8}{3}, \ldots$

In Exercises 57–60, determine whether the sequence is arithmetic or geometric, and write the $n$th term of the sequence.

**57.** $20, 10, 5, \frac{5}{2}, \ldots$    **58.** $100, 92, 84, 76, \ldots$

**59.** $\frac{8}{3}, \frac{10}{3}, 4, \frac{14}{3}, \ldots$    **60.** $378, -126, 42, -14, \ldots$

In Exercises 61 and 62, give an example of a sequence satisfying the given condition. (There is more than one correct answer.)

**61.** A sequence that converges to $\frac{3}{4}$

**62.** A sequence that converges to 100

**63. Compound Interest** Consider the sequence $\{A_n\}$, whose $n$th term is given by

$$A_n = P\left[1 + \frac{r}{12}\right]^n$$

where $P$ is the principal, $A_n$ is the amount of compound interest after $n$ months, and $r$ is the annual percentage rate. Write the first 10 terms of the sequence for $P = \$9000$ and $r = 0.06$.

**64. Compound Interest** Consider the sequence $\{A_n\}$, whose $n$th term is given by

$$A_n = P(1 + r)^n$$

where $P$ is the principal, $A_n$ is the amount of compound interest after $n$ years, and $r$ is the annual percentage rate. Write the first 10 terms of the sequence for $P = \$5000$ and $r = 0.08$.

**65. Individual Retirement Account** A deposit of $2000 is made each year in an account that earns 11% interest compounded annually. The balance after $n$ years is given by $A_n = 2000(11)[(1.1)^n - 1]$.

(a) Compute the first six terms of the sequence.

(b) Find the balance after 20 years by finding the 20th term of the sequence.

(T) (c) Use a symbolic algebra utility to find the balance after 40 years by finding the 40th term of the sequence.

**66. Investment** A deposit of $100 is made each month in an account that earns 6% interest, compounded monthly. The balance in the account after $n$ months is given by

$$A_n = 100(201)[(1.005)^n - 1].$$

(a) Compute the first six terms of this sequence.

(b) Find the balance after 5 years by computing the 60th term of the sequence.

(c) Find the balance after 20 years by computing the 240th term of the sequence.

**67. Population Growth** Consider an idealized population with the characteristic that each population member produces one offspring at the end of every time period. If each population member has a lifespan of three time periods and the population begins with 10 newborn members, then the table shown below gives the populations during the first five time periods.

| | Time period | | | | |
|---|---|---|---|---|---|
| Age Bracket | 1 | 2 | 3 | 4 | 5 |
| 0–1 | 10 | 10 | 20 | 40 | 70 |
| 1–2 | | 10 | 10 | 20 | 40 |
| 2–3 | | | 10 | 10 | 20 |
| Total | 10 | 20 | 40 | 70 | 130 |

The sequence for the total population has the property that

$$S_n = S_{n-1} + S_{n-2} + S_{n-3}, \quad n > 3.$$

Find the total population during the next five time periods.

**68. Carbon Dioxide** The average concentration levels of carbon dioxide ($CO_2$) in Earth's atmosphere for selected years since 1980, in parts per million of carbon dioxide, are shown in the table. *(Source: NOAA)*

| $n$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $a_n$ | 338.7 | 345.3 | 353.8 | 359.9 | 368.8 | 378.8 |

(T) (a) Use the *regression* feature of a graphing utility to find a model of the form $a_n = kn + b$ for the data. Let $n$ represent the year, with $n = 0$ corresponding to 1980. Use a graphing utility to plot the points and graph the model.

(b) Use the model to predict the average concentration level of $CO_2$ in the year 2015.

**69. Cost**  For a family of four, the average costs per week to buy food from 2000 through 2006 are shown in the table, where $a_n$ is the average cost in dollars and $n$ is the year, with $n = 0$ corresponding to 2000. *(Source: U.S. Department of Agriculture)*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $a_n$ | 161.3 | 168.0 | 171.0 | 174.6 | 184.2 | 187.1 | 190.4 |

(T) (a) Use the *regression* feature of a graphing utility to find a model of the form

$$a_n = kn + b, \quad n = 0, 1, 2, 3, 4, 5, 6$$

for the data. Use a graphing utility to plot the points and graph the model.

(b) Use the model to predict the cost in the year 2012.

**70. Cost**  The average costs per day for a hospital room from 1998 through 2004 are shown in the table, where $a_n$ is the average cost in dollars and $n$ is the year, with $n = 1$ corresponding to 1998. *(Source: Health Forum)*

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $a_n$ | 1067 | 1103 | 1149 | 1217 | 1290 | 1379 | 1450 |

(T) (a) Use the *regression* feature of a graphing utility to find a model of the form $a_n = kn + b$, $n = 1, 2, 3, 4, 5, 6, 7$, for the data. Use a graphing utility to plot the points and graph the model.

(b) Use the model to predict the cost in the year 2013.

**71. Federal Debt**  It took more than 200 years for the United States to accumulate a $1 trillion debt. Then it took just 8 years to get to $3 trillion. The federal debt during the years 1990 through 2005 is approximated by the model

$$a_n = 0.003n^3 - 0.07n^2 + 0.63n + 3.08,$$
$$n = 0, 1, 2, 3, \ldots, 15$$

where $a_n$ is the debt in trillions and $n$ is the year, with $n = 0$ corresponding to 1990. *(Source: U.S. Office of Management and Budget)*

(a) Write the terms of this finite sequence.

(b) Construct a bar graph that represents the sequence.

**72. Physical Science**  A ball is dropped from a height of 12 feet, and on each rebound it rises to $\frac{2}{3}$ its preceding height.

(a) Write an expression for the height of the $n$th rebound.

(b) Determine the convergence or divergence of this sequence. If it converges, find the limit.

**73. Budget Analysis**  A government program that currently costs taxpayers $1.3 billion per year is to be cut back by 15% per year.

(a) Write an expression for the amount budgeted for this program after $n$ years.

(b) Compute the budget amounts for the first 4 years.

(c) Determine the convergence or divergence of the sequence of reduced budgets. If the sequence converges, find its limit.

**74. Inflation Rate**  If the average price of a new car increases 2.5% per year and the average price is currently $28,400, then the average price after $n$ years is $P_n = \$28,400(1.025)^n$. Compute the average prices for the first 5 years of increases.

**75. Cost**  A well-drilling company charges $25 for drilling the first foot of a well, $25.10 for drilling the second foot, $25.20 for the third foot, and so on. Determine the cost of drilling a 100-foot well.

**76. Salary**  A person accepts a position with a company at a salary of $32,800 for the first year. The person is guaranteed a raise of 5% per year for the next 3 years. Determine the person's salary during the fourth year of employment.

**77. Sales**  The sales $a_n$ (in billions of dollars) of Wal-Mart from 1996 through 2005 are shown below as ordered pairs of the form $(n, a_n)$, where $n$ is the year, with $n = 1$ corresponding to 1996. *(Source: Wal-Mart Stores, Inc.)*

(1, 104.859), (2, 117.958), (3, 137.634), (4, 165.013), (5, 191.329), (6, 217.799), (7, 244.524), (8, 256.329), (9, 285.222), (10, 312.427)

(T) (a) Use the *regression* feature of a graphing utility to find a model of the form $a_n = bn^3 + cn^2 + dn + f$, $n = 1, 2, 3, \ldots, 10$, for the data. Graphically compare the points and the model.

(b) Use the model to predict the sales in the year 2012.

(B) **78. Biology**  Suppose that you have a single bacterium able to divide to form two new cells every half hour. At the end of the first half hour there are two individuals, at the end of the first hour there are four individuals, and so on.

(a) Write an expression for the $n$th term of the sequence.

(b) How many bacteria will there be after 10 hours? After 20 hours? *(Source: Adapted from Levine/Miller, Biology: Discovering Life, Second Edition)*

**79. Think About It**  Consider the sequence whose $n$th term $a_n$ is given by

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Demonstrate that the terms of this sequence approach $e$ by finding $a_1, a_{10}, a_{100}, a_{1000}$, and $a_{10,000}$.

**80. Extended Application**  To work an extended application analyzing the numbers of operating federal credit unions from 1991 through 2005, visit this text's website at *college.hmco.com*. *(Data Source: National Credit Union Administration)*

## Section 10.2

# Series and Convergence

- Write finite sums using sigma notation.
- Find the partial sums of series and determine the convergence or divergence of infinite series.
- Use the *n*th-Term Test for Divergence to show that series diverge.
- Find the *n*th partial sums of geometric series and determine the convergence or divergence of geometric series.
- Use geometric series to model and solve real-life problems.

## Sigma Notation

In this section, you will study infinite summations. The decimal representation of $\frac{1}{3}$ is a simple example of an infinite summation.

$$\frac{1}{3} = 0.33333 \ldots$$

$$= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots$$

$$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{3}{10^n}$$

The last notation is called **sigma notation** or **summation notation.**

**Algebra Review**

For help in rewriting expressions that use sigma notation, see Example 2 in the *Chapter 10 Algebra Review*, on page 721.

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**Sigma Notation**

The finite sum $a_1 + a_2 + a_3 + \cdots + a_n$ can be written as

$$\sum_{i=1}^{n} a_i.$$

The letter $i$ is the **index of summation,** and 1 and $n$ are the lower and upper limits of summation, respectively.

---

*Example 1*  **Using Sigma Notation**

| *Sum* | *Σ Notation* |
|---|---|
| **a.** $1 + 2 + 3 + 4 + 5 + 6$ | $\sum_{i=1}^{6} i$ |
| **b.** $3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots + 3\left(\frac{1}{2}\right)^n$ | $\sum_{k=0}^{n} 3\left(\frac{1}{2}\right)^k$ |

✓**CHECKPOINT 1**

Use sigma notation to write the sum

$$4\left(-\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) + 4\left(-\frac{1}{8}\right) + 4\left(\frac{1}{16}\right). \quad \blacksquare$$

## Infinite Series

The infinite summation

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

is called an **infinite series.** The **sequence of partial sums** of the series is denoted by

$$S_1 = a_1, \; S_2 = a_1 + a_2, \; S_3 = a_1 + a_2 + a_3, \ldots .$$

### Convergence and Divergence of an Infinite Series

Consider the infinite series $a_1 + a_2 + a_3 + \cdots$. If the sequence of partial sums $\{S_n\}$ converges to $S$, then the infinite series **converges** to $S$. This limit is denoted by

$$\lim_{n \to \infty} S_n = \sum_{n=1}^{\infty} a_n = S$$

and $S$ is called the **sum of the series.** If the limit of the sequence of partial sums $\{S_n\}$ does not exist, then the series **diverges.**

*Example 2*   **Determining Convergence and Divergence**

Determine whether each series converges or diverges.

**a.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$    **b.** $\displaystyle\sum_{n=1}^{\infty} 1$

**SOLUTION**

**a.** The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has the partial sums listed below.

$$S_1 = \frac{1}{2}, \quad S_2 = \frac{3}{4}, \quad S_3 = \frac{7}{8}, \quad S_4 = \frac{15}{16}, \ldots, \quad S_n = \frac{2^n - 1}{2^n}$$

Because the limit of $\{S_n\}$ is 1, it follows that the infinite series converges and its sum is 1. So, you can write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

**b.** The infinite series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$

diverges because the sequence of partial sums $\{S_n\}$ diverges.

✓**CHECKPOINT 2**

The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{4^n}$$

has partial sums

$$S_1 = \frac{1}{4},$$

$$S_2 = \frac{5}{16},$$

$$S_3 = \frac{21}{64}, \ldots,$$

$$S_n = \frac{1}{3}\left( \frac{4^n - 1}{4^n} \right).$$

Determine the convergence or divergence of the series. ■

The properties below are useful in determining the sums of infinite series.

---

**Properties of Infinite Series**

For the convergent infinite series

$$\sum_{n=1}^{\infty} a_n = A \quad \text{and} \quad \sum_{n=1}^{\infty} b_n = B$$

the properties below are true.

**1.** $\displaystyle\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n = cA$

**2.** $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$

---

### *DISCOVERY*

Determine the first eight terms of the sequence of partial sums for the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{n!}.$$

What is the limit of this sequence of partial sums?

*Example 3*   **Using Properties of Infinite Series**

Find the sum of each infinite series.

**a.** $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2^n}$

**b.** $\displaystyle\sum_{n=1}^{\infty} \left( \frac{1}{2^n} + \frac{1}{2^{n+1}} \right)$

**SOLUTION**   For each infinite series, use the result of Example 2(a).

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

**a.** Using Property 1 of infinite series, you can write

$$\sum_{n=1}^{\infty} \frac{5}{2^n} = 5 \sum_{n=1}^{\infty} \frac{1}{2^n} = 5(1) = 5.$$

**b.** Begin by noting that

$$\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{2} \left( \frac{1}{2^n} \right) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}(1) = \frac{1}{2}.$$

Now, you can use Property 2 of infinite series to write

$$\sum_{n=1}^{\infty} \left( \frac{1}{2^n} + \frac{1}{2^{n+1}} \right) = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = 1 + \frac{1}{2} = \frac{3}{2}.$$

✓**CHECKPOINT 3**

Use the properties of infinite series to find the sum of each infinite series.

**a.** $\displaystyle\sum_{n=1}^{\infty} \frac{4}{2^n}$     **b.** $\displaystyle\sum_{n=1}^{\infty} \left( \frac{1}{2^n} - \frac{1}{2^{n+1}} \right)$

## The *n*th-Term Test for Divergence

The two primary questions regarding an infinite series are as shown.

**1.** Does the series converge or does it diverge?

**2.** If the series converges, to what value does it converge?

A simple test for divergence gives a partial answer to the first question.

**nth-Term Test for Divergence**

Consider the infinite series $\sum_{n=1}^{\infty} a_n$. If

$$\lim_{n \to \infty} a_n \neq 0$$

then the series diverges.

*Example 4* **Testing for Divergence**

Use the *n*th-Term Test to determine whether each series diverges.

**a.** $\sum_{n=1}^{\infty} 2^n$   **b.** $\sum_{n=1}^{\infty} \frac{1}{2^n}$   **c.** $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$

**SOLUTION**

**a.** By the *n*th-Term Test, the infinite series

$$\sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + 16 + \cdots$$

diverges because

$$\lim_{n \to \infty} 2^n = \infty.$$

**b.** The *n*th-Term Test tells you nothing about the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

because

$$\lim_{n \to \infty} \frac{1}{2^n} = 0.$$

From Example 2(a), you know that this series converges. The point here is that you cannot deduce this from the *n*th-Term Test.

**c.** The infinite series

$$\sum_{n=1}^{\infty} \frac{n!}{2n! + 1} = \frac{1}{3} + \frac{2}{5} + \frac{6}{13} + \frac{24}{49} + \frac{120}{241} + \cdots$$

diverges because

$$\lim_{n \to \infty} \frac{n!}{2n! + 1} = \lim_{n \to \infty} \frac{1}{2 + (1/n!)} = \frac{1}{2 + 0} = \frac{1}{2}.$$

✓ **CHECKPOINT 4**

Use the *n*th-Term Test to determine whether each series diverges.

**a.** $\sum_{n=1}^{\infty} \frac{2^n}{2^{n+1} + 1}$

**b.** $\sum_{n=1}^{\infty} \frac{n^2}{n^2 + 1}$ ■

## Geometric Series

If $a$ is a nonzero real number, then the infinite series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdot \cdot \cdot + ar^n + \cdot \cdot \cdot$$     Geometric series

is called a **geometric series** with ratio $r$.

---

**$n$th Partial Sum of a Geometric Series**

The $n$th partial sum of the geometric series $\sum_{n=0}^{\infty} ar^n$ is

$$S_n = \frac{a(1 - r^{n+1})}{1 - r}, \quad r \neq 1.$$

---

*Example 5*  **Finding an $n$th Partial Sum**

Find the third, fifth, and tenth partial sums of the geometric series

$$\sum_{n=0}^{\infty} 3 \left( \frac{1}{4} \right)^n = 3 + \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdot \cdot \cdot .$$

**SOLUTION**     For this geometric series, $a = 3$ and $r = \frac{1}{4}$. Because the index begins with $n = 0$, the $n$th partial sum is

$$S_n = \frac{a(1 - r^{n+1})}{1 - r}$$

$$= \frac{3[1 - (1/4)^{n+1}]}{1 - (1/4)}$$

$$= \frac{3[1 - (1/4)^{n+1}]}{3/4}$$

$$= 4 \left[ 1 - \left( \frac{1}{4} \right)^{n+1} \right]$$

$$= 4 - \left( \frac{1}{4} \right)^n .$$

Using this formula, you can find the third, fifth, and tenth partial sums as shown.

$$S_3 = 4 - \left( \tfrac{1}{4} \right)^3 \approx 3.984$$     Third partial sum

$$S_5 = 4 - \left( \tfrac{1}{4} \right)^5 \approx 3.999$$     Fifth partial sum

$$S_{10} = 4 - \left( \tfrac{1}{4} \right)^{10} \approx 4.000$$     Tenth partial sum

✓ **CHECKPOINT 5**

Find the 5th, 50th, and 500th partial sums of the geometric series

$$\sum_{n=0}^{\infty} 5 \left( \frac{1}{10} \right)^n . \quad \blacksquare$$

When applying the formula for the $n$th partial sum of a geometric series, be sure to check that the index begins with $n = 0$. If it begins with some other number, you will have to adjust $S_n$ accordingly. Here is an example.

$$\sum_{n=1}^{10} ar^n = -a + \sum_{n=0}^{10} ar^n = -a + \frac{a(1 - r^{11})}{1 - r}$$

The same type of adjustment is used in the next example.

**Example 6** **Finding an Annuity Balance** Ⓡ

A deposit of $50 is made every month for 2 years in a savings account that pays 6%, compounded monthly. What is the balance in the account at the end of 2 years?

**SOLUTION** Using the formula for compound interest from Section 4.2, after 24 months, the money that was deposited the first month will have become

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A_{24} = 50\left(1 + \frac{0.06}{12}\right)^{24} = 50(1.005)^{24}.$$

Similarly, after 23 months, the money deposited the second month will have become

$$A_{23} = 50\left(1 + \frac{0.06}{12}\right)^{23} = 50(1.005)^{23}.$$

Continuing this process, you can find that the total balance resulting from the 24 deposits will be

$$A = A_1 + A_2 + \cdots + A_{24}$$

$$= \sum_{n=1}^{24} A_n = \sum_{n=1}^{24} 50(1.005)^n.$$

Noting that the index begins with $n = 1$, you can use the formula for the $n$th partial sum to find the balance.

$$A = \sum_{n=1}^{24} 50(1.005)^n$$

$$= -50 + \sum_{n=0}^{24} 50(1.005)^n$$

$$= -50 + \frac{50(1 - 1.005^{25})}{1 - 1.005} \approx \$1277.96$$

The growth of this account is shown graphically in Figure 10.2.

**Growth of a Savings Account**

$$A = -50 + \frac{50(1 - 1.005^{t+1})}{1 - 1.005}$$
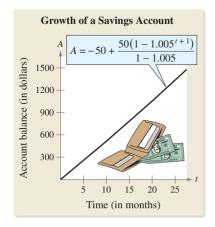


**FIGURE 10.2**

✓**CHECKPOINT 6**

A deposit of $20 is made every month for 4 years in an account that pays 3% compounded monthly. What is the balance in the account at the end of the 4 years? ▪

**Convergence of an Infinite Geometric Series**

An infinite geometric series given by

$$\sum_{n=0}^{\infty} ar^n$$

diverges if $|r| \geq 1$. If $|r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad |r| < 1.$$

**Example 7**   **Determining Convergence and Divergence**

Decide whether each series converges or diverges.

**a.** $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$    **b.** $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$    **c.** $\sum_{n=1}^{\infty} \frac{4}{3^n}$

**SOLUTION**

**a.** For this infinite geometric series, $a = 1$ and $r = -\frac{1}{2}$. Because $|r| < 1$, it follows that the series converges. Moreover, because the index begins with $n = 0$, you can apply the formula for the sum of an infinite geometric series to conclude that

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{a}{1 - r} = \frac{1}{1 - (-1/2)}$$
$$= \frac{1}{3/2}$$
$$= \frac{2}{3}.$$

**b.** For this infinite geometric series, $a = 1$ and $r = \frac{3}{2}$. Because $|r| > 1$, it follows that the series diverges.

**c.** By rewriting this infinite geometric series as

$$\sum_{n=1}^{\infty} 4\left(\frac{1}{3}\right)^n$$

you can see that $a = 4$ and $r = \frac{1}{3}$. Because $|r| < 1$, the series converges. To find the sum of the series, note that the index begins with $n = 1$, then adjust the formula for the sum as shown.

$$\sum_{n=1}^{\infty} 4\left(\frac{1}{3}\right)^n = -4 + \sum_{n=0}^{\infty} 4\left(\frac{1}{3}\right)^n$$
$$= -4 + \frac{4}{1 - (1/3)}$$
$$= -4 + \frac{4}{2/3}$$
$$= -4 + 6$$
$$= 2$$

So, the series converges to 2.

**✓ CHECKPOINT 7**

Decide whether each series converges or diverges.

**a.** $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$

**b.** $\sum_{n=0}^{\infty} 4\left(\frac{3}{2}\right)^n$

**c.** $\sum_{n=1}^{\infty} \frac{5}{4^n}$   ∎

## Applications

*Example 8* **Modeling Market Stabilization** ®

Microwave ovens were first marketed in the early 1970s. From that time until the mid-1980s, the number of units sold each year increased. From the mid-1980s to the mid-1990s, the annual sales stabilized to about 11 million units per year.

A manufacturer sells 10,000 units of a product each year. In any given year, each unit has a 10% chance of breaking. That is, after 1 year you expect that only 9000 of the previous year's 10,000 units will still be in use. During the next year, this number will drop by an additional 10% to 8100, and so on. How many units will be in use after 20 years? Is the number of units in use stabilizing? If so, what is the stabilization point?

**SOLUTION**  You can model this situation with a geometric series, as shown.
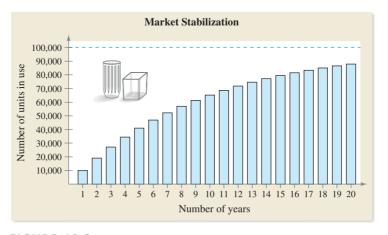
| End of Year | Number of Units in Use |
|---|---|
| 0 | 10,000 |
| 1 | $10{,}000 + 10{,}000(0.9)$ |
| 2 | $10{,}000 + 10{,}000(0.9) + 10{,}000(0.9)^2$ |
| 3 | $10{,}000 + 10{,}000(0.9) + 10{,}000(0.9)^2 + 10{,}000(0.9)^3$ |

After 20 years, the number of units in use will be

$$\sum_{n=0}^{20} 10{,}000(0.9)^n = \frac{10{,}000[1 - (0.9)^{21}]}{1 - 0.9} \approx 89{,}058.$$

As indicated in Figure 10.3, the number of units in use is approaching a stabilization point of

$$\sum_{n=0}^{\infty} 10{,}000(0.9)^n = \frac{10{,}000}{1 - 0.9} = 100{,}000 \text{ units.}$$

**STUDY TIP**

Note that in Figure 10.3, the number of years represents how many years have passed since the beginning of year zero. For instance, at the end of year zero, 1 year has passed, at the end of year one, 2 years have passed, and so on.
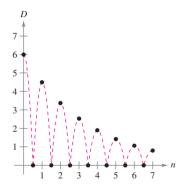


**FIGURE 10.3**

✓**CHECKPOINT 8**

If the manufacturer in Example 8 sells 10,000 units and any given unit has a 25% chance of breaking, find the number of units that will still be in use after 40 years. ◼

**FIGURE 10.4**

### Example 9
**MAKE A DECISION    Modeling a Bouncing Ball**    Ⓡ

A ball is dropped from a height of 6 feet and begins to bounce. The height of each bounce is $\frac{3}{4}$ that of the preceding bounce, as shown in Figure 10.4. Will the total vertical distance traveled by the ball be more than 40 feet?

**SOLUTION**    When the ball hits the ground the first time, it has traveled a distance of

$$D_1 = 6.$$

Between the first and second times it hits the ground, it travels an additional distance of

$$D_2 = \underbrace{6\left(\frac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\frac{3}{4}\right)}_{\text{Down}} = 12\left(\frac{3}{4}\right).$$

Between the second and third times the ball hits the ground, it travels an additional distance of

$$D_3 = \underbrace{6\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}_{\text{Down}} = 12\left(\frac{3}{4}\right)^2.$$

By continuing this process, you can determine that the total vertical distance is

$$D = 6 + 12\left(\frac{3}{4}\right) + 12\left(\frac{3}{4}\right)^2 + \cdots$$

$$= -6 + 12 + 12\left(\frac{3}{4}\right) + 12\left(\frac{3}{4}\right)^2 + \cdots$$

$$= -6 + \sum_{n=0}^{\infty} 12\left(\frac{3}{4}\right)^n$$

$$= -6 + \frac{12}{1 - (3/4)}$$

$$= -6 + 48$$

$$= 42 \text{ feet.}$$

Yes, the total vertical distance traveled by the ball will be more than 40 feet.

**✓ CHECKPOINT 9**

Find the total vertical distance the ball travels in Example 9 if it is dropped from 20 feet and bounces $\frac{3}{4}$ of the height of the preceding bounce. ■

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**CONCEPT CHECK**

1. Consider the infinite series $\sum\limits_{n=1}^{\infty} a_n$ given that $\lim\limits_{n \to \infty} a_n \neq 0$. Does the series converge or diverge?

2. Give the formula for the $n$th partial sum of the geometric series $\sum\limits_{n=0}^{\infty} ar^n$.

3. Can the $n$th-Term Test be used to determine whether a series converges?

4. For what values of $r$ will an infinite geometric series given by $\sum\limits_{n=0}^{\infty} ar^n$ converge?

| **Skills Review 10.2** | The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.5, 3.6, and 10.1. |
|---|---|

In Exercises 1 and 2, add the fractions.

**1.** $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

**2.** $1 + \frac{3}{4} + \frac{4}{6} + \frac{5}{8}$

In Exercises 3–6, evaluate the expression.

**3.** $\dfrac{1 - \left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}}$

**4.** $\dfrac{3\left[1 - \left(\frac{1}{3}\right)^4\right]}{1 - \frac{1}{3}}$

**5.** $\dfrac{2\left[1 - \left(\frac{1}{4}\right)^3\right]}{1 - \frac{1}{4}}$

**6.** $\dfrac{\frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^5\right]}{1 - \frac{1}{2}}$

In Exercises 7–10, find the limit.

**7.** $\displaystyle\lim_{n\to\infty} \frac{3n}{4n + 1}$

**8.** $\displaystyle\lim_{n\to\infty} \frac{3n}{n^2 + 1}$

**9.** $\displaystyle\lim_{n\to\infty} \frac{n!}{n! - 3}$

**10.** $\displaystyle\lim_{n\to\infty} \frac{2n! + 1}{4n! - 1}$

## Exercises 10.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, write the first five terms of the sequence of partial sums.

**1.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

**2.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2^{n-1}} = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \cdots$

**3.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{2^{n-1}} = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} - \cdots$

**4.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \cdots$

In Exercises 5–12, verify that the infinite series diverges.

**5.** $\displaystyle\sum_{n=0}^{\infty} 3\left(\frac{3}{2}\right)^n = 3 + \frac{9}{2} + \frac{27}{4} + \frac{81}{8} + \cdots$

**6.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n = 1 + \frac{4}{3} + \frac{16}{9} + \frac{64}{27} + \cdots$

**7.** $\displaystyle\sum_{n=0}^{\infty} 1000(1.055)^n = 1000 + 1055 + 1113.025 + \cdots$

**8.** $\displaystyle\sum_{n=0}^{\infty} 2(-1.03)^n = 2 - 2.06 + 2.1218 - \cdots$

**9.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n + 1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$

**10.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n + 3} = \frac{1}{5} + \frac{2}{7} + \frac{3}{9} + \frac{4}{11} + \cdots$

**11.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2 + 1} = \frac{1}{2} + \frac{4}{5} + \frac{9}{10} + \frac{16}{17} + \cdots$

**12.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 1}} = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} + \frac{4}{\sqrt{17}} + \cdots$

In Exercises 13–16, verify that the geometric series converges.

**13.** $\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n = 2 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32} + \frac{81}{128} + \cdots$

**14.** $\displaystyle\sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n = 2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots$

**15.** $\displaystyle\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots$

**16.** $\displaystyle\sum_{n=0}^{\infty} (-0.6)^n = 1 - 0.6 + 0.36 - 0.216 + \cdots$

**(T)** In Exercises 17–20, use a symbolic algebra utility to find the sum of the convergent series.

**17.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

**18.** $\displaystyle\sum_{n=0}^{\infty}2\left(\frac{2}{3}\right)^n = 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$

**19.** $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

**20.** $\displaystyle\sum_{n=0}^{\infty}2\left(-\frac{2}{3}\right)^n = 2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \cdots$

In Exercises 21–30, find the sum of the convergent series.

**21.** $\displaystyle\sum_{n=0}^{\infty}4\left(\frac{1}{4}\right)^n = 4 + 1 + \frac{1}{4} + \frac{1}{16} + \cdots$

**22.** $\displaystyle\sum_{n=0}^{\infty}6\left(\frac{4}{5}\right)^n = 6 + \frac{24}{5} + \frac{96}{25} + \frac{384}{125} + \cdots$

**23.** $1 + 0.1 + 0.01 + 0.001 + \cdots$

**24.** $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$

**25.** $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \cdots$

**26.** $4 - 2 + 1 - \frac{1}{2} + \cdots$

**27.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2^n} - \frac{1}{3^n}\right)$

**28.** $\displaystyle\sum_{n=0}^{\infty}\left[(0.7)^n + (0.9)^n\right]$

**29.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{3^n} + \frac{1}{4^n}\right)$

**30.** $\displaystyle\sum_{n=0}^{\infty}\left[(0.4)^n - (0.8)^n\right]$

In Exercises 31–40, determine the convergence or divergence of the series. Use a symbolic algebra utility to verify your result.

**31.** $\displaystyle\sum_{n=1}^{\infty}\frac{n + 10}{10n + 1}$

**32.** $\displaystyle\sum_{n=0}^{\infty}\frac{4}{2^n}$

**33.** $\displaystyle\sum_{n=1}^{\infty}\frac{n! + 1}{n!}$

**34.** $\displaystyle\sum_{n=1}^{\infty}\frac{n + 1}{2n - 1}$

**35.** $\displaystyle\sum_{n=1}^{\infty}\frac{3n - 1}{2n + 1}$

**36.** $\displaystyle\sum_{n=0}^{\infty}\frac{1}{4^n}$

**37.** $\displaystyle\sum_{n=0}^{\infty}(1.075)^n$

**38.** $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{100}$

**39.** $\displaystyle\sum_{n=0}^{\infty}\frac{3}{4^n}$

**40.** $\displaystyle\sum_{n=0}^{\infty}n!$

In Exercises 41–44, the repeating decimal is expressed as a geometric series. Find the sum of the geometric series and write the decimal as the ratio of two integers.

**41.** $0.\overline{4} = 0.4 + 0.04 + 0.004 + 0.0004 + \cdots$

**42.** $0.\overline{9} = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots$

**43.** $0.\overline{81} = 0.81 + 0.0081 + 0.000081 + \cdots$

**44.** $0.\overline{21} = 0.21 + 0.0021 + 0.000021 + \cdots$

**45. Sales**  A company produces a new product for which it estimates the annual sales to be 8000 units. Suppose that in any given year 10% of the units (regardless of age) will become inoperative.

(a)  How many units will be in use after $n$ years?

(b)  Find the market stabilization level of the product.

**46. Sales**  Repeat Exercise 45 with the assumption that 25% of the units will become inoperative each year.

**47. Physical Science**  A ball is dropped from a height of 16 feet. Each time it drops $h$ feet, it rebounds $0.81h$ feet. Find the total vertical distance traveled by the ball.

**48. Physical Science**  The ball in Exercise 47 takes the times listed below for each fall. ($t$ is measured in seconds.)

| | |
|---|---|
| $s_1 = -16t^2 + 16$ | $s_1 = 0$ if $t = 1$ |
| $s_2 = -16t^2 + 16(0.81)$ | $s_2 = 0$ if $t = 0.9$ |
| $s_3 = -16t^2 + 16(0.81)^2$ | $s_3 = 0$ if $t = (0.9)^2$ |
| $s_4 = -16t^2 + 16(0.81)^3$ | $s_4 = 0$ if $t = (0.9)^3$ |
| $\vdots$ | $\vdots$ |
| $s_n = -16t^2 + 16(0.81)^{n-1}$ | $s_n = 0$ if $t = (0.9)^{n-1}$ |

Beginning with $s_2$, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is given by

$$t = 1 + 2\sum_{n=1}^{\infty}(0.9)^n.$$

Find this total time.

**(T) 49. Annuity**  A deposit of $100 is made at the beginning of each month for 5 years in an account that pays 10% interest, compounded monthly. Use a symbolic algebra utility to find the balance $A$ in the account at the end of the 5 years.

$$A = 100\left(1 + \frac{0.10}{12}\right) + \cdots + 100\left(1 + \frac{0.10}{12}\right)^{60}$$

**50. Annuity**  A deposit of $P$ dollars is made every month for $t$ years in an account that pays an annual interest rate of $r\%$, compounded monthly. Let $N = 12t$ be the total number of deposits. Show that the balance in the account after $t$ years is

$$A = P\left[\left(1 + \frac{r}{12}\right)^N - 1\right]\left(1 + \frac{12}{r}\right), \quad t > 0.$$

**51. Consumer Trends: Multiplier Effect** The annual spending by tourists in a resort city is 100 million dollars. Approximately 75% of that revenue is again spent in the resort city, and of that amount approximately 75% is again spent in the resort city. If this pattern continues, write the geometric series that gives the total amount of spending generated by the 100 million dollars and find the sum of the series.

**52. Consumer Trends: Multiplier Effect** Repeat Exercise 51 assuming the percent of the revenue that is spent in the city is 60%.

**53. Depreciation** A company buys a machine for $225,000 that depreciates at a rate of 30% per year. Find a formula for the value of the machine after $n$ years. What is its value after 5 years?

**54. Depreciation** Repeat Exercise 53 assuming the machine depreciates at a rate of 25% per year.

**55. Salary** You accept a job that pays a salary of $40,000 the first year. During the next 39 years, you will receive a 4% raise each year. What would be your total compensation over the 40-year period?

**56. Salary** You go to work at a company that pays $0.01 for the first day, $0.02 for the second day, $0.04 for the third day, and so on. If the daily wage keeps doubling, what would your total income be for working (a) 29 days, (b) 30 days, and (c) 31 days?

**57. Probability: Coin Toss** A fair coin is tossed until a head appears. The probability that the first head appears on the $n$th toss is given by $P = \left(\frac{1}{2}\right)^n$, where $n \geq 1$. Show that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1.$$

**(T) 58. Probability: Coin Toss** Use a symbolic algebra utility to estimate the expected number of tosses required until the first head occurs in the experiment in Exercise 57.

**Probability** In Exercises 59 and 60, the random variable $n$ represents the number of units of a product sold per day in a store. The probability distribution of $n$ is given by $P(n)$. Find the probability that two units are sold in a given day [$P(2)$] and show that

$$P(0) + P(1) + P(2) + P(3) + \cdots = 1.$$

**59.** $P(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n$      **60.** $P(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$

**61. Profit** The annual revenues for eBay from 2001 through 2006 can be approximated by the model

$$a_n = 540.7e^{0.42n}, \quad n = 1, 2, 3, 4, 5, 6$$

where $a_n$ is the annual revenue (in millions of dollars) and $n$ is the year, with $n = 1$ corresponding to 2001. Use the formula for the sum of a geometric series to approximate the total revenue earned during this 6-year period. *(Source: eBay, Inc.)*

**62. Environment** A factory is polluting a river such that at every mile down river from the factory an environmental expert finds 15% less pollutant than at the preceding mile. If the pollutant's concentration is 500 ppm at the factory, what is its concentration 12 miles down river?

**63. Physical Science** In a certain brand of CD player, after the STOP function is activated, the disc, during each second after the first second, makes 85% fewer revolutions than it made during the preceding second. In coming to rest, how many revolutions does the disc make if it makes 5.5 revolutions during the first second after the STOP function is activated?

**(B) 64. Finance: Annuity** The simplest kind of annuity is a straight-line annuity, which pays a fixed amount per month until the annuitant dies. Suppose that, when he turns 65, Bob wants to purchase a straight-line annuity that has a premium of $100,000 and pays $880 per month. Use sigma notation to represent each scenario below, and give the numerical amount that the summation represents. *(Source: Adapted from Garman/Forgue, Personal Finance, Eighth Edition)*

(a) Suppose Bob dies 10 months after he takes out the annuity. How much will he have collected up to that point?

(b) Suppose Bob lives the average number of months beyond age 65 for a man (168 months). How much more or less than the $100,000 will he have collected?

**(T)** In Exercises 65–70, use a symbolic algebra utility to evaluate the summation.

**65.** $\displaystyle\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n$

**66.** $\displaystyle\sum_{n=1}^{\infty} 2n^3 \left(\frac{1}{5}\right)^n$

**67.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

**68.** $\displaystyle\sum_{n=1}^{\infty} n \left(\frac{4}{11}\right)^n$

**69.** $\displaystyle\sum_{n=1}^{\infty} e^2 \left(\frac{1}{e}\right)^n$

**70.** $\displaystyle\sum_{n=1}^{\infty} \ln 2 \left(\frac{1}{8}\right)^{2n}$

**True or False?** In Exercises 71 and 72, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**71.** If $\displaystyle\lim_{n\to\infty} a_n = 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

**72.** If $|r| < 1$, then $\displaystyle\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$.

## Section 10.3

# *p*-Series and the Ratio Test

- Determine the convergence or divergence of *p*-series.
- Use the Ratio Test to determine the convergence or divergence of series.

### *p*-Series

In Section 10.2, you studied geometric series. In this section you will study another common type of series called a *p*-series.

---

**Definition of *p*-Series**

Let $p$ be a positive constant. An infinite series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is called a *p*-series. If $p = 1$, then the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is called the **harmonic series.**

---

### Example 1 Classifying Infinite Series

Classify each infinite series.

**a.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$  **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  **c.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$

**SOLUTION**

**a.** The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

is a *p*-series with $p = 3$.

**b.** The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \cdots$$

is a *p*-series with $p = \frac{1}{2}$.

**c.** The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$$

is *not* a *p*-series. It is a geometric series.

✓**CHECKPOINT 1**

Classify each infinite series.

**a.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\pi}$

**b.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

**c.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ ■

Some infinite *p*-series converge and others diverge. With the test below, you can determine the convergence or divergence of a *p*-series.

---

**Test for Convergence of a *p*-Series**

Consider the *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots .$$

1. The series diverges if $0 < p \leq 1$.

2. The series converges if $p > 1$.

---

## DISCOVERY

It may seem surprising that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges even though its individual terms approach zero. Verify that this series diverges by showing that

$$1 + \frac{1}{2} > \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$$

$$\frac{1}{5} + \cdots + \frac{1}{8} > \frac{1}{2}$$

$$\frac{1}{9} + \cdots + \frac{1}{16} > \frac{1}{2}.$$

---

## ✓ CHECKPOINT 2

Determine whether each *p*-series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2.5}}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/10}}$ ▪

---

**Example 2**  **Determining Convergence or Divergence**

Determine whether each *p*-series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$

### SOLUTION

a. For the *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.9}} = \frac{1}{1^{0.9}} + \frac{1}{2^{0.9}} + \frac{1}{3^{0.9}} + \cdots$$

$p = 0.9$. Because $p \leq 1$, you can conclude that the series diverges.

b. For the *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

$p = 1$, which means that the series is the *harmonic* series. Because $p \leq 1$, you can conclude that the series diverges.

c. For the *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.1}} = \frac{1}{1^{1.1}} + \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \cdots$$

$p = 1.1$. Because $p > 1$, you can conclude that the series converges.

In Example 2, notice that the *p*-Series Test tells you only whether the series diverges or converges. It does not give a formula for the sum of a convergent *p*-series. To approximate such a sum, you can use a computer to evaluate several partial sums. More is said about this on the next page.

### TECHNOLOGY

**Approximating the Sum of a *p*-Series**

(T) It can be shown that the sum $S$ of a convergent *p*-series differs from its *n*th partial sum by no more than

$$\frac{1}{(p-1)n^{p-1}}.$$     Maximum error

When $p$ is greater than or equal to 3, this means that you can approximate the sum of the convergent *p*-series by adding several of the terms. For instance, because

$$\sum_{n=1}^{10} \frac{1}{n^3} \approx 1.19753$$     Tenth partial sum

and

$$\frac{1}{(2)(10^2)} = 0.005$$

you can conclude that the sum of the infinite *p*-series with $p = 3$ is between 1.19753 and $(1.19753 + 0.005)$ or 1.20253. For convergent *p*-series in which $p$ is less than 3, you need to add more and more terms of the series to obtain a reasonable approximation of the sum. For instance, when $p = 2$, you can use a computer to find the sums below.

$$\sum_{n=1}^{10} \frac{1}{n^2} \approx 1.54977$$     Tenth partial sum

$$\sum_{n=1}^{100} \frac{1}{n^2} \approx 1.63498$$     100th partial sum

$$\sum_{n=1}^{1000} \frac{1}{n^2} \approx 1.64393$$     1000th partial sum

Because

$$\frac{1}{(1)(1000)} = 0.001$$

you can conclude that the partial sum $S_{1000}$ is within 0.001 of the actual sum of the series.

When $p$ is close to 1, approximating the sum of the series becomes difficult. For instance, consider the partial sums below.

$$\sum_{n=1}^{10} \frac{1}{n^{1.1}} \approx 2.680155$$     Tenth partial sum

$$\sum_{n=1}^{100} \frac{1}{n^{1.1}} \approx 4.278024$$     100th partial sum

$$\sum_{n=1}^{1000} \frac{1}{n^{1.1}} \approx 5.572827$$     1000th partial sum

Because $1/[(0.1)(1000^{0.1})] \approx 5$, you can see that even the partial sum $S_{1000}$ is not very close to the actual sum of the series.

## The Ratio Test

At this point, you have studied two convergence tests: one for a geometric series and one for a *p*-series. The next test is more general: it can be applied to infinite series that do not happen to be geometric series or *p*-series.

**STUDY TIP**

Although the Ratio Test is listed with an index that begins with $n = 1$, it can be applied to an infinite series with a beginning index of $n = 0$ (or any other integer).

### The Ratio Test

Let $\sum\limits_{n=1}^{\infty} a_n$ be an infinite series with nonzero terms.

1. The series converges if $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| < 1$.

2. The series diverges if $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| > 1$   or   $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \infty$.

3. The test is inconclusive if $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$.

The Ratio Test is particularly useful for series that converge rapidly. Series involving factorial or exponential functions are frequently of this type.

### *Example 3*   Using the Ratio Test

Determine the convergence or divergence of the infinite series

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = \frac{1}{1} + \frac{2}{1} + \frac{4}{2} + \frac{8}{6} + \frac{16}{24} + \frac{32}{120} + \cdots .$$

**STUDY TIP**

A step frequently used in applications of the Ratio Test involves simplifying quotients of factorials. In Example 3, for instance, notice that

$$\frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!}$$

$$= \frac{1}{n+1}.$$

**SOLUTION**   Using the Ratio Test with $a_n = 2^n/n!$, you obtain

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[ \frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right]$$

$$= \lim_{n\to\infty} \left[ \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right]$$

$$= \lim_{n\to\infty} \frac{2}{n+1}$$

$$= 0.$$

Because this limit is less than 1, you can apply the Ratio Test to conclude that the series converges. Using a computer, you can approximate the sum of the series to be $S \approx S_{10} \approx 7.39$.

### ✓ CHECKPOINT 3

Determine the convergence or divergence of the infinite series

$$\sum_{n=0}^{\infty} \frac{3^n}{n!}. \quad \blacksquare$$

Example 3 tells you something about the rates at which the sequences $\{2^n\}$ and $\{n!\}$ increase as $n$ approaches infinity. For example, in the table below, you can see that although the factorial sequence $\{n!\}$ has a slow start, it quickly overpowers the exponential sequence $\{2^n\}$.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $2^n$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| $n!$ | 1 | 1 | 2 | 6 | 24 | 120 | 720 | 5040 | 40,320 | 362,880 |

From this table, you can also see that the sequence $\{n\}$ approaches infinity more slowly than the sequence $\{2^n\}$. This is further demonstrated in Example 4.

### Example 4    Using the Ratio Test

Determine the convergence or divergence of the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64} + \cdots .$$

**SOLUTION**    Using the Ratio Test with $a_n = n/2^n$, you obtain

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left( \frac{n+1}{2^{n+1}} \div \frac{n}{2^n} \right)$$

$$= \lim_{n\to\infty} \left( \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right)$$

$$= \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Because this limit is less than 1, you can apply the Ratio Test to conclude that the series converges. Using a computer, you can determine that the sum of the series is $S = 2$.

When applying the Ratio Test, remember that if the limit of

$$\left| \frac{a_{n+1}}{a_n} \right|$$

as $n\to\infty$ is 1, then the test does not tell you whether the series converges or diverges. This type of result often occurs with series that converge or diverge slowly. For instance, when you apply the Ratio Test to the harmonic series in which

$$a_n = \frac{1}{n}$$

you obtain

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

So, from the Ratio Test, you cannot conclude that the harmonic series diverges. (The Ratio Test is also inconclusive for any $p$-series.) From the $p$-Series Test, you know that it diverges.

✓ **CHECKPOINT 4**

Determine the convergence or divergence of the infinite series

$$\sum_{n=1}^{\infty} \frac{5^n}{n^2}. \blacksquare$$

**Example 5** **Using the Ratio Test**

Determine the convergence or divergence of the infinite series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} = \frac{2}{1} + \frac{4}{4} + \frac{8}{9} + \frac{16}{16} + \frac{32}{25} + \frac{64}{36} + \cdots.$$

**SOLUTION** Using the Ratio Test with $a_n = 2^n/n^2$, you obtain

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[ \frac{2^{n+1}}{(n+1)^2} \div \frac{2^n}{n^2} \right]$$

$$= \lim_{n\to\infty} \left[ \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} \right]$$

$$= \lim_{n\to\infty} 2 \left( \frac{n}{n+1} \right)^2$$

$$= 2.$$

Because this limit is greater than 1, you can apply the Ratio Test to conclude that the series diverges.

**✓ CHECKPOINT 5**

Determine the convergence or divergence of the infinite series

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}. \quad \blacksquare$$

---

**Summary of Tests of Series**

| Test | Series | Converges | Diverges |
|------|--------|-----------|----------|
| $n$th-Term | $\sum_{n=1}^{\infty} a_n$ | No test | $\lim_{n\to\infty} a_n \neq 0$ |
| Geometric | $\sum_{n=0}^{\infty} ar^n$ | $|r| < 1$ | $|r| \geq 1$ |
| $p$-Series | $\sum_{n=1}^{\infty} \frac{1}{n^p}$ | $p > 1$ | $0 < p \leq 1$ |
| Ratio | $\sum_{n=1}^{\infty} a_n$ | $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ | $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ |

---

**CONCEPT CHECK**

1. For what value of $p$ is $\sum_{n=1}^{\infty} \frac{1}{n^p}$ a harmonic series?

2. For what value(s) of $p$ does the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

3. For what value(s) of $p$ does the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverge?

4. Can the Ratio Test be used to determine the convergence of any $p$-series?

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 3.6, 10.1, and 10.2.

In Exercises 1–4, simplify the expression.

**1.** $\dfrac{n!}{(n+1)!}$

**2.** $\dfrac{(n+1)!}{n!}$

**3.** $\dfrac{3^{n+1}}{n+1} \cdot \dfrac{n}{3^n}$

**4.** $\dfrac{(n+1)^2}{(n+1)!} \cdot \dfrac{n!}{n^2}$

In Exercises 5–8, find the limit.

**5.** $\lim\limits_{n\to\infty} \dfrac{(n+1)^2}{n^2}$

**6.** $\lim\limits_{n\to\infty} \dfrac{5^{n+1}}{5^n}$

**7.** $\lim\limits_{n\to\infty} \left( \dfrac{5}{n+1} \div \dfrac{5}{n} \right)$

**8.** $\lim\limits_{n\to\infty} \left( \dfrac{(n+1)^3}{3^{n+1}} \div \dfrac{n^3}{3^n} \right)$

In Exercises 9 and 10, decide whether the series is geometric.

**9.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{4^n}$

**10.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^4}$

## Exercises 10.3

In Exercises 1–8, determine whether the series is a $p$-series.

**1.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^2}$

**2.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$

**3.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{3^n}$

**4.** $\displaystyle\sum_{n=1}^{\infty} n^{-3/4}$

**5.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^n}$

**6.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n+1}$

**7.** $1 + \dfrac{1}{\sqrt[3]{2}} + \dfrac{1}{\sqrt[3]{3}} + \dfrac{1}{\sqrt[3]{4}} + \cdots$

**8.** $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \cdots$

In Exercises 9–18, determine the convergence or divergence of the $p$-series.

**9.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^3}$

**10.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^{1/3}}$

**11.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{\sqrt[5]{n}}$

**12.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^{4/3}}$

**13.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^{1.03}}$

**14.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^{\pi/2}}$

**15.** $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \cdots$

**16.** $1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \dfrac{1}{25} + \cdots$

**17.** $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \cdots$

**18.** $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \cdots$

In Exercises 19–32, use the Ratio Test to determine the convergence or divergence of the series.

**19.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{3^n}{n!}$

**20.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{n!}{3^n}$

**21.** $\displaystyle\sum_{n=1}^{\infty} n\left(\dfrac{3}{4}\right)^n$

**22.** $\displaystyle\sum_{n=1}^{\infty} n\left(\dfrac{3}{2}\right)^n$

**23.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{n}{4^n}$

**24.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{n^2}{2^n}$

**25.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{2^n}{n^5}$

**26.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n e^{-n}$

**27.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n 2^n}{n!}$

**28.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{4n}{n!}$

**29.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{4^n}{3^n+1}$

**30.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{3^n}{n+1}$

**31.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{n5^n}{n!}$

**32.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{2n!}{n^5}$

In Exercises 33–36, approximate the sum of the convergent series using the indicated number of terms. Estimate the maximum error of your approximation.

**33.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$, four terms

**34.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$, four terms

**35.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, 10 terms

**36.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{10}}$, three terms
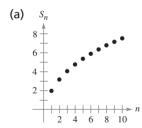
In Exercises 37–40, verify that the Ratio Test is inconclusive for the *p*-series.
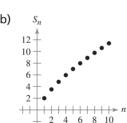
**37.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

**38.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$
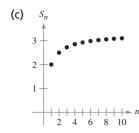
**39.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$
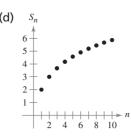
**40.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$
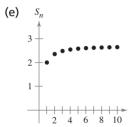
In Exercises 41–46, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a)–(f).] Determine the convergence or divergence of the series.

(a) $S_n$



(b) $S_n$



(c) $S_n$



(d) $S_n$



(e) $S_n$



(f) $S_n$
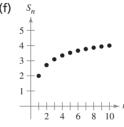


**41.** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n^3}}$

**42.** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n}$

**43.** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^5}}$

**44.** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt[5]{n^2}}$

**45.** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$

**46.** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$

In Exercises 47–64, test the series for convergence or divergence using any appropriate test from this chapter. Identify the test used and explain your reasoning. If the series is convergent, find the sum whenever possible.

**47.** $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n+1}$

**48.** $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n}$

**49.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

**50.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[3]{n}}$

**51.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$

**52.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$

**53.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^n}$

**54.** $\displaystyle\sum_{n=2}^{\infty} \ln n$

**55.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3}\right)$

**56.** $\displaystyle\sum_{n=1}^{\infty} \frac{n3^n}{n!}$

**57.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$

**58.** $\displaystyle\sum_{n=1}^{\infty} n(0.4)^n$

**59.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^{n-1}}$

**60.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$

**61.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{2^{n+1}+1}$

**62.** $\displaystyle\sum_{n=1}^{\infty} 2e^{-n}$

**63.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^{n-1}}$

**64.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

(T) In Exercises 65 and 66, use a computer to confirm the sum of the convergent series.

**65.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

**66.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{24}$

## Mid-Chapter Quiz

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, write the first five terms of the sequence.

**1.** $a_n = \left(\dfrac{-1}{4}\right)^n$     **2.** $a_n = \dfrac{n+1}{n+3}$     **3.** $a_n = 5(-1)^n$     **4.** $a_n = \dfrac{n-2}{n!}$

In Exercises 5–8, determine the convergence or divergence of the sequence. If the sequence converges, find its limit.

**5.** $a_n = \dfrac{3}{\sqrt{n}}$     **6.** $a_n = \dfrac{n}{2n+3}$     **7.** $a_n = \dfrac{2}{(n+1)!}$     **8.** $a_n = \dfrac{(-1)^n}{2}$

In Exercises 9–11, write an expression for the $n$th term of the sequence. (There is more than one correct answer.)

**9.** $0, \frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \ldots$     **10.** $-3, \sqrt{3}, -\sqrt[3]{3}, \sqrt[4]{3}, \ldots$     **11.** $\frac{1}{2}, 2, \frac{1}{2}, 2, \ldots$

In Exercises 12 and 13, write the first five terms of the sequence of partial sums.

**12.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{n-1}{n!} = 0 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{8} + \dfrac{1}{30} + \cdots$

**13.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\dfrac{n}{2^{n-1}} = 1 - 1 + \dfrac{3}{4} - \dfrac{1}{2} + \dfrac{5}{16} - \cdots$

In Exercises 14–19, test the series for convergence or divergence using any appropriate test from this chapter.

**14.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{2n^2-1}{n^2+1}$     **15.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{e}{3^n}$     **16.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{e^n}{2}$

**17.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{\sqrt[3]{n^5}}$     **18.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{n}{(n-1)!}$     **19.** $\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{2}{3}\right)^n n!$

In Exercises 20–22, find the sum of the convergent series.

**20.** $\displaystyle\sum_{n=0}^{\infty} 4\left(\dfrac{2}{3}\right)^n$     **21.** $\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{1}{3^n} - \dfrac{1}{3^{n+1}}\right)$

**22.** $5 + 0.5 + 0.05 + 0.005 + \cdots$

**23.** A deposit of \$1000 is made in an account that earns 4.5% interest, compounded quarterly. Find a sequence that represents the quarterly balances. Then use the sequence to determine the balance in the account after 2 years.

**24.** The annual research and development expenditures at universities and colleges for the years 2000 through 2005 can be approximated by the model

$a_n = 30.34e^{0.0864n}, \quad n = 0, 1, 2, 3, 4, 5$

where $a_n$ is the annual expenditure (in billions of dollars) and $n$ is the year, with $n = 0$ corresponding to 2000. Use the formula for the sum of a geometric series to approximate the total research and development expenditures during this 6-year period. *(Source: U.S. National Science Foundation)*

## Section 10.4

# Power Series and Taylor's Theorem

- Recognize power series.
- Find the radii of convergence of power series.
- Use Taylor's Theorem to find power series for functions.
- Use the basic list of power series to find power series for functions.

## Power Series

In the preceding two sections, you studied infinite series whose terms are constants. In this section, you will study infinite series that have variable terms. Specifically, you will study a type of infinite series that is called a **power series.** Informally, you can think of a power series as a "very long" polynomial.

**STUDY TIP**

The index of a power series usually begins with $n = 0$. In such cases, it is agreed that $(x - c)^0 = 1$ even if $x = c$.

### Definition of Power Series

If $x$ is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series.** More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots +$$

$$a_n(x - c)^n + \cdots$$

is called a **power series centered at $c$,** where $c$ is a constant.

### Example 1  Power Series

**a.** The following power series is centered at 0.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

**b.** The following power series is centered at 1.

$$\sum_{n=1}^{\infty} \frac{(x - 1)^n}{n} = (x - 1) + \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} + \cdots$$

**c.** The following power series is centered at $-1$.

$$\sum_{n=1}^{\infty} \frac{(x + 1)^n}{n} = (x + 1) + \frac{(x + 1)^2}{2} + \frac{(x + 1)^3}{3} + \cdots$$

**✓CHECKPOINT 1**

Identify the center of each power series.

**a.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x - 2)^n}{n^2}$

**b.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x + 3)^n}{3^n}$

**c.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$  ∎

## Radius of Convergence of a Power Series

A power series in $x$ can be viewed as a function of $x$

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

where the *domain of f* is the set of all $x$ for which the power series converges. Determining this domain is one of the primary problems associated with power series. Of course, every power series converges at its center $c$ because

$$f(c) = \sum_{n=0}^{\infty} a_n(c - c)^n$$
$$= a_0(1) + 0 + 0 + \cdot\cdot\cdot = a_0.$$

So, $c$ is always in the domain of $f$. In fact, the domain of a power series can take three basic forms: a single point, an interval centered at $c$, or the entire real line, as shown in Figure 10.5.
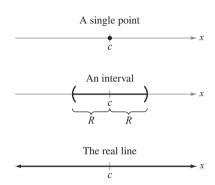


**FIGURE 10.5**    Three Types of Domains

### Convergence of a Power Series

For a power series centered at $c$, precisely one of the following is true.

1. The series converges only at $c$.
2. There exists a positive real number $R$ such that the series converges for $|x - c| < R$ and diverges for $|x - c| > R$.
3. The series converges for all $x$.

The number $R$ is the **radius of convergence** of the power series. If the series converges only at $c$, then $R = 0$, and if the series converges for all $x$, then $R = \infty$.

In the second case, the series converges in the interval

$$(c - R, c + R)$$

and diverges in the intervals $(-\infty, c - R)$ and $(c + R, \infty)$. Determining the convergence or divergence at the endpoints $c - R$ and $c + R$ can be difficult, and, except for simple cases, the endpoint question is left open. To find the radius of convergence of a power series, use the Ratio Test, as illustrated in Examples 2 and 3.

*Example 2*  **Finding the Radius of Convergence**

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

**SOLUTION**   For this power series, $a_n = 1/n!$. So, you have

$$\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x}{n+1} \right|$$

$$= 0.$$

So, by the Ratio Test, this series converges for all $x$, and the radius of convergence is $R = \infty$.

✓ **CHECKPOINT 2**

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$ ■

*Example 3*  **Finding the Radius of Convergence**

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}.$$

**SOLUTION**   For this power series, $a_n = (-1)^n/2^n$. So, you have

$$\lim_{n \to \infty} \left| \frac{a_{n+1}(x+1)^{n+1}}{a_n(x+1)^n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}(x+1)^{n+1}/2^{n+1}}{(-1)^n(x+1)^n/2^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(-1)(x+1)}{2} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x+1}{2} \right|$$

$$= \left| \frac{x+1}{2} \right|.$$

**DISCOVERY**

Interval: $(-3, 1)$
Radius: $R = 2$



**FIGURE 10.6**

By the Ratio Test, this series will converge as long as $|(x+1)/2| < 1$ or $|x+1| < 2$. So, the radius of convergence is $R = 2$. Because the series is centered at $x = -1$, it will converge in the interval $(-3, 1)$, as shown in Figure 10.6.
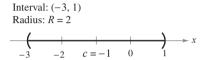
✓ **CHECKPOINT 3**

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3^n}.$ ■

## Taylor and Maclaurin Series

The problem of finding a power series for a given function is answered by **Taylor's Theorem,** named after the English mathematician Brook Taylor (1685–1731). This theorem shows how to use derivatives of a function $f$ to write the power series for $f$.

**Taylor's Theorem**

If $f$ is represented by a power series centered at $c$, then the power series has the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!}$$

$$= f(c) + \frac{f'(c)(x-c)}{1!} + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots.$$

*Example 4*    **Finding a Maclaurin Series**

Find the power series for $f(x) = e^x$, centered at 0. What is the radius of convergence of the series?

**SOLUTION**    Begin by finding several derivatives of $f$ and evaluating each at $c = 0$.

| | | |
|---|---|---|
| $f(x) = e^x$ | $f(0) = 1$ | Write original function. |
| $f'(x) = e^x$ | $f'(0) = 1$ | Find first derivative. |
| $f''(x) = e^x$ | $f''(0) = 1$ | Find second derivative. |
| $f'''(x) = e^x$ | $f'''(0) = 1$ | Find third derivative. |
| $f^{(4)}(x) = e^x$ | $f^{(4)}(0) = 1$ | Find fourth derivative. |
| $f^{(5)}(x) = e^x$ | $f^{(5)}(0) = 1$ | Find fifth derivative. |

From this pattern, you can see that $f^{(n)}(0) = 1$. So, by Taylor's Theorem,

$$e^x = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

From Example 2, you know that the radius of convergence is $R = \infty$. In other words, it converges for all values of $x$.

✓ **CHECKPOINT 4**

Find the power series for $f(x) = e^{-x}$, centered at 0. What is the radius of convergence? ∎

*Example 5* **Finding a Taylor Series**

Find the power series for $f(x) = 1/x$, centered at 1. Then use the result to evaluate $f(\frac{1}{2})$.

**SOLUTION** Successive differentiation of $f(x)$ produces the pattern below.

| | | |
|---|---|---|
| $f(x) = x^{-1}$ | $f(1) = 1 = 0!$ | Write original function. |
| $f'(x) = -x^{-2}$ | $f'(1) = -1 = -(1!)$ | Find first derivative. |
| $f''(x) = 2x^{-3}$ | $f''(1) = 2 = 2!$ | Find second derivative. |
| $f'''(x) = -6x^{-4}$ | $f'''(1) = -6 = -(3!)$ | Find third derivative. |
| $f^{(4)}(x) = 24x^{-5}$ | $f^{(4)}(1) = 24 = 4!$ | Find fourth derivative. |
| $f^{(5)}(x) = -120x^{-6}$ | $f^{(5)}(1) = -120 = -(5!)$ | Find fifth derivative. |

From this pattern, you can see that $f^{(n)}(1) = (-1)^n n!$. So, by Taylor's Theorem, you can write

$$\frac{1}{x} = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \cdots$$
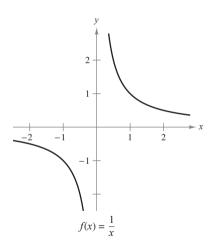
$$= 1 - (x-1) + \frac{2!(x-1)^2}{2!} - \frac{3!(x-1)^3}{3!} + \frac{4!(x-1)^4}{4!} - \cdots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots$$
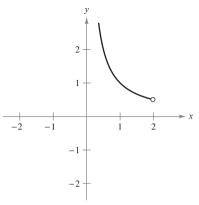
$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

To evaluate the series when $x = \frac{1}{2}$, you can substitute $\frac{1}{2}$ for $x$ and use the formula for the sum of a geometric series.

$$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2} - 1\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{1 - (1/2)}$$

$$= 2$$



$f(x) = \dfrac{1}{x}$

Domain: all $x \neq 0$



$f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

Domain: $0 < x < 2$

**FIGURE 10.7**

✓**CHECKPOINT 5**

Find the power series for $f(x) = \ln x$, centered at 1. ∎

In Example 5, the radius of convergence of the series is $R = 1$, and its interval of convergence is

$(0, 2)$.                                    Interval of convergence

(It is possible to show that the series diverges when $x = 0$ and when $x = 2$.) Figure 10.7 compares the graph of $f(x) = 1/x$ and the graph of the Taylor series for $f$. In the figure, note that the domains are different. In other words, the power series in Example 5 represents $f$ only in the interval $(0, 2)$.

**DISCOVERY**

*Example 6*  **Finding a Maclaurin Series**

Find the Maclaurin series for each function.

**a.** $f(x) = e^{x^2}$    **b.** $f(x) = e^{-x^2}$

**SOLUTION**

**a.** To use Taylor's Theorem directly, you would have to calculate successive derivatives of $f(x) = e^{x^2}$. By calculating the first two

$$f'(x) = 2xe^{x^2}$$

and

$$f''(x) = (4x^2 + 2)e^{x^2}$$

you can see that this task would be very tedious. Fortunately, there is a simpler way to find the power series. From Example 4, you already know that the Maclaurin series for $e^x$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

So, to find the Maclaurin series for $e^{x^2}$, you can simply substitute $x^2$ for $x$ in the series for $e^x$. Doing this produces

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots$$

$$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} .$$

**b.** Using a similar approach, you can find the Maclaurin series for $e^{-x^2}$ by substituting $-x^2$ for $x$ in the series for $e^x$. Doing this produces

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} .$$

✓**CHECKPOINT 6**

Use the results from Example 6 to find the Maclaurin series for each function.

**a.** $f(x) = e^{2x}$    **b.** $g(x) = e^{-2x}$  ◼

Why are power series useful? The reason is that power series share many of the desirable properties of polynomials—they can be easily differentiated and easily integrated. This means that if you wanted to integrate a function such as $f(x) = e^{x^2}$ (which does not have an elementary antiderivative), you could represent the function with a power series and then integrate the power series.

## A Basic List of Power Series

Example 6 illustrates an important point in determining power series representations of functions. Although Taylor's Theorem is applicable to a wide variety of functions, it is often tedious to use because of the complexity of finding derivatives. The most practical use of Taylor's Theorem is in developing power series for a *basic list* of elementary functions. Then, from the basic list, you can determine power series for other functions by the operations of addition, subtraction, multiplication, division, differentiation, integration, and composition with known power series.

**Power Series for Elementary Functions**

$$\frac{1}{x} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4 - \cdots + (-1)^n(x - 1)^n + \cdots, \quad 0 < x < 2$$

$$\frac{1}{x + 1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots, \quad\quad\quad -1 < x < 1$$

$$\ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x - 1)^n}{n} + \cdots, \quad 0 < x \leq 2$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots, \quad\quad\quad -\infty < x < \infty$$

$$(1 + x)^k = 1 + kx + \frac{k(k - 1)x^2}{2!} + \frac{k(k - 1)(k - 2)x^3}{3!} + \frac{k(k - 1)(k - 2)(k - 3)x^4}{4!} + \cdots, \quad -1 < x < 1$$

The last series in the list above is called a **binomial series.** Example 7 illustrates the use of such a series.

### *Example 7*   Using the Basic List of Power Series

Find the power series for

$$g(x) = \sqrt[3]{1 + x}$$

centered at zero.

**SOLUTION**   Using the binomial series

$$(1 + x)^k = 1 + kx + \frac{k(k - 1)x^2}{2!} + \frac{k(k - 1)(k - 2)x^3}{3!} + \cdots$$

with $k = \frac{1}{3}$, you can write

$$(1 + x)^{1/3} = 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \cdots$$

which converges for $-1 < x < 1$.

### ✓ CHECKPOINT 7

Use the basic list of power series to find the power series for $g(x) = \sqrt{1 + x}$, centered at zero. ▪

*Example 8*   **Using the Basic List of Power Series**

Find the power series for each function.

**a.** $f(x) = 1 + 2e^x$, centered at 0

**b.** $f(x) = e^{2x+1}$, centered at 0

**c.** $f(x) = \ln 2x$, centered at 1

**STUDY TIP**

Determining power series representations of functions is necessary to solve certain real-life problems, such as finding the expected value for a coin-tossing experiment. You will see such applications in Section 10.5.

**SOLUTION**

**a.** To find the power series for this function, you can multiply the series for $e^x$ by 2 and add 1.

$$1 + 2e^x = 1 + 2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

$$= 1 + 2 + 2x + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \frac{2x^4}{4!} + \cdots$$

$$= 1 + \sum_{n=0}^{\infty} \frac{2x^n}{n!}$$

**b.** To find the power series for $e^{2x+1} = e^{2x}e$, you can substitute $2x$ for $x$ in the series for $e^x$ and multiply the result by $e$.

$$e^{2x+1} = e^{2x}e$$

$$= e\left[1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots\right]$$

$$= e\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

**✓ CHECKPOINT 8**

Use the basic list of power series to find the power series for each function.

**a.** $f(x) = 1 - e^{-x}$, centered at 0

**b.** $f(x) = e^{-x+2}$, centered at 0

**c.** $f(x) = \ln x$, centered at 1 ▪

**c.** To find the power series for $\ln 2x$ centered at 1, use the properties of logarithms.

$$\ln 2x = \ln 2 + \ln x$$

$$= \ln 2 + (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$$

**CONCEPT CHECK**

1. Is a power series a finite series or an infinite series?

2. For a power series centered at $c$, will the series converge at $c$?

3. The domain of a power series can take three basic forms. Identify each form.

4. To find a power series for a given function, what theorem can you use?

Skills Review 10.4   The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.4, 2.6, and 10.1.

In Exercises 1–4, find $f(g(x))$ and $g(f(x))$.

**1.** $f(x) = x^2$,   $g(x) = x - 1$

**2.** $f(x) = 3x$,   $g(x) = 2x + 1$

**3.** $f(x) = \sqrt{x + 4}$,   $g(x) = x^2$

**4.** $f(x) = e^x$,   $g(x) = x^2$

In Exercises 5–8, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

**5.** $f(x) = 5e^x$

**6.** $f(x) = \ln x$

**7.** $f(x) = 3e^{2x}$

**8.** $f(x) = \ln 2x$

In Exercises 9 and 10, simplify the expression.

**9.** $\dfrac{3^n}{n!} \div \dfrac{3^{n+1}}{(n+1)!}$

**10.** $\dfrac{n!}{(n+2)!} \div \dfrac{(n+1)!}{(n+3)!}$

# Exercises 10.4

In Exercises 1–4, write the first five terms of the power series.

**1.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

**2.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-2)^n}{3^n}$

**3.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x+1)^n}{n!}$

**4.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n-1)!}$

In Exercises 5–24, find the radius of convergence of the series.

**5.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

**6.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$

**7.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

**8.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} nx^n$

**9.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

**10.** $\displaystyle\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$

**11.** $\displaystyle\sum_{n=0}^{\infty} n! \left(\frac{x}{2}\right)^n$

**12.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$

**14.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-4)^n}{3^n}$

**15.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$

**16.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)3^{n+1}}$

**17.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

**18.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

**20.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

**22.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

**23.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

**24.** $\displaystyle\sum_{n=0}^{\infty} \frac{n! x^n}{(n+1)!}$

In Exercises 25–32, apply Taylor's Theorem to find the power series (centered at c) for the function, and find the radius of convergence.

| Function | Center |
|---|---|
| **25.** $f(x) = e^x$ | $c = 1$ |
| **26.** $f(x) = e^{-x}$ | $c = 1$ |
| **27.** $f(x) = e^{3x}$ | $c = 0$ |
| **28.** $f(x) = e^{-3x}$ | $c = 0$ |
| **29.** $f(x) = \dfrac{1}{x + 1}$ | $c = 0$ |
| **30.** $f(x) = \dfrac{1}{2 - x}$ | $c = 0$ |
| **31.** $f(x) = \sqrt{x}$ | $c = 1$ |
| **32.** $f(x) = \sqrt{x}$ | $c = 4$ |

In Exercises 33–36, apply Taylor's Theorem to find the binomial series (centered at c = 0) for the function, and find the radius of convergence.

**33.** $f(x) = \dfrac{1}{(1 + x)^2}$

**34.** $f(x) = \sqrt{1 + x}$

**35.** $f(x) = \dfrac{1}{\sqrt{1 - x}}$

**36.** $f(x) = \sqrt[4]{1 + x}$

In Exercises 37–40, find the radius of convergence of (a) f(x), (b) f′(x), (c) f″(x), and (d) ∫f(x) dx.

**37.** $f(x) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

**38.** $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n5^n}$

**39.** $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x + 1)^{n+1}}{n + 1}$

**40.** $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x - 1)^n}{n}$

In Exercises 41–52, find the power series for the function using the suggested method. Use the basic list of power series for elementary functions on page 699.

**41.** Use the power series for $e^x$.

$f(x) = e^{x^3}$

**42.** Use the series found in Exercise 41.

$f(x) = e^{-x^3}$

**43.** Differentiate the series found in Exercise 41.

$f(x) = 3x^2 e^{x^3}$

**44.** Use the power series for $e^x$ and $e^{-x}$.

$f(x) = \dfrac{e^x + e^{-x}}{2}$

**45.** Use the power series for $\dfrac{1}{x + 1}$.

$f(x) = \dfrac{1}{x^4 + 1}$

**46.** Use the power series for $\dfrac{1}{x + 1}$.

$f(x) = \dfrac{2x}{x + 1}$

**47.** Use the power series for $\dfrac{1}{x + 1}$.

$f(x) = \ln(x^2 + 1)$

**48.** Integrate the series for $\dfrac{1}{x + 1}$.

$f(x) = \ln(x + 1)$

**49.** Integrate the series for $\dfrac{1}{x}$.

$f(x) = \ln x$

**50.** Differentiate the series found in Exercise 44.

$f(x) = \dfrac{e^x - e^{-x}}{2}$

**51.** Differentiate the series for $-\dfrac{1}{x}$.

$f(x) = \dfrac{1}{x^2}$

**52.** Differentiate the power series for $e^x$ term by term and use the resulting series to show that

$$\frac{d}{dx}[e^x] = e^x.$$

In Exercises 53 and 54, use the Taylor series for the exponential function to approximate the expression to four decimal places.

**53.** $e^{1/2}$                      **54.** $e^{-1}$

Ⓣ In Exercises 55–58, use a symbolic algebra utility and 50 terms of the series to approximate the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x - 1)^n}{n}, \quad 0 < x \leq 2.$$

**55.** $f(0.5)$ (Actual sum is ln 0.5.)

**56.** $f(1.5)$ (Actual sum is ln 1.5.)

**57.** $f(0.1)$ (Actual sum is ln 0.1.)

**58.** $f(1.95)$ (Actual sum is ln 1.95.)

## Section 10.5

# Taylor Polynomials

■ Find Taylor polynomials for functions.
■ Use Taylor polynomials to determine the maximum errors of approximations and to approximate definite integrals.
■ Use Taylor polynomials to model probabilities.

## Taylor Polynomials and Approximation

In Section 10.4, you saw that it is sometimes possible to obtain an *exact* power series representation of a function. For example, the function

$$f(x) = e^{-x}$$

can be represented exactly by the power series

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^n}{n!} + \cdots.$$

The problem with using this power series is that the exactness of its representation depends on the summation of an infinite number of terms. In practice, this is not feasible, and you must be content with a finite summation that approximates the function rather than representing it exactly. For instance, consider the sequence of partial sums below.

$$S_0(x) = 1 \qquad\qquad\qquad \text{Zeroth-degree Taylor polynomial}$$

$$S_1(x) = 1 - x \qquad\qquad\qquad \text{First-degree Taylor polynomial}$$

$$S_2(x) = 1 - x + \frac{x^2}{2!} \qquad\qquad \text{Second-degree Taylor polynomial}$$

$$S_3(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \qquad\qquad \text{Third-degree Taylor polynomial}$$

$$S_4(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \qquad \text{Fourth-degree Taylor polynomial}$$

$$S_5(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \qquad \text{Fifth-degree Taylor polynomial}$$

$$\vdots$$

$$S_n(x) = 1 - x + \frac{x^2}{2!} - \cdots + \frac{(-1)^n x^n}{n!} \qquad n\text{th-degree Taylor polynomial}$$
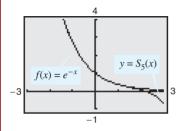
Each of these polynomial approximations of $e^{-x}$ is a **Taylor polynomial** for $e^{-x}$. As $n$ approaches infinity, the graphs of these Taylor polynomials become closer and closer approximations of the graph of

$$f(x) = e^{-x}.$$

For example, Figure 10.8 shows the graphs of $S_1$, $S_2$, $S_3$, and $S_4$. Notice in the figure that the closer $x$ is to the center of convergence ($x = 0$ in this case), the better the polynomial approximates $e^{-x}$. This conclusion is reinforced by the table below. From Section 10.4, you know that the power series for $e^{-x}$ converges for *all* $x$. From the figure and table, however, you can see that the farther $x$ is from zero, the more terms you need to obtain a good approximation.

| $x$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $S_1 = 1 - x$ | 1 | 0.5000 | 0 | $-0.5000$ | $-1.0$ |
| $S_2 = 1 - x + \dfrac{x^2}{2!}$ | 1 | 0.6250 | 0.5000 | 0.6250 | 1.0 |
| $S_3 = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!}$ | 1 | 0.6042 | 0.3333 | 0.0625 | $-0.3333$ |
| $S_4 = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$ | 1 | 0.6068 | 0.3750 | 0.2734 | 0.3333 |
| $e^{-x}$ | 1 | 0.6065 | 0.3679 | 0.2231 | 0.1353 |



**FIGURE 10.8**

*Example 1*  **Finding Taylor Polynomials**  (T)

Find a Taylor polynomial that is a reasonable approximation of

$$f(x) = e^{-x^2}$$
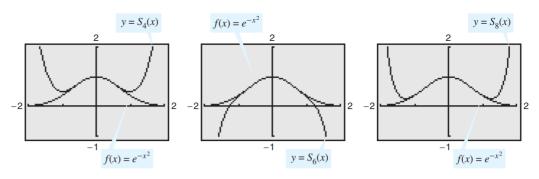
in the interval $[-1, 1]$. What degree of Taylor polynomial should you use?

**SOLUTION**   To begin, you need to define what constitutes a "reasonable" approximation. Suppose you decide that you want the values of $S_n(x)$ and $e^{-x^2}$ to differ by no more than 0.01 in the interval $[-1, 1]$. To answer the question, you can compute Taylor polynomials of higher and higher degree and then graphically compare them with $f(x) = e^{-x^2}$. For instance, the fourth-, sixth-, and eighth-degree Taylor polynomials for $f$ are as shown.

$$S_4(x) = 1 - x^2 + \frac{x^4}{2} \qquad \text{\color{magenta}Fourth-degree Taylor polynomial}$$

$$S_6(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} \qquad \text{\color{magenta}Sixth-degree Taylor polynomial}$$

$$S_8(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} \qquad \text{\color{magenta}Eighth-degree Taylor polynomial}$$

To compare $S_4(x)$ and $e^{-x^2}$ graphically, use a graphing utility to graph both equations in the same viewing window, as shown in Figure 10.9. Then, use the *trace* feature of the graphing utility to compare the $y$-values at $-1$ and 1. When you do this, you will find that the $y$-values differ by more than 0.01.

Next, perform the same comparison with $S_6(x)$ and $e^{-x^2}$. With these two graphs, the $y$-values still differ by more than 0.01 in the interval $[-1, 1]$. Finally, by graphically comparing $S_8(x)$ and $e^{-x^2}$, you can determine that their graphs differ by less than 0.01 in the interval $[-1, 1]$. To convince yourself of this, try evaluating $e^{-x^2}$ and $S_8(x)$ when $x = \pm 1$. You should obtain the values 0.368 and 0.375, which differ by 0.007.



**FIGURE 10.9**

✓ **CHECKPOINT 1**

Find the 12th-degree Taylor polynomial for $f(x) = e^{2x^3}$, centered at zero. Then use a graphing utility to graph the function and the polynomial in the same viewing window. ■

## Taylor's Theorem with Remainder

In Example 1, you studied a graphical procedure for determining how well a Taylor polynomial approximates a function $f$. The theorem below gives you an analytic procedure for determining this.

---

**Taylor's Theorem with Remainder**

Let $f$ have derivatives up through order $n + 1$ for every $x$ in an interval $I$ containing $c$. Then, for all $x$ in $I$, $f(x) = S_n(x) + R_n$, where $S_n(x)$ is the $n$th-degree Taylor polynomial for $f$ and where $R_n$ is given by

$$R_n = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}$$

for some number $z$ between $c$ and $x$. The value $R_n$ is called the **$n$th remainder** of $f(x)$.

---

Although this theorem appears to give a formula for the exact remainder, note that it does not specify which value of $z$ should be used to find $R_n$. In other words, the practical application of this theorem lies not in calculating $R_n$, but in finding bounds for $R_n$.

### Example 2    Using a Taylor Polynomial Approximation

Approximate $e^{-0.75}$ using a fourth-degree Taylor polynomial and determine the maximum error of the approximation.

**SOLUTION**    Using the fourth-degree Taylor polynomial for $e^{-x}$, you obtain the approximation as shown.

$$e^{-x} \approx 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{-0.75} \approx 1 - 0.75 + \frac{(0.75)^2}{2!} - \frac{(0.75)^3}{3!} + \frac{(0.75)^4}{4!} \approx 0.474$$

By Taylor's Theorem with Remainder, the error in this approximation is

$$R_4 = -\frac{e^{-z}}{5!}(0.75)^5, \quad 0 \le z \le 0.75.$$

Because $e^{-z}$ has a maximum value of 1 in the interval $[0, 0.75]$, it follows that

$$|R_4| \le \frac{1}{5!}(0.75)^5 \approx 0.002.$$

So, the approximation is off by at most 0.002.

---

### ✓ CHECKPOINT 2

Use the fourth-degree Taylor polynomial from Example 2 to approximate $e^{-0.5}$ and determine the maximum error of the approximation. ■

Earlier in the text, you looked at different ways to approximate a definite integral: the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. The next example shows how you can use a power series to approximate a definite integral.

### Example 3    Approximating a Definite Integral

Use the eighth-degree Taylor polynomial for $e^{-x^2}$ to approximate the definite integral.

$$\int_0^1 e^{-x^2} dx$$

**SOLUTION**    The eighth-degree Taylor polynomial for $e^{-x^2}$ is

$$S_8(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}.$$

Using this polynomial, you can write the approximation as shown.

$$\int_0^1 e^{-x^2} dx \approx \int_0^1 S_8(x) dx$$
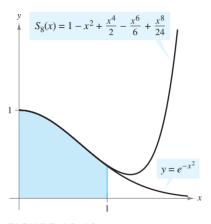
$$= \int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\right) dx$$

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{5(2)} - \frac{x^7}{7(6)} + \frac{x^9}{9(24)}\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216}$$

$$\approx 0.747$$

This result is shown graphically in Figure 10.10. Try comparing this result with that obtained using a symbolic integration utility. When such a utility approximated the integral, it returned a value of 0.746824. So, the approximation has an error of less than 0.001.



$S_8(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}$

$y = e^{-x^2}$

**FIGURE 10.10**

### ✓ CHECKPOINT 3

Use the 12th-degree Taylor polynomial from Checkpoint 1 to approximate

$$\int_0^1 e^{2x^3} dx.$$ ■

When using a Taylor polynomial to approximate a function on an interval, remember that as you move farther away from the center of the Taylor polynomial, you must use a polynomial of higher and higher degree to obtain an indicated accuracy. For definite integrals, it helps to locate the center of the Taylor polynomial at the midpoint of the interval of integration. For instance, if you want to approximate the definite integral

$$\int_0^2 e^{-x^2} dx$$

you could use a Taylor polynomial centered at 1. If you try this with an eighth-degree Taylor polynomial, you will obtain an approximation of 0.882.

## Applications of Probability

Many applications of probability involve experiments in which the sample space is

$$S = \{0, 1, 2, 3, 4, \ldots\}. \qquad \text{Sample space}$$

As is always true in probability, the sum of the probabilities of the various outcomes is 1. That is, if $P(n)$ is the probability that $n$ will occur, then

$$\sum_{n=0}^{\infty} P(n) = P(0) + P(1) + P(2) + P(3) + \cdots$$

$$= 1.$$

The next example shows how this concept is used in a coin-tossing experiment.

### Example 4  Finding Probabilities  Ⓡ

A fair coin is tossed until a head turns up. The possible outcomes are

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}$$
$$\{H, TH, TTH, TTTH, TTTTH, \ldots\}$$

where the random variable assigned to each outcome is the number of tails that have consecutively appeared before a head is tossed (which automatically ends the experiment). Show that the sum of the probabilities in this experiment is 1.
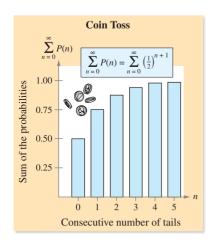
**SOLUTION**  The probability that no tails will appear is $\frac{1}{2}$. The probability that exactly one tail will appear is $\left(\frac{1}{2}\right)^2$. The probability that exactly two tails will appear is $\left(\frac{1}{2}\right)^3$, and so on. So, the sum of the probabilities is given by a geometric series.

$$\sum_{n=0}^{\infty} P(n) = P(0) + P(1) + P(2) + P(3) + \cdots$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \qquad \text{Geometric series}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^n \qquad a = \tfrac{1}{2} \text{ and } r = \tfrac{1}{2}$$

$$= \frac{a}{1 - r} \qquad \text{Sum of a geometric series}$$

$$= \frac{1/2}{1 - 1/2}$$

$$= 1$$

So, the sum of the probabilities is 1. This result is shown graphically in Figure 10.11.



**Coin Toss**

$$\sum_{n=0}^{\infty} P(n) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{n+1}$$

**FIGURE 10.11**

### ✓CHECKPOINT 4

Use the geometric series in Example 4 to find the probability that exactly 10 tails will appear. How many times must the coin be tossed until the probability that exactly one head appears is less than $\frac{1}{2}\%$? ■

The **expected value** of a random variable whose range is $\{0, 1, 2, \ldots\}$ is

$$\text{Expected value} = \sum_{n=0}^{\infty} nP(n) = 0P(0) + 1P(1) + 2P(2) + \cdots.$$

This formula is demonstrated in Example 5.

*Example 5* **Finding an Expected Value** ®

Find the expected value for the coin-tossing experiment in Example 4.

**SOLUTION** Using the probabilities from Example 4, you can write

$$\text{Expected value} = \sum_{n=0}^{\infty} nP(n)$$

$$= \sum_{n=0}^{\infty} n\left(\tfrac{1}{2}\right)^{n+1}$$

$$= 0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1}{2}\right)^3 + 3\left(\tfrac{1}{2}\right)^4 + \cdots.$$

This series is not geometric, so to find its sum you need to resort to a different tactic. Using the binomial series from Section 10.4, you can write

$$(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

which implies that

$$\left(1 - \tfrac{1}{2}\right)^{-2} = 1 + 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right)^2 + 4\left(\tfrac{1}{2}\right)^3 + \cdots$$

$$= \left(\tfrac{1}{2}\right)^{-2} = 4.$$

So, you can conclude that the expected value is

$$\text{Expected value} = 1\left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1}{2}\right)^3 + 3\left(\tfrac{1}{2}\right)^4 + \cdots$$

$$= \left(\tfrac{1}{2}\right)^2\left[1 + 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right)^2 + 4\left(\tfrac{1}{2}\right)^3 + \cdots\right]$$

$$= \tfrac{1}{4}(4) = 1.$$

This means that if the experiment were conducted many times, the average number of tails that would occur would be 1.

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An experiment consists of tossing a fair coin repeatedly until the coin finally lands "heads up." Each time the experiment is performed, the number of tosses necessary to obtain "heads" is counted. If this experiment is performed hundreds of times, the average number of tosses per experiment will be 2.

✓ **CHECKPOINT 5**

Find the expected value for the probability distribution represented by

$$\sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^{n-1}. \quad ∎$$

**CONCEPT CHECK**

1. A Taylor polynomial for $e^{-x}$ is $S_3(x) = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!}$. What is the degree of the polynomial?

2. Without calculating, which Taylor polynomial do you think would give a better approximation of $f(x) = e^{-x}$: $S_1(x) = 1 - x$ or $S_3(x) = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!}$?

3. Describe the accuracy of the $n$th-degree Taylor polynomial of $f$ centered at $c$ as the distance between $c$ and $x$ increases.

4. In general, how does the accuracy of a Taylor polynomial change as the degree of the polynomial is increased?

In Exercises 1–6, find a power series representation for the function.

**1.** $f(x) = e^{3x}$

**2.** $f(x) = e^{-3x}$

**3.** $f(x) = \dfrac{4}{x}$

**4.** $f(x) = \ln 5x$

**5.** $f(x) = (1 + x)^{1/4}$

**6.** $f(x) = \sqrt{1 + x}$

In Exercises 7–10, evaluate the definite integral.

**7.** $\displaystyle\int_0^1 (1 - x + x^2 - x^3 + x^4)\,dx$

**8.** $\displaystyle\int_0^{1/2} \left(1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{27}\right)dx$

**9.** $\displaystyle\int_1^2 \left[(x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}\right]dx$

**10.** $\displaystyle\int_1^{3/2} [1 - (x - 1) + (x - 1)^2 - (x - 1)^3]\,dx$

## Exercises 10.5

In Exercises 1–12, find the Taylor polynomials (centered at zero) of degrees (a) 1, (b) 2, (c) 3, and (d) 4.

**1.** $f(x) = e^x$

**2.** $f(x) = e^{-x}$

**3.** $f(x) = e^{2x}$

**4.** $f(x) = e^{3x}$

**5.** $f(x) = \ln(x + 1)$

**6.** $f(x) = \ln(2x + 1)$

**7.** $f(x) = \sqrt{x + 1}$

**8.** $f(x) = \sqrt[3]{x + 1}$

**9.** $f(x) = \dfrac{1}{(x + 1)^2}$

**10.** $f(x) = \dfrac{1}{(x + 1)^3}$

**11.** $f(x) = \dfrac{x}{x + 1}$

**12.** $f(x) = \dfrac{4}{x + 1}$

(S) In Exercises 13 and 14, use a spreadsheet to complete the table using the indicated Taylor polynomials as approximations of the function $f$.

**13.**

| $x$ | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
|---|---|---|---|---|---|
| $f(x) = e^{x/2}$ | | | | | |
| $1 + \dfrac{x}{2}$ | | | | | |
| $1 + \dfrac{x}{2} + \dfrac{x^2}{8}$ | | | | | |
| $1 + \dfrac{x}{2} + \dfrac{x^2}{8} + \dfrac{x^3}{48}$ | | | | | |
| $1 + \dfrac{x}{2} + \dfrac{x^2}{8} + \dfrac{x^3}{48} + \dfrac{x^4}{384}$ | | | | | |

**14.**

| $x$ | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ |
|---|---|---|---|---|
| $f(x) = \ln(x^2 + 1)$ | | | | |
| $x^2$ | | | | |
| $x^2 - \dfrac{x^4}{2}$ | | | | |
| $x^2 - \dfrac{x^4}{2} + \dfrac{x^6}{3}$ | | | | |
| $x^2 - \dfrac{x^4}{2} + \dfrac{x^6}{3} - \dfrac{x^8}{4}$ | | | | |

(T) In Exercises 15 and 16, use a symbolic differentiation utility to find the Taylor polynomials (centered at zero) of degrees (a) 2, (b) 4, (c) 6, and (d) 8.

**15.** $f(x) = \dfrac{1}{1 + x^2}$

**16.** $f(x) = e^{-x^2}$
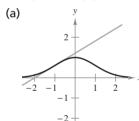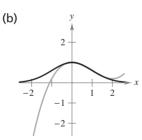
(T) In Exercises 17 and 18, use a symbolic differentiation utility to find the fourth-degree Taylor polynomial (centered at zero).
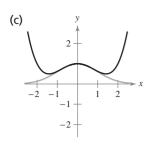
**17.** $f(x) = \dfrac{1}{\sqrt[3]{x + 1}}$
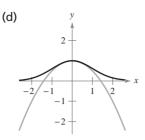
**18.** $f(x) = xe^x$

In Exercises 19–22, match the Taylor polynomial approximation of the function $f(x) = e^{-x^2/2}$ with its graph. [The graphs are labeled (a)–(d).] Use a graphing utility to verify your results.

(a)



(b)



(c)



(d)



**19.** $y = -\frac{1}{2}x^2 + 1$

**20.** $y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

**21.** $y = e^{-1/2}[(x + 1) + 1]$

**22.** $y = e^{-1/2}\left[\frac{1}{3}(x - 1)^3 - (x - 1) + 1\right]$

In Exercises 23–26, use a sixth-degree Taylor polynomial centered at $c$ for the function $f$ to obtain the required approximation.

| Function | Approximation |
|---|---|
| **23.** $f(x) = e^{-x}, \quad c = 0$ | $f\left(\frac{1}{2}\right)$ |
| **24.** $f(x) = x^2 e^{-x}, \quad c = 0$ | $f\left(\frac{1}{4}\right)$ |
| **25.** $f(x) = \ln x, \quad c = 2$ | $f\left(\frac{3}{2}\right)$ |
| **26.** $f(x) = \sqrt{x}, \quad c = 4$ | $f(5)$ |

In Exercises 27–30, use a sixth-degree Taylor polynomial centered at zero to approximate the integral.

| Function | Approximation |
|---|---|
| **27.** $f(x) = e^{-x^2}$ | $\int_0^1 e^{-x^2} \, dx$ |
| **28.** $f(x) = \ln(x^2 + 1)$ | $\int_{-1/4}^{1/4} \ln(x^2 + 1) \, dx$ |
| **29.** $f(x) = \dfrac{1}{\sqrt{1 + x^2}}$ | $\int_0^{1/2} \dfrac{1}{\sqrt{1 + x^2}} \, dx$ |
| **30.** $f(x) = \dfrac{1}{\sqrt[3]{1 + x^2}}$ | $\int_0^{1/2} \dfrac{1}{\sqrt[3]{1 + x^2}} \, dx$ |

In Exercises 31 and 32, determine the degree of the Taylor polynomial centered at $c$ required to approximate $f$ in the given interval to an accuracy of $\pm 0.001$.

| Function | Interval |
|---|---|
| **31.** $f(x) = e^x, \quad c = 1$ | $[0, 2]$ |
| **32.** $f(x) = \dfrac{1}{x}, \quad c = 1$ | $\left[1, \dfrac{3}{2}\right]$ |

In Exercises 33 and 34, determine the maximum error guaranteed by Taylor's Theorem with Remainder when the fifth-degree Taylor polynomial is used to approximate $f$ in the given interval.

**33.** $f(x) = e^{-x}, \quad [0, 1], \quad$ centered at 0

**34.** $f(x) = \dfrac{1}{x}, \quad \left[1, \dfrac{3}{2}\right], \quad$ centered at 1

**35. Profit**   Let $n$ be a random variable representing the number of units of a certain commodity sold per day in a certain store. The probability distribution of $n$ is shown in the table.

| $n$ | 0 | 1 | 2 | 3 | 4, . . . |
|---|---|---|---|---|---|
| $P(n)$ | $\frac{1}{2}$ | $\left(\frac{1}{2}\right)^2$ | $\left(\frac{1}{2}\right)^3$ | $\left(\frac{1}{2}\right)^4$ | $\left(\frac{1}{2}\right)^5$, . . . |

(a) Show that
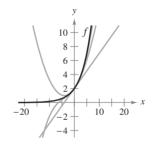
$$\sum_{n=0}^{\infty} P(n) = 1.$$

(b) Find the expected value of the random variable $n$.

(c) If there is a \$10 profit on each unit sold, what is the expected daily profit on this commodity?

**36. Profit**   Repeat Exercise 35 for the probability distribution for $n$ that is shown in the table below.

| $n$ | 0 | 1 | 2 | 3, . . . |
|---|---|---|---|---|
| $P(n)$ | $\frac{1}{2}\left(\frac{2}{3}\right)$ | $\frac{1}{2}\left(\frac{2}{3}\right)^2$ | $\frac{1}{2}\left(\frac{2}{3}\right)^3$ | $\frac{1}{2}\left(\frac{2}{3}\right)^4$, . . . |

**37. Graphical Reasoning**   The figure shows first-, second-, and third-degree polynomial approximations $P_1$, $P_2$, and $P_3$ of a function $f$. Label the graphs of $P_1$, $P_2$, and $P_3$.

# Newton's Method

■ Use Newton's Method to approximate the zeros of functions.
■ Use Newton's Method to approximate points of intersection of graphs.

## Newton's Method

Finding the zeros of a function is a fundamental problem in algebra and in calculus. Sometimes this problem can be solved algebraically. For instance, you can find the zeros of the quadratic function

$$f(x) = 2x^2 - 13x + 1$$

by applying the Quadratic Formula to the equation

$$2x^2 - 13x + 1 = 0.$$

In real life, however, you often encounter functions whose zeros cannot be found so easily. In some cases, you can use various approximation methods to find the zeros. One such method, called **Newton's Method,** is described in this section.

Consider a function $f$, such as the one whose graph is shown in Figure 10.12(a). An actual zero of the function is $x = c$. To approximate this zero, choose $x_1$ close to $c$ and form the first-degree Taylor polynomial centered at $x_1$.

$$S_1(x) = f(x_1) + f'(x_1)(x - x_1) \qquad \text{First-degree Taylor polynomial}$$

Graphically, you can interpret this polynomial as the equation of the tangent line to the graph of $f$ at the point $(x_1, f(x_1))$. Newton's Method is based on the assumption that this tangent line will cross the $x$-axis at about the same point as the graph of $f$ crosses the $x$-axis. With this assumption, set $S_1(x)$ equal to zero, solve for $x$

$$f(x_1) + f'(x_1)(x - x_1) = 0$$

$$x - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and use the resulting value as a new, and hopefully better, approximation of the actual zero $c$. So, from the approximation $x_1$, you form a second approximation
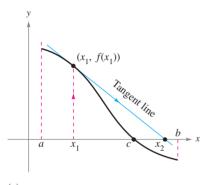
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \qquad \text{Second estimate}$$

If you want to obtain an even better approximation, you can use $x_2$ to calculate $x_3$,
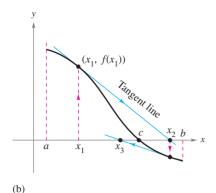
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}. \qquad \text{Third estimate [see Figure 10.12(b)].}$$

Repeated application of this process is called Newton's Method.

For many functions, just a few iterations of Newton's Method will produce approximations having very small errors.

(a)

(b)

**FIGURE 10.12** The *x*-intercept of the tangent line approximates the zero of *f*.

**Newton's Method**

Let $c$ be a zero of $f$ and let $f$ be differentiable on an open interval containing $c$. To approximate $c$, use the following steps.

1. Make an initial approximation $x_1$ that is close to $c$. (A graph is helpful.)

2. Determine a new approximation using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. If $|x_n - x_{n+1}|$ is within the desired accuracy, let $x_{n+1}$ serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration.**

*Example 1* **Using Newton's Method**

Use three iterations of Newton's Method to approximate a zero of

$$f(x) = x^2 - 2.$$
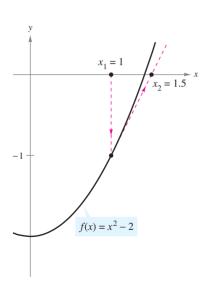
Use $x_1 = 1$ as the initial guess.

**SOLUTION** The first derivative of $f$ is $f'(x) = 2x$. So, the iterative formula for Newton's Method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 2}{2x_n}.$$

The calculations for three iterations are shown in the table. The first iteration is depicted graphically in Figure 10.13.



**FIGURE 10.13**

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 1.000000 | $-1.000000$ | 2.000000 | $-0.500000$ | 1.500000 |
| 2 | 1.500000 | 0.250000 | 3.000000 | 0.083333 | 1.416667 |
| 3 | 1.416667 | 0.006945 | 2.833334 | 1.002451 | 1.414216 |
| 4 | 1.414216 | | | | |

So, the approximation is $x = 1.414216$. For this particular function, you can easily determine the exact zero to be $\sqrt{2} \approx 1.414214$. So, after only three iterations of Newton's Method, you can obtain an approximation that is within 0.000002 of an actual root.

✓**CHECKPOINT 1**

Use three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 3$. Use $x_1 = 1$ as the initial guess. ■

$f(x) = 2x^3 + x^2 - x + 1$

**FIGURE 10.14**

### Example 2 | Using Newton's Method

Use Newton's Method to approximate the zeros of $f(x) = 2x^3 + x^2 - x + 1$. Continue the iterations until two successive approximations differ by less than 0.001.

**SOLUTION**    Begin by sketching a graph of $f$, as shown in Figure 10.14. From the graph, you can observe that the function has only one zero, which occurs near $x_1 = -1.2$. Next, differentiate $f$ and form the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n^3 + x_n^2 - x_n + 1}{6x_n^2 + 2x_n - 1}.$$

The calculations are shown in the table.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | $-1.20000$ | $0.18400$ | $5.24000$ | $0.03511$ | $-1.23511$ |
| 2 | $-1.23511$ | $-0.00771$ | $5.68276$ | $-0.00136$ | $-1.23375$ |
| 3 | $-1.23375$ | $-0.00001$ | $5.66533$ | $-0.00000$ | $-1.23375$ |
| 4 | $-1.23375$ | | | | |

So, you can estimate the zero of $f$ to be $-1.23375$.

### Example 3 | Using Newton's Method

Use Newton's Method to approximate the zeros of $f(x) = e^x + x$. Continue the iterations until two successive approximations differ by less than 0.001.

**SOLUTION**    Begin by sketching a graph of $f$, as shown in Figure 10.15. From the graph, you can observe that the function has only one zero, which occurs near $x_1 = -0.5$. Next, differentiate $f$ and form the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} + x_n}{e^{x_n} + 1}.$$

The calculations are shown in the table.



$f(x) = e^x + x$

**FIGURE 10.15**

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | $-0.500000$ | $0.106531$ | $1.606531$ | $0.066311$ | $-0.566311$ |
| 2 | $-0.566311$ | $0.001305$ | $1.567616$ | $0.000832$ | $-0.567143$ |
| 3 | $-0.567143$ | | | | |

So, you can estimate the zero of $f$ to be $-0.567143$.

**Programming Newton's Method**
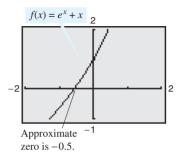
(T) Technology can be used in a variety of ways to approximate zeros of functions. Earlier in the text, you saw how the *zoom* feature of a graphing utility could be used to approximate a zero graphically.

Another way to use technology to approximate the zeros of a function is to program Newton's Method into a graphing utility. Appendix E lists the programs for several models of graphing utilities.

Before running the program, enter the function into the utility. Then graph the function and estimate one of its zeros. When you run the program, you will be prompted to enter the estimate as $x$. After two successive approximations differ by less than 0.0000000001, the program will display the refined approximation. For instance, to use the program to approximate the zero of the function in Example 3

$$f(x) = e^x + x$$

begin by graphing the function, as shown below. Then run the program and enter $-0.5$ as the initial approximation. The calculator should display a refined approximation of $-0.5671432904$ using four iterations.

$f(x) = e^x + x$

Approximate zero is $-0.5$.

## Convergence of Newton's Method
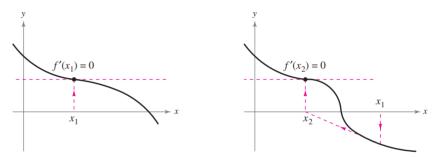
When, as in Examples 1, 2, and 3, the approximations approach a zero of a function, Newton's Method is said to **converge.** You should know, however, that Newton's Method does not always converge. Two situations in which it may not converge are as shown.

**1.** If $f'(x_n) = 0$ for some $n$. (See Figure 10.16.)

**2.** If $\lim_{n \to \infty} x_n$ does not exist. (See Figure 10.17.)
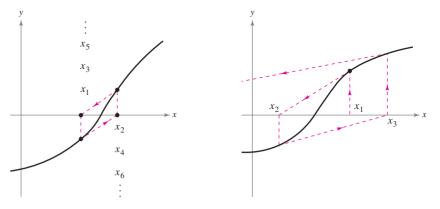
The type of problem illustrated in Figure 10.16 can usually be overcome with a better choice of $x_1$. The problem illustrated in Figure 10.17, however, is usually more serious. For instance, Newton's Method does not converge for any choice of $x_1$ (other than the actual zero) for the function

$$f(x) = x^{1/3}.$$
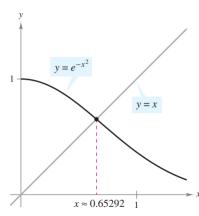
Using a graphing utility, try running the program discussed at the left to approximate the zero of this function. If you do, you will find that the program fails.



**FIGURE 10.16**   Newton's Method fails to converge if $f'(x_n) = 0$.



**FIGURE 10.17**   Newton's Method fails to converge for every $x$-value other than the actual zero of $f$.

**FIGURE 10.18**

✓ **CHECKPOINT 4**

Estimate the point of intersection
of the graphs of $y = e^{-x}$ and
$y = x$. Use Newton's Method and
continue the iterations until two
successive approximations differ
by less than 0.001. ∎

*Example 4*  **Finding a Point of Intersection**

Estimate the point of intersection of the graphs of

$$y = e^{-x^2} \quad \text{and} \quad y = x$$

as shown in Figure 10.18. Use Newton's Method and continue the iterations until
two successive approximations differ by less than 0.0001.

**SOLUTION**   The point of intersection of the two graphs occurs when $e^{-x^2} = x$,
which implies that

$$0 = x - e^{-x^2}.$$

To use Newton's Method, let

$$f(x) = x - e^{-x^2}$$

and employ the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n - e^{-x_n^2}}{1 + 2x_n e^{-x_n^2}}.$$

The table shows three iterations of Newton's Method beginning with an initial
approximation of $x_1 = 0.5$.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\dfrac{f(x_n)}{f'(x_n)}$ | $x_n - \dfrac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | 0.500000 | $-0.27880$ | 1.77880 | $-0.15673$ | 0.65673 |
| 2 | 0.65673 | 0.00706 | 1.85331 | 0.00381 | 0.65292 |
| 3 | 0.65292 | 0.00000 | 1.85261 | 0.00000 | 0.65292 |
| 4 | 0.65292 | | | | |

So, you can estimate that the point of intersection occurs when $x \approx 0.65292$.

**CONCEPT CHECK**

1. What is each successive application of the procedure for Newton's
   Method called?
2. Explain why Newton's Method fails when $f(x) = x^2 - 4x$ and $x_1 = 2$.
3. When using Newton's Method to find the zero of a function, will any
   initial approximation work?
4. Under what conditions will Newton's Method fail?

**Skills Review 10.6**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.2, 1.2, 2.2, and 4.3.

In Exercises 1–4, evaluate $f$ and $f'$ at the given $x$-value.

**1.** $f(x) = x^2 - 2x - 1$, $x = 2.4$

**2.** $f(x) = x^3 - 2x^2 + 1$, $x = -0.6$

**3.** $f(x) = e^{2x} - 2$, $x = 0.35$

**4.** $f(x) = e^{x^2} - 7x + 3$, $x = 1.4$

In Exercises 5–8, solve for $x$.

**5.** $|x - 5| \leq 0.1$      **6.** $|4 - 5x| \leq 0.01$      **7.** $\left|2 - \dfrac{x}{3}\right| \leq 0.01$      **8.** $|2x + 7| \leq 0.01$

In Exercises 9 and 10, find the point(s) of intersection of the graphs of the two equations.

**9.** $y = x^2 - x - 2$, $y = 2x - 1$

**10.** $y = x^2$, $y = x + 1$
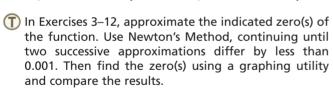
## Exercises 10.6
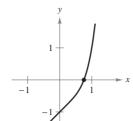
In Exercises 1 and 2, complete two iterations of Newton's Method for the function using the given initial estimate.

**1.** $f(x) = x^2 - 3$, $x_1 = 1.7$    **2.** $f(x) = 2x^2 - 3$, $x_1 = 1$

Ⓣ In Exercises 3–12, approximate the indicated zero(s) of the function. Use Newton's Method, continuing until two successive approximations differ by less than 0.001. Then find the zero(s) using a graphing utility and compare the results.

**3.** $f(x) = x^3 + x - 1$
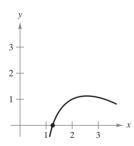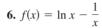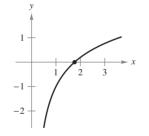


**4.** $f(x) = x^5 + x - 1$
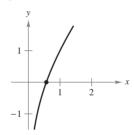


**5.** $y = 5\sqrt{x - 1} - 2x$



**6.** $f(x) = \ln x - \dfrac{1}{x}$



**7.** $f(x) = \ln x + x$



**8.** $y = e^{3x}(3 - x) - 1$



**9.** $f(x) = e^{-x^2} - x^2$



**10.** $y = x^4 + x^3 - 1$



**11.** $f(x) = x^3 - 27x - 27$



**12.** $y = x^3 - x^2 + 3$
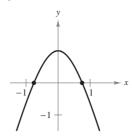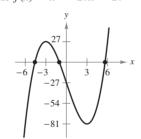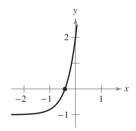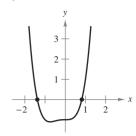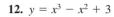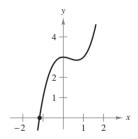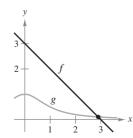
In Exercises 13–16, approximate, to three decimal places, the x-value of the point of intersection of the graphs.

**13.** $f(x) = 3 - x$

$g(x) = \dfrac{1}{x^2 + 1}$



**14.** $f(x) = 2x + 1$

$g(x) = \sqrt{x + 4}$



**15.** $f(x) = x$

$g(x) = e^{-x}$



**16.** $f(x) = -x$

$g(x) = \ln x$
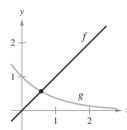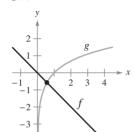


(T) In Exercises 17–30, use a graphing utility to approximate all the real zeros of the function by Newton's Method. Graph the function to make the initial estimate of a zero.

**17.** $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

**18.** $f(x) = \frac{1}{2}x^3 - \frac{11}{3}x^2 + \frac{10}{3}x + \frac{32}{3}$

**19.** $f(x) = \frac{1}{2}x^4 - 3x - 3$

**20.** $f(x) = -x^4 + 5x^2 - 5$

**21.** $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

**22.** $f(x) = 0.16e^{x^2} - 4.1x^3 + 6.8$

**23.** $f(x) = 3\sqrt{x - 1} - x$

**24.** $f(x) = x^4 - 10x^2 - 11$

**25.** $f(x) = x - e^{-x}$

**26.** $f(x) = x - 3 + \ln x$

**27.** $f(x) = x^2 - \ln x - \frac{3}{2}$
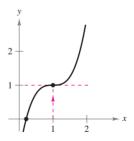
**28.** $f(x) = x + \sin(x + 1)$

**29.** $f(x) = x^3 - \cos x$
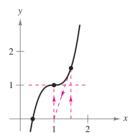
**30.** $f(x) = \sin \pi x + x - 1$

In Exercises 31–34, apply Newton's Method using the indicated initial estimate. Then explain why the method fails.
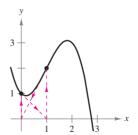
**31.** $y = 2x^3 - 6x^2 + 6x - 1, \quad x_1 = 1$
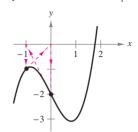


**32.** $y = 4x^3 - 12x^2 + 12x - 3, \quad x_1 = \frac{3}{2}$



**33.** $y = -x^3 + 3x^2 - x + 1, \quad x_1 = 1$



**34.** $y = x^3 - 2x - 2, \quad x_1 = 0$



In Exercises 35 and 36, use Newton's Method to obtain a general rule for approximating the indicated radical.

**35.** $\sqrt{a}$  [*Hint:* Consider $f(x) = x^2 - a$.]

**36.** $\sqrt[n]{a}$  [*Hint:* Consider $f(x) = x^n - a$.]

In Exercises 37–40, use the results of Exercises 35 and 36 to approximate the indicated radical to three decimal places.

**37.** $\sqrt{7}$

**38.** $\sqrt{5}$

**39.** $\sqrt[4]{6}$

**40.** $\sqrt[3]{15}$
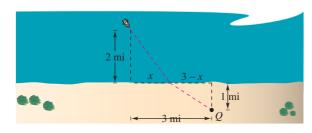
**41.** Use Newton's Method to show that the equation $x_{n+1} = x_n(2 - ax_n)$ can be used to approximate $1/a$ if $x_1$ is an initial guess of the reciprocal of $a$. Note that this method of approximating reciprocals uses only the operations of multiplication and subtraction. [*Hint:* Consider $f(x) = (1/x) - a$.]

**42.** Use the result of Exercise 41 to approximate (a) $\frac{1}{3}$ and (b) $\frac{1}{11}$ to three decimal places.

In Exercise 43–50, some typical problems from previous chapters are given. In each case, use Newton's Method to approximate the solution.

**43. Minimum Distance** Find the point on the graph of $f(x) = 4 - x^2$ that is closest to the point $(1, 0)$.

**44. Minimum Distance** Find the point on the graph of $f(x) = x^2$ that is closest to the point $(4, -3)$.

**45. Minimum Time** You are in a boat 2 miles from the nearest point on the coast (see figure). You are to go to a point $Q$, which is 3 miles down the coast and 1 mile inland. You can row at 3 miles per hour and walk at 4 miles per hour. Toward what point on the coast should you row in order to reach point $Q$ in the least time?



**46. Average Cost** A company estimates that the cost in dollars of producing $x$ units of a product is given by

$$C = 0.0001x^3 + 0.02x^2 + 0.4x + 800.$$

Find the production level that minimizes the average cost per unit.

**47. Advertising Costs** A company that produces portable digital audio players estimates that the profit for selling a particular model is

$$P = -76x^3 + 4830x^2 - 320,000, \quad 0 \le x \le 60$$

where $P$ is the profit in dollars and $x$ is the advertising expense in 10,000s of dollars (see figure). According to this model, find the smaller of the two advertising amounts that yield a profit $P$ of $2,500,000.
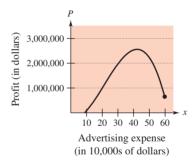


Figure for 47

**48. Medicine** The concentration $C$ of an antibiotic in the bloodstream $t$ hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{50 + t^3}.$$

When is the concentration the greatest?

**49. Cost** The ordering and transportation cost $C$ of the components used in manufacturing a product is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \ge 1$$

where $C$ is measured in thousands of dollars and $x$ is the order size in hundreds. Find the order size that minimizes the cost.

**50. Forestry** The value of a tract of timber is given by

$$V(t) = 100,000e^{0.8\sqrt{t}}$$

where $t$ is the time in years, with $t = 0$ corresponding to 2000. If money earns interest at a rate of $r = 10\%$ compounded continuously, then the present value of the timber at any time $t$ is given by

$$A(t) = V(t)e^{-0.10t}.$$

Assume the cost of maintenance of the timber to be a constant cash flow at the rate of $1000 per year. Then the total present value of this cost for $t$ years is given by

$$C(t) = \int_0^t 1000e^{-0.10u}\, du$$

and the net present value of the tract of timber is given by

$$P(t) = A(t) - C(t).$$

Find the year when the timber should be harvested to maximize the present value function $P$.

**True or False?** In Exercises 51 and 52, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**51.** The zeros of $f(x) = \dfrac{p(x)}{q(x)}$ coincide with the zeros of $p(x)$.

**52.** The roots of $\sqrt{f(x)} = 0$ coincide with the roots of $f(x) = 0$.

## Algebra Review

### *Example 1*   Simplifying Factorial Expressions

Simplify each expression.

**a.** $\dfrac{8!}{2! \cdot 6!}$    **b.** $\dfrac{2! \cdot 6!}{3! \cdot 5!}$    **c.** $\dfrac{n!}{(n-1)!}$    **d.** $\dfrac{(2n+2)!}{(2n+4)!}$

**e.** $\dfrac{x^{n+1}}{(n+1)!} \div \dfrac{x^n}{n!}$    **f.** $\dfrac{2^{n+1}x^{n+1}}{(n+1)!} \div \dfrac{2^n x^n}{n!}$

**SOLUTION**

**a.** $\dfrac{8!}{2! \cdot 6!} = \dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$    <span style="color:magenta">Factor.</span>

$\qquad = \dfrac{7 \cdot 8}{1 \cdot 2}$    <span style="color:magenta">Divide out like factors.</span>

$\qquad = \dfrac{56}{2}$    <span style="color:magenta">Multiply.</span>

$\qquad = 28$    <span style="color:magenta">Divide.</span>

**b.** $\dfrac{2! \cdot 6!}{3! \cdot 5!} = \dfrac{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$    <span style="color:magenta">Factor.</span>

$\qquad = \dfrac{6}{3}$    <span style="color:magenta">Divide out like factors.</span>

$\qquad = 2$    <span style="color:magenta">Divide.</span>

**c.** $\dfrac{n!}{(n-1)!} = \dfrac{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n}{1 \cdot 2 \cdot 3 \cdots (n-1)}$    <span style="color:magenta">Factor.</span>

$\qquad\qquad = n$    <span style="color:magenta">Divide out like factors.</span>

**d.** $\dfrac{(2n+2)!}{(2n+4)!} = \dfrac{(2n+2)!}{(2n+2)!(2n+3)(2n+4)}$    <span style="color:magenta">Factor.</span>

$\qquad\qquad = \dfrac{1}{(2n+3)(2n+4)}$    <span style="color:magenta">Divide out like factors.</span>

**e.** $\dfrac{x^{n+1}}{(n+1)!} \div \dfrac{x^n}{n!} = \dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}$    <span style="color:magenta">Multiply by reciprocal.</span>

$\qquad\qquad = \dfrac{x \cdot x^n}{n!(n+1)} \cdot \dfrac{n!}{x^n}$    <span style="color:magenta">Factor.</span>

$\qquad\qquad = \dfrac{x}{n+1}$    <span style="color:magenta">Divide out like factors.</span>

**f.** $\dfrac{2^{n+1}x^{n+1}}{(n+1)!} \div \dfrac{2^n x^n}{n!} = \dfrac{2^{n+1}x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^n x^n}$    <span style="color:magenta">Multiply by reciprocal.</span>

$\qquad\qquad = \dfrac{2 \cdot 2^n \cdot x \cdot x^n \cdot n!}{n!(n+1) \cdot 2^n \cdot x^n}$    <span style="color:magenta">Factor.</span>

$\qquad\qquad = \dfrac{2x}{n+1}$    <span style="color:magenta">Divide out like factors.</span>

| Example 2 | **Rewriting Expressions with Sigma Notation** |

Rewrite each expression.

**a.** $\displaystyle\sum_{i=1}^{5} 3i$    **b.** $\displaystyle\sum_{k=3}^{6} (1 + k^2)$    **c.** $\displaystyle\sum_{i=0}^{8} \frac{1}{i!}$

**d.** $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^n$    **e.** $\displaystyle\sum_{n=1}^{20} 6(1.01)^n$    **f.** $\displaystyle\sum_{n=1}^{10} \left(n + \frac{1}{3^n}\right)$

**SOLUTION**

**a.** $\displaystyle\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$

$\qquad = 3(1 + 2 + 3 + 4 + 5)$

$\qquad = 3(15) = 45$

**b.** $\displaystyle\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2)$

$\qquad\qquad = 10 + 17 + 26 + 37 = 90$

**c.** $\displaystyle\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$\qquad = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40,320}$

$\qquad \approx 2.71828$

**d.** $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^{n+1}$

$\qquad\qquad = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n$

$\qquad\qquad = \frac{3}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

**e.** $\displaystyle\sum_{n=1}^{20} 6(1.01)^n = 6(1.01)^1 + 6(1.01)^2 + \cdots + 6(1.01)^{20}$

$\qquad\qquad = -6 + 6 + 6(1.01)^1 + 6(1.01)^2 + \cdots + 6(1.01)^{20}$

$\qquad\qquad = -6 + 6(1.01)^0 + 6(1.01)^1 + 6(1.01)^2 + \cdots + 6(1.01)^{20}$

$\qquad\qquad = -6 + \sum_{n=0}^{20} 6(1.01)^n$

**f.** $\displaystyle\sum_{n=1}^{10} \left(n + \frac{1}{3^n}\right) = \sum_{n=1}^{10} n + \sum_{n=1}^{10} \frac{1}{3^n}$

$\qquad\qquad = \sum_{n=1}^{10} n + \sum_{n=1}^{10} \left(\frac{1}{3}\right)^n$

$\qquad\qquad = \sum_{n=1}^{10} n - 1 + \sum_{n=0}^{10} \left(\frac{1}{3}\right)^n$

# Chapter Summary and Study Strategies

After studying this chapter, you should have acquired the following skills.
The exercise numbers are keyed to the Review Exercises that begin on page 724.
Answers to odd-numbered Review Exercises are given in the back of the text.*

| Section 10.1 | Review Exercises |
|---|---|
| ■ Find the terms of sequences. | *1–4* |
| ■ Determine the convergence or divergence of sequences and find the limits of convergent sequences. | *5–12* |
| ■ Find the $n$th terms of sequences. | *13–16* |
| ■ Use sequences to answer questions about real-life situations. | *17–20* |

| Section 10.2 | |
|---|---|
| ■ Find the terms of sequences. | *21–24* |
| $\quad S_1 = a_1,\quad S_2 = a_1 + a_2,\quad S_3 = a_1 + a_2 + a_3,\ \ldots\ .$ | |
| ■ Determine the convergence or divergence of infinite series. | *25–28* |
| ■ Use the $n$th-Term Test to show that series diverge. | *29–32* |
| $\quad$ The series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges if $\displaystyle\lim_{n\to\infty} a_n \neq 0$. | |
| ■ Find the $n$th partial sums of geometric series. | *33–36* |
| $\quad S_n = \dfrac{a(1 - r^{n+1})}{1 - r},\quad r \neq 1.$ | |
| ■ Determine the convergence or divergence of geometric series. | *37–42* |
| $\quad$ If $|r| \geq 1$, the series *diverges*. | |
| $\quad$ If $|r| < 1$, the series *converges* and its sum is $\dfrac{a}{1 - r}$. | |
| ■ Use geometric series to model real-life situations. | *43, 44* |
| ■ Use sequences to solve real-life problems. | *45–50* |

| Section 10.3 | |
|---|---|
| ■ Determine the convergence or divergence of $p$-series. | *51–54* |
| $\quad$ The series diverges if $0 < p \leq 1$. | |
| $\quad$ The series converges if $p > 1$. | |
| ■ Match sequences with their graphs. | *55–58* |
| ■ Approximate the sums of convergent $p$-series. | *59–62* |
| ■ Use the Ratio Test to determine the convergence or divergence of series. | *63–68* |
| $\quad$ The series converges if $\displaystyle\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1.$ | |
| $\quad$ The series diverges if $\displaystyle\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| > 1.$ | |

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\* Use a wide range of valuable study aids to help you master the material in this chapter. The *Student Solutions Guide* includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at *college.hmco.com/info/larsonapplied* offers algebra help and a *Graphing Technology Guide*. The *Graphing Technology Guide* contains step-by-step commands and instructions for a wide variety of graphing calculators, including the most recent models.

## Section 10.4

- Find the radii of convergence of power series.
- Use Taylor's Theorem to find power series for functions.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!} = f(c) + \frac{f'(c)(x-c)}{1!} + \frac{f''(c)(x-c)^2}{2!} + \cdots$$

- Use the basic list of power series to find power series for functions.

## Section 10.5

- Find Taylor polynomials for functions.
- Use Taylor polynomials to approximate the values of functions at points.
- Determine the maximum errors of approximations using Taylor polynomials.
- Approximate definite integrals using Taylor polynomials.
- Use Taylor polynomials to model probabilities.

## Section 10.6

- Use Newton's Method to approximate the zeros of functions.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Use Newton's Method to approximate points of intersection of graphs.

## Study Strategies

- **Using the List of Basic Power Series**   To be efficient at finding Taylor series or Taylor polynomials, learn how to use the basic list below.

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots, \quad 0 < x < 2$$
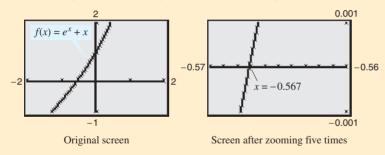
$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots, \quad -1 < x < 1$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots, \quad 0 < x \leq 2$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots, \quad -\infty < x < \infty$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots, \quad -1 < x < 1$$

- **Using Technology to Approximate Zeros**   Newton's Method is only one way that technology can be used to approximate the zeros of a function. Another way is to use the *zoom* and *trace* features of a graphing utility, as shown below. (Compare this with the procedure described on page 715.)



Original screen

Screen after zooming five times

## Review Exercises

In Exercises 1–4, write the first five terms of the sequence. (Begin with $n = 1$.)

**1.** $a_n = \left(\dfrac{3}{2}\right)^n$

**2.** $a_n = \dfrac{n - 2}{n^2 + 2}$

**3.** $a_n = \dfrac{4^n}{n!}$

**4.** $a_n = \dfrac{(-1)^n}{n^3}$

In Exercises 5–8, determine the convergence or divergence of the given sequence. If the sequence converges, find its limit.

**5.** $a_n = \dfrac{n + 1}{n^2}$

**6.** $a_n = \dfrac{5n + 2}{n}$

**7.** $a_n = \dfrac{n^3}{n^2 + 1}$

**8.** $a_n = 10e^{-n}$

(T) In Exercises 9–12, determine the convergence or divergence of the given sequence. If the sequence converges, use a symbolic algebra utility to find its limit.

**9.** $a_n = 5 + \dfrac{1}{3^n}$

**10.** $a_n = \dfrac{n}{\sqrt{n^2 + 1}}$

**11.** $a_n = \dfrac{1}{n^{4/3}}$

**12.** $a_n = \dfrac{(n - 1)!}{(n + 1)!}$

In Exercises 13–16, write an expression for the $n$th term of the sequence.

**13.** $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \cdot \cdot \cdot$

**14.** $\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \cdot \cdot \cdot$

**15.** $\frac{1}{3}, -\frac{2}{9}, \frac{4}{27}, -\frac{8}{81}, \cdot \cdot \cdot$
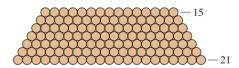
**16.** $1, \frac{2}{5}, \frac{4}{25}, \frac{8}{125}, \frac{16}{625}, \cdot \cdot \cdot$

**17. Sales** A mail order company sells $15,000 worth of products during its first year (see figure). The company's goal is to increase sales by $10,000 each year for the next 9 years.

**Mail Order Products**

Sales (in thousands of dollars) vs. Year

(a) Write an expression for the amount of sales during the $n$th year.

(b) Find the total sales for the first 5 years that the mail order company is in business.

**18. Number of Logs** Logs are stacked in a pile, as shown in the figure. The top row has 15 logs and the bottom row has 21 logs.

— 15

— 21

(a) Write an expression for the number of logs in the $n$th row.

(b) Use the expression in part (a) to verify the number of logs in each row.

(B) **19. Finance: Compound Interest** A deposit of $1 is made in an account that earns 7% interest, compounded annually. Find the first 10 terms of the sequence that represents the account balance. *(Source: Adapted from Garman/Forgue, Personal Finance, Eighth Edition)*

**20. Compound Interest** A deposit of $10,000 is made in an account that earns 5% interest, compounded quarterly. The balance in the account after $n$ quarters is given by

$$A_n = 10{,}000\left(1 + \frac{0.05}{4}\right)^n, \quad n = 1, 2, 3, \ldots$$

(a) Write the first eight terms of the sequence.

(T) (b) Use a symbolic algebra utility to find the balance in the account after 10 years by finding the 40th term of the sequence.

In Exercises 21–24, find the first five terms of the sequence of partial sums for the infinite series.

**21.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

**22.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$

**23.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!}$

**24.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

In Exercises 25–28, determine the convergence or divergence of the infinite series.

**25.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 1}{n(n + 1)}$

**26.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$

**27.** $\displaystyle\sum_{n=0}^{\infty} 2(0.25)^{n+1}$

**28.** $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{n}$

In Exercises 29–32, use the $n$th-Term Test to verify that the series diverges.

**29.** $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n + 5}$

**30.** $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{1 - n^3}$

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2 + 1}$

**32.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{4n^2 + 1}}$

In Exercises 33–36, find the sum of the series.

**33.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n$

**34.** $\displaystyle\sum_{n=0}^{\infty}(0.82)^n$

**35.** $\displaystyle\sum_{n=0}^{\infty}2\left(\frac{2}{3}\right)^n$

**36.** $\displaystyle\sum_{n=0}^{\infty}2\left(\frac{1}{\sqrt{3}}\right)^n$

In Exercises 37–42, decide whether the series converges or diverges. If it converges, find its sum.

**37.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5}{4}\right)^n$

**38.** $\displaystyle\sum_{n=0}^{\infty}3\left(-\frac{4}{3}\right)^n$

**39.** $\displaystyle\sum_{n=0}^{\infty}\frac{1}{4}(4)^n$

**40.** $\displaystyle\sum_{n=0}^{\infty}4\left(-\frac{1}{4}\right)^n$

**41.** $\displaystyle\sum_{n=0}^{\infty}[(0.5)^n+(0.2)^n]$

**42.** $\displaystyle\sum_{n=0}^{\infty}[(1.5)^n+(0.2)^n]$

**43. Salary**  You accept a job that pays a salary of $32,000 the first year. During the next 39 years, you will receive a 5.5% raise each year. What would be your total compensation over the 40-year period?

**44. Market Stabilization**  A company estimates the annual sales of a new product to be 8000 units. Each year, 15% of the units that have been sold become inoperative. So, after 1 year 8000 units are in use, after 2 years $[8000+0.85(8000)]$ units are in use, and so on. How many units will be in use after $n$ years?

**Multiplier Effect**  In Exercises 45–48, use the following information. A tax rebate has been given to property owners by the state government with the anticipation that each property owner spends approximately $p\%$ of the rebate, and in turn each recipient of this amount spends $p\%$ of what they receive, and so on. For the given tax rebate, find the total amount put back into the state's economy, if this effect continues without end.

| Tax rebate | $p\%$ |
|---|---|
| **45.** $500 | 75% |
| **46.** $250 | 80% |
| **47.** $600 | 72.5% |
| **48.** $450 | 77.5% |

**49. Compound Interest**  The holder of a winning $1,000,000 lottery ticket has the choice of receiving a lump sum payment of $500,000 or receiving an annuity of $40,000 for 25 years. Find the interest rate necessary if the winner wants to deposit the $500,000 in a savings account in order to have $1,000,000 in 25 years. Assume that the interest is compounded quarterly.

**50. Depreciation**  A company buys a machine for $120,000. During the next 5 years it will depreciate at a rate of 30% per year. (That is, at the end of each year the depreciated value will be 70% of what it was at the beginning of the year.)

(a) Find the formula for the $n$th term of a sequence that gives the value $V$ of the machine $t$ full years after it was purchased.

(b) Find the depreciated value of the machine at the end of 5 full years.

In Exercises 51–54, determine the convergence or divergence of the $p$-series.

**51.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$
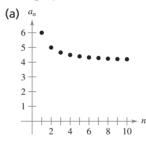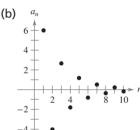
**52.** $\displaystyle\sum_{n=1}^{\infty}2n^{-2/3}$
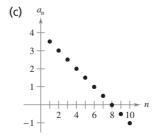
**53.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt[4]{n}}$
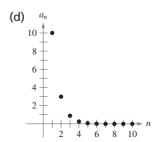
**54.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^e}$

In Exercises 55–58, match the sequence with its graph. [The graphs are labeled (a)–(d).]

(a) 

(b) 

(c) 

(d) 

**55.** $a_n=4+\dfrac{2}{n}$

**56.** $a_n=4-\dfrac{1}{2}n$

**57.** $a_n=10(0.3)^{n-1}$

**58.** $a_n=6\left(-\frac{2}{3}\right)^{n-1}$

In Exercises 59–62, approximate the sum of the given convergent series using the indicated number of terms. Include an estimate of the maximum error for your approximation.

**59.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^6}$, four terms

**60.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$, six terms

**61.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{5/4}}$, six terms

**62.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{12}}$, four terms

In Exercises 63–68, use the Ratio Test to determine the convergence or divergence of the series.

**63.** $\displaystyle\sum_{n=1}^{\infty} \frac{n4^n}{n!}$

**64.** $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{4^n}$

**65.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n}$

**66.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}$

**67.** $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{n!}$

**68.** $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{1 - 4^n}$

In Exercises 69–74, find the radius of convergence of the power series.

**69.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$

**70.** $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

**71.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(x - 2)^n}{(n + 1)^2}$

**72.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x - 2)^n}{n}$

**73.** $\displaystyle\sum_{n=0}^{\infty} n!(x - 3)^n$

**74.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x - 2)^n}{2^n}$

Ⓣ In Exercises 75–80, use a symbolic differentiation utility to apply Taylor's Theorem to find the power series for $f(x)$ centered at $c$.

**75.** $f(x) = e^{-0.5x}$, $c = 0$

**76.** $f(x) = e^{-x/3}$, $c = 0$

**77.** $f(x) = \dfrac{1}{\sqrt{x}}$, $c = 1$

**78.** $f(x) = \dfrac{1}{x}$, $c = -1$

**79.** $f(x) = \sqrt[4]{1 + x}$, $c = 0$

**80.** $f(x) = \dfrac{1}{(1 + x)^3}$, $c = 0$

In Exercises 81–88, use the basic list of power series for elementary functions on page 699 to find the series representation of $f(x)$.

**81.** $f(x) = \ln(x + 2)$

**82.** $f(x) = e^{2x+1}$

**83.** $f(x) = (1 + x^2)^2$

**84.** $f(x) = \dfrac{1}{x^3 + 1}$

**85.** $f(x) = x^2 e^x$

**86.** $f(x) = \dfrac{e^x}{x^2}$

**87.** $f(x) = \dfrac{x^2}{x + 1}$

**88.** $f(x) = \dfrac{\sqrt{x}}{x + 1}$

In Exercises 89–92, use a sixth-degree Taylor polynomial to approximate the function in the indicated interval.

**89.** $f(x) = \dfrac{1}{(x + 3)^2}$, $[-1, 1]$

**90.** $f(x) = e^{x+1}$, $[-2, 2]$

**91.** $f(x) = \ln(x + 2)$, $[0, 2]$

**92.** $f(x) = \sqrt{x + 2}$, $[-1, 1]$

In Exercises 93–96, use a sixth-degree Taylor polynomial centered at $c$ for the function $f$ to obtain the desired approximation.

**93.** $f(x) = e^{x^2}$, $c = 1$, approximate $f(1.25)$

**94.** $f(x) = \dfrac{1}{\sqrt{x}}$, $c = 1$, approximate $f(1.15)$

**95.** $f(x) = \ln(1 + x)$, $c = 1$, approximate $f(1.5)$

**96.** $f(x) = e^{x-1}$, $c = 0$, approximate $f(1.75)$

In Exercises 97 and 98, determine the maximum error guaranteed by Taylor's Theorem when the fifth-degree polynomial is used to approximate $f$ in the indicated interval.

**97.** $f(x) = \dfrac{2}{x}$, $\left[1, \dfrac{3}{2}\right]$

$P_5(x) = 2[1 - (x - 1) + (x - 1)^2 - (x - 1)^3$
$\qquad\qquad + (x - 1)^4 - (x - 1)^5]$

**98.** $f(x) = e^{-2x}$, $[0, 1]$

$P_5(x) = 1 - 2x + \dfrac{(2x)^2}{2!} - \dfrac{(2x)^3}{3!} + \dfrac{(2x)^4}{4!} - \dfrac{(2x)^5}{5!}$

In Exercises 99–102, use a sixth-degree Taylor polynomial centered at zero to approximate the definite integral.

**99.** $\displaystyle\int_0^{0.3} \sqrt{1 + x^3}\, dx$

**100.** $\displaystyle\int_0^{0.5} e^{-x^2}\, dx$

**101.** $\displaystyle\int_0^{0.75} \ln(x^2 + 1)\, dx$

**102.** $\displaystyle\int_0^{0.5} \sqrt{1 + x}\, dx$

**Production** In Exercises 103 and 104, let $n$ be a random variable representing the number of units produced per hour from one of three machines. The probability distribution of $n$ is $P(n)$. Find the expected value of $n$, and determine the expected production costs if each unit costs \$25.00 to make.

**103.** $P(n) = 2\left(\frac{1}{3}\right)^{n+1}$

**104.** $P(n) = \frac{1}{2}\left(\frac{2}{3}\right)^{n+1}$

In Exercises 105–108, use Newton's Method to approximate to three decimal places the zero(s) of the function.

**105.** $f(x) = x^3 - 3x - 1$

**106.** $f(x) = x^3 + 2x + 1$

**107.** $f(x) = \ln 3x + x$

**108.** $f(x) = e^x - 3$

Ⓣ In Exercises 109–112, use a program similar to the one discussed on page 715 to use Newton's Method to approximate to three decimal places the $x$-value of the point of intersection of the graphs of the equations.

**109.** $f(x) = x^5$, $g(x) = x + 3$

**110.** $f(x) = 2 - x$, $g(x) = x^5 + 2$

**111.** $f(x) = x^3$, $g(x) = e^{-x}$

**112.** $f(x) = 2x^2$, $g(x) = 5e^{-x}$

# Chapter Test

Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, (a) write the first five terms of the sequence, (b) determine the convergence or divergence of the sequence, and (c) if the sequence converges, find its limit.

**1.** $a_n = \left(\dfrac{3}{5}\right)^n$     **2.** $a_n = \dfrac{n^2}{3n^2 + 4}$     **3.** $a_n = \dfrac{(-1)^{n+1}}{6}$     **4.** $a_n = \dfrac{4n}{(n-1)!}$

**5.** Write an expression for the $n$th term of the sequence $\frac{1}{2}, -\frac{2}{5}, \frac{3}{10}, -\frac{4}{17}, \frac{5}{26}, \ldots$ .

In Exercises 6–9, test the series for convergence or divergence using any appropriate test from this chapter.

**6.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{4^n}{n!}$     **7.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{n+1}{n-3}$     **8.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{2}{5^n}$     **9.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{\sqrt[3]{n}}{\sqrt{n}}$

**10.** Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{2}{5^n} - \dfrac{1}{7^{n+1}}\right)$.

In Exercises 11–13, (a) write the first five terms and (b) find the radius of convergence of the power series.

**11.** $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}\left(\dfrac{x}{3}\right)^n$     **12.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{(n+1)!}$     **13.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n(x-3)^n}{(n+4)^2}$

**14.** Use Taylor's Theorem to find the power series (centered at 0) for $f(x) = e^{-4x}$.

**15.** Use the basic list of power series for elementary functions on page 699 to find the power series for $f(x) = (1+x)^{2/3}$.

**16.** Approximate $\ln 1.25$ using a fifth-degree Taylor polynomial centered at 1. Then use a calculator to evaluate $\ln 1.25$ and compare the results.

**(T) 17.** Use a ninth-degree Taylor polynomial for $e^{x^3}$ to approximate the value of $\int_0^1 e^{x^3}\, dx$. Then evaluate the integral using a graphing utility and compare the results.

**18.** The annual average yields for 1-year U.S. Treasury bonds from 2000 through 2005 are shown in the table, where $a_n$ is the annual average yield (in percent) and $n$ is the year, with $n = 0$ corresponding to 2000. *(Source: U.S. Treasury)*

| $n$ | 0 | 1 | 2 |
|-----|------|------|------|
| $a_n$ | 6.11 | 3.49 | 2.00 |

| $n$ | 3 | 4 | 5 |
|-----|------|------|------|
| $a_n$ | 1.24 | 1.89 | 3.62 |

Table for 18

**(T)** (a) Use the *regression* feature of a graphing utility to find a model of the form

$$a_n = bn^2 + cn + d, \quad n = 0, 1, 2, 3, 4, 5$$

for the data.

(b) Use the model you found in part (a) and Newton's Method to find the year in which the average yield was a minimum, and then find the minimum. Compare your result with the minimum average yield listed in the table.

**(T) 19.** Use Newton's Method to approximate the zero of $f(x) = x^3 + x - 3$ to three decimal places. Then find the zero using a graphing utility and compare the results.

# Appendices

Appendices D and E are located on the website that accompanies this text at *college.hmco.com*.

# A  Alternative Introduction to the Fundamental Theorem of Calculus

■ Approximate the areas of regions using Riemann sums.
■ Evaluate definite integrals.

In this appendix, a summation process is used to provide an alternative development of the definite integral. It is intended that this supplement follow Section 5.3 in the text. If used, this appendix should replace the material preceding Example 2 in Section 5.4. Example 1 below shows how the area of a region in the plane can be approximated by the use of rectangles.

### Example 1  Using Rectangles to Approximate the Area of a Region
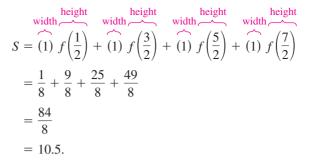
Use the four rectangles indicated in Figure A.1 to approximate the area of the region lying between the graph of

$$f(x) = \frac{x^2}{2}$$

and the $x$-axis, between $x = 0$ and $x = 4$.

**SOLUTION**  You can find the heights of the rectangles by evaluating the function $f$ at each of the midpoints of the subintervals

$$[0, 1], \quad [1, 2], \quad [2, 3], \quad [3, 4].$$

Because the width of each rectangle is 1, the sum of the areas of the four rectangles is

$$S = \overset{\text{width}}{\overbrace{(1)}} \ \overset{\text{height}}{\overbrace{f\left(\frac{1}{2}\right)}} + \overset{\text{width}}{\overbrace{(1)}} \ \overset{\text{height}}{\overbrace{f\left(\frac{3}{2}\right)}} + \overset{\text{width}}{\overbrace{(1)}} \ \overset{\text{height}}{\overbrace{f\left(\frac{5}{2}\right)}} + \overset{\text{width}}{\overbrace{(1)}} \ \overset{\text{height}}{\overbrace{f\left(\frac{7}{2}\right)}}$$

$$= \frac{1}{8} + \frac{9}{8} + \frac{25}{8} + \frac{49}{8}$$

$$= \frac{84}{8}$$

$$= 10.5.$$

So, you can approximate the area of the region to be 10.5 square units.



**FIGURE A.1**

$f(x) = \dfrac{x^2}{2}$
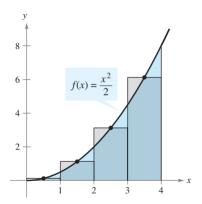
### STUDY TIP

The approximation technique used in Example 1 is called the **Midpoint Rule.** The Midpoint Rule is discussed further in Section 5.6.

The procedure shown in Example 1 can be generalized. Let $f$ be a continuous function defined on the closed interval $[a, b]$. To begin, partition the interval into $n$ subintervals, each of width $\Delta x = (b - a)/n$, as shown.

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

In each subinterval $[x_{i-1}, x_i]$, choose an arbitrary point $c_i$ and form the sum

$$S = f(c_1)\,\Delta x + f(c_2)\,\Delta x + \cdots + f(c_{n-1})\,\Delta x + f(c_n)\,\Delta x.$$

This type of summation is called a **Riemann sum,** and is often written using summation notation as shown below.

$$S = \sum_{i=1}^{n} f(c_i)\,\Delta x, \quad x_{i-1} \le c_i \le x_i$$

For the Riemann sum in Example 1, the interval is $[a, b] = [0, 4]$, the number of subintervals is $n = 4$, the width of each subinterval is $\Delta x = 1$, and the point $c_i$ in each subinterval is its midpoint. So, you can write the approximation in Example 1 as

$$\begin{aligned}
S &= \sum_{i=1}^{n} f(c_i)\,\Delta x \\
&= \sum_{i=1}^{4} f(c_i)(1) \\
&= \frac{1}{8} + \frac{9}{8} + \frac{25}{8} + \frac{49}{8} \\
&= \frac{84}{8}.
\end{aligned}$$

### Example 2  Using a Riemann Sum to Approximate Area

Use a Riemann sum to approximate the area of the region bounded by the graph of $f(x) = -x^2 + 2x$ and the $x$-axis, for $0 \le x \le 2$. In the Riemann sum, let $n = 6$ and choose $c_i$ to be the left endpoint of each subinterval.

**SOLUTION**    Subdivide the interval $[0, 2]$ into six subintervals, each of width

$$\begin{aligned}
\Delta x &= \frac{2 - 0}{6} \\
&= \frac{1}{3}
\end{aligned}$$



$f(x) = -x^2 + 2x$

**FIGURE A.2**

as shown in Figure A.2. Because $c_i$ is the left endpoint of each subinterval, the Riemann sum is given by

$$\begin{aligned}
S &= \sum_{i=1}^{n} f(c_i)\,\Delta x \\
&= \left[ f(0) + f\!\left(\frac{1}{3}\right) + f\!\left(\frac{2}{3}\right) + f(1) + f\!\left(\frac{4}{3}\right) + f\!\left(\frac{5}{3}\right) \right]\!\left(\frac{1}{3}\right) \\
&= \left[ 0 + \frac{5}{9} + \frac{8}{9} + 1 + \frac{8}{9} + \frac{5}{9} \right]\!\left(\frac{1}{3}\right) \\
&= \frac{35}{27} \text{ square units}.
\end{aligned}$$
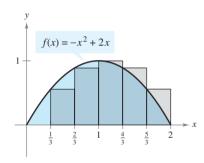
Example 2 illustrates an important point. If a function $f$ is continuous and nonnegative over the interval $[a, b]$, then the Riemann sum

$$S = \sum_{i=1}^{n} f(c_i)\,\Delta x$$

can be used to approximate the area of the region bounded by the graph of $f$ and the $x$-axis, between $x = a$ and $x = b$. Moreover, for a given interval, as the number of subintervals increases, the approximation to the actual area will improve. This is illustrated in the next two examples by using Riemann sums to approximate the area of a triangle.

### Example 3    Approximating the Area of a Triangle

Use a Riemann sum to approximate the area of the triangular region bounded by the graph of $f(x) = 2x$ and the $x$-axis, $0 \le x \le 3$. Use a partition of six subintervals and choose $c_i$ to be the left endpoint of each subinterval.
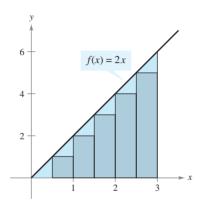


**FIGURE A.3**

**SOLUTION**    Subdivide the interval $[0, 3]$ into six subintervals, each of width

$$\Delta x = \frac{3 - 0}{6}$$

$$= \frac{1}{2}$$

as shown in Figure A.3. Because $c_i$ is the left endpoint of each subinterval, the Riemann sum is given by

$$S = \sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$= \left[ f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) \right]\left(\frac{1}{2}\right)$$

$$= [0 + 1 + 2 + 3 + 4 + 5]\left(\frac{1}{2}\right)$$

$$= \frac{15}{2} \text{ square units.}$$

The approximations in Examples 2 and 3 are called **left Riemann sums,** because $c_i$ was chosen to be the left endpoint of each subinterval. If the right endpoints had been used in Example 3, the **right Riemann sum** would have been $\frac{21}{2}$. Note that the exact area of the triangular region in Example 3 is

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(3)(6) = 9 \text{ square units.}$$

So, the left Riemann sum gives an approximation that is less than the actual area, and the right Riemann sum gives an approximation that is greater than the actual area.

In Example 4, you will see that the approximation improves as the number of subintervals increases.

### Example 4   Increasing the Number of Subintervals

Let $f(x) = 2x$, $0 \le x \le 3$. Use a computer to determine the left and right Riemann sums for $n = 10$, $n = 100$, and $n = 1000$ subintervals.

**SOLUTION**   A basic computer program for this problem is as shown.

```
10      INPUT; N
20      DELTA=3/N
30      LSUM=0: RSUM=0
40      FOR I=1 TO N
50      LC=(I-1)*DELTA: RC=I*DELTA
60      LSUM=LSUM+2*LC*DELTA: RSUM=RSUM+2*RC*DELTA
70      NEXT
80      PRINT "LEFT RIEMANN SUM:"; LSUM
90      PRINT "RIGHT RIEMANN SUM:"; RSUM
100     END
```

Running this program for $n = 10$, $n = 100$, and $n = 1000$ gave the results shown in the table.

| $n$ | Left Riemann sum | Right Riemann sum |
|---|---|---|
| 10 | 8.100 | 9.900 |
| 100 | 8.910 | 9.090 |
| 1000 | 8.991 | 9.009 |

From the results of Example 4, it appears that the Riemann sums are approaching the limit 9 as $n$ approaches infinity. It is this observation that motivates the definition of a **definite integral.** In this definition, consider the partition of $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b - a)/n$, as shown.

$$a = x_0 < x_1 < x_2 < \cdot\cdot\cdot < x_{n-1} < x_n = b$$

Moreover, consider $c_i$ to be an arbitrary point in the $i$th subinterval $[x_{i-1}, x_i]$. To say that the number of subintervals $n$ approaches infinity is equivalent to saying that the width, $\Delta x$, of the subintervals approaches zero.

**Definition of Definite Integral**

If $f$ is a continuous function defined on the closed interval $[a, b]$, then the **definite integral of $f$ on $[a, b]$** is

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^n f(c_i)\,\Delta x$$

$$= \lim_{n \to \infty} \sum_{i=1}^n f(c_i)\,\Delta x.$$

If $f$ is continuous and nonnegative on the interval $[a, b]$, then the definite integral of $f$ on $[a, b]$ gives the area of the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$.

Evaluation of a definite integral by its limit definition can be difficult. However, there are times when a definite integral can be solved by recognizing that it represents the area of a common type of geometric figure.

### *Example 5*  The Areas of Common Geometric Figures

Sketch the region corresponding to each of the definite integrals. Then evaluate each definite integral using a geometric formula.

**a.** $\displaystyle\int_1^3 4\,dx$    **b.** $\displaystyle\int_0^3 (x + 2)\,dx$    **c.** $\displaystyle\int_{-2}^2 \sqrt{4 - x^2}\,dx$

**SOLUTION**    A sketch of each region is shown in Figure A.4.

**a.** The region associated with this definite integral is a rectangle of height 4 and width 2. Moreover, because the function $f(x) = 4$ is continuous and nonnegative on the interval $[1, 3]$, you can conclude that the area of the rectangle is given by the definite integral. So, the value of the definite integral is
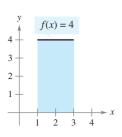
$$\int_1^3 4\,dx = 4(2) = 8 \text{ square units.}$$

**b.** The region associated with this definite integral is a trapezoid with an altitude of 3 and parallel bases of lengths 2 and 5. The formula for the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, and so you have

$$\int_0^3 (x + 2)\,dx = \frac{1}{2}(3)(2 + 5)$$

$$= \frac{21}{2} \text{ square units.}$$
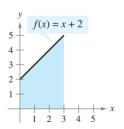
**c.** The region associated with this definite integral is a semicircle of radius 2. So, the area is $\frac{1}{2}\pi r^2$, and you have

$$\int_{-2}^2 \sqrt{4 - x^2}\,dx = \frac{1}{2}\pi(2^2) = 2\pi \text{ square units.}$$
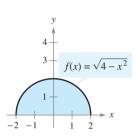
(a) $\displaystyle\int_1^3 4\,dx$

Rectangle

(b) $\displaystyle\int_0^3 (x + 2)\,dx$

Trapezoid

(c) $\displaystyle\int_{-2}^2 \sqrt{4 - x^2}\,dx$

Semicircle

**FIGURE A.4**

For some simple functions, it is possible to evaluate definite integrals by the Riemann sum definition. In the next example, you will use the fact that the sum of the first $n$ integers is given by the formula

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{See Exercise 29.}$$

to compute the area of the triangular region in Examples 3 and 4.

### Example 6  Evaluating a Definite Integral by Its Definition

Evaluate $\displaystyle\int_0^3 2x \, dx$.

**SOLUTION**   Let $\Delta x = (b - a)/n = 3/n$, and choose $c_i$ to be the right endpoint of each subinterval, $c_i = 3i/n$. Then you have

$$\int_0^3 2x \, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} 2\left(i\frac{3}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n \to \infty} \frac{18}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n \to \infty} \left(\frac{18}{n^2}\right)\left(\frac{n(n+1)}{2}\right)$$

$$= \lim_{n \to \infty} \left(9 + \frac{9}{n}\right).$$

This limit can be evaluated in the same way that you calculated horizontal asymptotes in Section 3.6. In particular, as $n$ approaches infinity, you see that $9/n$ approaches 0, and the limit above is 9. So, you can conclude that

$$\int_0^3 2x \, dx = 9.$$



**FIGURE A.5**

From Example 6, you can see that it can be difficult to evaluate the definite integral of even a simple function by using Riemann sums. A computer can help in calculating these sums for large values of $n$, but this procedure would only give an approximation of the definite integral. Fortunately, the **Fundamental Theorem of Calculus** provides a technique for evaluating definite integrals using antiderivatives, and for this reason it is often thought to be the most important theorem in calculus. In the remainder of this appendix, you will see how derivatives and integrals are related via the Fundamental Theorem of Calculus.

To simplify the discussion, assume that $f$ is a continuous nonnegative function defined on the interval $[a, b]$. Let $A(x)$ be the area of the region under the graph of $f$ from $a$ to $x$, as indicated in Figure A.5. The area under the shaded region in Figure A.6 is $A(x + \Delta x) - A(x)$.
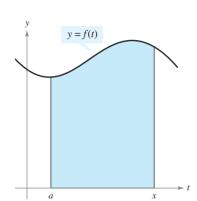


**FIGURE A.6**
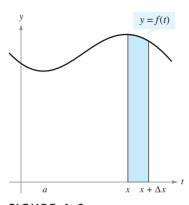
If $\Delta x$ is small, then this area is approximated by the area of the rectangle of height $f(x)$ and width $\Delta x$. So, you have

$$A(x + \Delta x) - A(x) \approx f(x) \, \Delta x.$$

Dividing by $\Delta x$ produces

$$f(x) \approx \frac{A(x + \Delta x) - A(x)}{\Delta x}.$$

By taking the limit as $\Delta x$ approaches 0, you can see that

$$f(x) = \lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x}$$

$$= A'(x)$$

and you can establish the fact that the area function $A(x)$ is an antiderivative of $f$. Although it was assumed that $f$ is continuous and nonnegative, this development is valid if the function $f$ is simply continuous on the closed interval $[a, b]$. This result is used in the proof of the Fundamental Theorem of Calculus.

---

**Fundamental Theorem of Calculus**

If $f$ is a continuous function on the closed interval $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where $F$ is any function such that $F'(x) = f(x)$.

---

**PROOF**   From the discussion above, you know that

$$\int_a^x f(x) \, dx = A(x)$$

and in particular

$$A(a) = \int_a^a f(x) \, dx = 0$$

and

$$A(b) = \int_a^b f(x) \, dx.$$

If $F$ is *any* antiderivative of $f$, then you know that $F$ differs from $A$ by a constant. That is, $A(x) = F(x) + C$. So

$$\int_a^b f(x) \, dx = A(b) - A(a)$$

$$= [F(b) + C] - [F(a) + C]$$

$$= F(b) - F(a).$$

# Appendix A
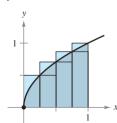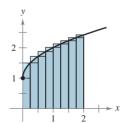
In Exercises 1–6, use the left Riemann sum and the right Riemann sum to approximate the area of the region using the indicated number of subintervals.
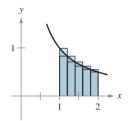
**1.** $y = \sqrt{x}$
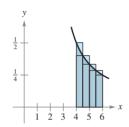


**2.** $y = \sqrt{x} + 1$
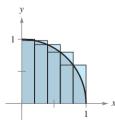


**3.** $y = \dfrac{1}{x}$
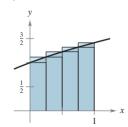


**4.** $y = \dfrac{1}{x - 2}$



**5.** $y = \sqrt{1 - x^2}$



**6.** $y = \sqrt{x + 1}$



**7.** Repeat Exercise 1 using the midpoint Riemann sum.

**8.** Repeat Exercise 2 using the midpoint Riemann sum.

**9.** Consider a triangle of area 2 bounded by the graphs of $y = x$, $y = 0$, and $x = 2$.

(a) Sketch the graph of the region.

(b) Divide the interval $[0, 2]$ into $n$ equal subintervals and show that the endpoints are

$$0 < 1\left(\frac{2}{n}\right) < \cdots < (n - 1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right).$$

(c) Show that the left Riemann sum is

$$S_L = \sum_{i=1}^{n} \left[(i - 1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right).$$

(d) Show that the right Riemann sum is

$$S_R = \sum_{i=1}^{n} \left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right).$$

(e) Complete the table below.

| $n$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| Left sum, $S_L$ | | | | |
| Right sum, $S_R$ | | | | |

(f) Show that $\lim\limits_{n \to \infty} S_L = \lim\limits_{n \to \infty} S_R = 2$.

**10.** Consider a trapezoid of area 4 bounded by the graphs of $y = x$, $y = 0$, $x = 1$, and $x = 3$.

(a) Sketch the graph of the region.

(b) Divide the interval $[1, 3]$ into $n$ equal subintervals and show that the endpoints are

$$1 < 1 + 1\left(\frac{2}{n}\right) < \cdots < 1 + (n - 1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right).$$

(c) Show that the left Riemann sum is

$$S_L = \sum_{i=1}^{n} \left[1 + (i - 1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right).$$

(d) Show that the right Riemann sum is

$$S_R = \sum_{i=1}^{n} \left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right).$$

(e) Complete the table below.

| $n$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| Left sum, $S_L$ | | | | |
| Right sum, $S_R$ | | | | |
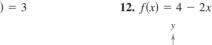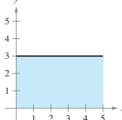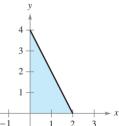
(f) Show that $\lim\limits_{n \to \infty} S_L = \lim\limits_{n \to \infty} S_R = 4$.

In Exercises 11–18, set up a definite integral that yields the area of the region. (Do not evaluate the integral.)

**11.** $f(x) = 3$



**12.** $f(x) = 4 - 2x$
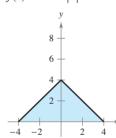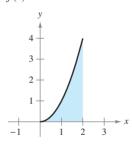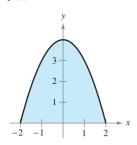
**13.** $f(x) = 4 - |x|$
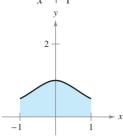


**14.** $f(x) = x^2$
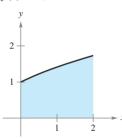


**15.** $f(x) = 4 - x^2$
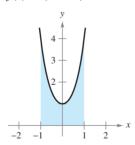


**16.** $f(x) = \dfrac{1}{x^2 + 1}$



**17.** $f(x) = \sqrt{x + 1}$



**18.** $f(x) = (x^2 + 1)^2$



In Exercises 19–28, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral ($a > 0, r > 0$).

**19.** $\displaystyle\int_0^3 4 \, dx$

**20.** $\displaystyle\int_{-a}^{a} 4 \, dx$

**21.** $\displaystyle\int_0^4 x \, dx$

**22.** $\displaystyle\int_0^4 \dfrac{x}{2} \, dx$

**23.** $\displaystyle\int_0^2 (2x + 5) \, dx$

**24.** $\displaystyle\int_0^5 (5 - x) \, dx$

**25.** $\displaystyle\int_{-1}^{1} \left(1 - |x|\right) dx$

**26.** $\displaystyle\int_{-a}^{a} \left(a - |x|\right) dx$

**27.** $\displaystyle\int_{-3}^{3} \sqrt{9 - x^2} \, dx$
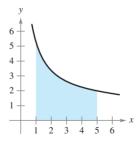
**28.** $\displaystyle\int_{-r}^{r} \sqrt{r^2 - x^2} \, dx$

**29.** Show that $\displaystyle\sum_{i=1}^{n} i = \dfrac{n(n + 1)}{2}$.   (*Hint:* Add the two sums below.)

$$S = 1 + 2 + 3 + \cdots + (n - 2) + (n - 1) + n$$
$$S = n + (n - 1) + (n - 2) + \cdots + 3 + 2 + 1$$

**30.** Use the Riemann sum definition of the definite integral and the result of Exercise 29 to evaluate $\int_1^2 x \, dx$.

In Exercises 31 and 32, use the figure to fill in the blank with the symbol <, >, or =.



**31.** The interval $[1, 5]$ is partitioned into $n$ subintervals of equal width $\Delta x$, and $x_i$ is the left endpoint of the $i$th subinterval.

$$\sum_{i=1}^{n} f(x_i) \, \Delta x \quad \blacksquare \quad \int_1^5 f(x) \, dx$$

**32.** The interval $[1, 5]$ is partitioned into $n$ subintervals of equal width $\Delta x$, and $x_i$ is the right endpoint of the $i$th subinterval.

$$\sum_{i=1}^{n} f(x_i) \, \Delta x \quad \blacksquare \quad \int_1^5 f(x) \, dx$$

# B   Formulas

## B.1   Differentiation and Integration Formulas

**Use differentiation and integration tables to supplement differentiation and integration techniques.**

### Differentiation Formulas

**1.** $\dfrac{d}{dx}[cu] = cu'$

**2.** $\dfrac{d}{dx}[u \pm v] = u' \pm v'$

**3.** $\dfrac{d}{dx}[uv] = uv' + vu'$

**4.** $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{vu' - uv'}{v^2}$

**5.** $\dfrac{d}{dx}[c] = 0$

**6.** $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$

**7.** $\dfrac{d}{dx}[x] = 1$

**8.** $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$

**9.** $\dfrac{d}{dx}[e^u] = e^u u'$

**10.** $\dfrac{d}{dx}[\sin u] = (\cos u)u'$

**11.** $\dfrac{d}{dx}[\cos u] = -(\sin u)u'$

**12.** $\dfrac{d}{dx}[\tan u] = (\sec^2 u)u'$

**13.** $\dfrac{d}{dx}[\cot u] = -(\csc^2 u)u'$

**14.** $\dfrac{d}{dx}[\sec u] = (\sec u \tan u)u'$

**15.** $\dfrac{d}{dx}[\csc u] = -(\csc u \cot u)u'$

### Integration Formulas

**Forms Involving $u^n$**

**1.** $\displaystyle\int u^n \, du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$

**2.** $\displaystyle\int \dfrac{1}{u} \, du = \ln |u| + C$

**Forms Involving $a + bu$**

**3.** $\displaystyle\int \dfrac{u}{a + bu} \, du = \dfrac{1}{b^2}(bu - a \ln|a + bu|) + C$

**4.** $\displaystyle\int \dfrac{u}{(a + bu)^2} \, du = \dfrac{1}{b^2}\left(\dfrac{a}{a + bu} + \ln|a + bu|\right) + C$

**5.** $\displaystyle\int \dfrac{u}{(a + bu)^n} \, du = \dfrac{1}{b^2}\left[\dfrac{-1}{(n - 2)(a + bu)^{n-2}} + \dfrac{a}{(n - 1)(a + bu)^{n-1}}\right] + C, \quad n \neq 1, 2$

**6.** $\displaystyle\int \dfrac{u^2}{a + bu} \, du = \dfrac{1}{b^3}\left[-\dfrac{bu}{2}(2a - bu) + a^2 \ln|a + bu|\right] + C$

**7.** $\displaystyle\int \dfrac{u^2}{(a + bu)^2} \, du = \dfrac{1}{b^3}\left(bu - \dfrac{a^2}{a + bu} - 2a \ln|a + bu|\right) + C$

**8.** $\displaystyle\int \frac{u^2}{(a + bu)^3}\, du = \frac{1}{b^3}\left[\frac{2a}{a + bu} - \frac{a^2}{2(a + bu)^2} + \ln|a + bu|\right] + C$

**9.** $\displaystyle\int \frac{u^2}{(a + bu)^n}\, du = \frac{1}{b^3}\left[\frac{-1}{(n - 3)(a + bu)^{n-3}} + \frac{2a}{(n - 2)(a + bu)^{n-2}}\right.$

$$\left. - \frac{a^2}{(n - 1)(a + bu)^{n-1}}\right] + C, \quad n \neq 1, 2, 3$$

**10.** $\displaystyle\int \frac{1}{u(a + bu)}\, du = \frac{1}{a}\ln\left|\frac{u}{a + bu}\right| + C$

**11.** $\displaystyle\int \frac{1}{u(a + bu)^2}\, du = \frac{1}{a}\left(\frac{1}{a + bu} + \frac{1}{a}\ln\left|\frac{u}{a + bu}\right|\right) + C$

**12.** $\displaystyle\int \frac{1}{u^2(a + bu)}\, du = -\frac{1}{a}\left(\frac{1}{u} + \frac{b}{a}\ln\left|\frac{u}{a + bu}\right|\right) + C$

**13.** $\displaystyle\int \frac{1}{u^2(a + bu)^2}\, du = -\frac{1}{a^2}\left[\frac{a + 2bu}{u(a + bu)} + \frac{2b}{a}\ln\left|\frac{u}{a + bu}\right|\right] + C$

**Forms Involving $\sqrt{a + bu}$**

**14.** $\displaystyle\int u^n \sqrt{a + bu}\, du = \frac{2}{b(2n + 3)}\left[u^n(a + bu)^{3/2} - na\int u^{n-1}\sqrt{a + bu}\, du\right]$

**15.** $\displaystyle\int \frac{1}{u\sqrt{a + bu}}\, du = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}}\right| + C, \quad a > 0$

**16.** $\displaystyle\int \frac{1}{u^n\sqrt{a + bu}}\, du = \frac{-1}{a(n - 1)}\left[\frac{\sqrt{a + bu}}{u^{n-1}} + \frac{(2n - 3)b}{2}\int \frac{1}{u^{n-1}\sqrt{a + bu}}\, du\right], \quad n \neq 1$

**17.** $\displaystyle\int \frac{\sqrt{a + bu}}{u}\, du = 2\sqrt{a + bu} + a\int \frac{1}{u\sqrt{a + bu}}\, du$

**18.** $\displaystyle\int \frac{\sqrt{a + bu}}{u^n}\, du = \frac{-1}{a(n - 1)}\left[\frac{(a + bu)^{3/2}}{u^{n-1}} + \frac{(2n - 5)b}{2}\int \frac{\sqrt{a + bu}}{u^{n-1}}\, du\right], \quad n \neq 1$

**19.** $\displaystyle\int \frac{u}{\sqrt{a + bu}}\, du = -\frac{2(2a - bu)}{3b^2}\sqrt{a + bu} + C$

**20.** $\displaystyle\int \frac{u^n}{\sqrt{a + bu}}\, du = \frac{2}{(2n + 1)b}\left(u^n\sqrt{a + bu} - na\int \frac{u^{n-1}}{\sqrt{a + bu}}\, du\right)$

**Forms Involving $u^2 - a^2$, $a > 0$**

**21.** $\displaystyle\int \frac{1}{u^2 - a^2}\, du = -\int \frac{1}{a^2 - u^2}\, du$

$$= \frac{1}{2a}\ln\left|\frac{u - a}{u + a}\right| + C$$

**22.** $\displaystyle\int \frac{1}{(u^2 - a^2)^n}\, du = \frac{-1}{2a^2(n - 1)}\left[\frac{u}{(u^2 - a^2)^{n-1}} + (2n - 3)\int \frac{1}{(u^2 - a^2)^{n-1}}\, du\right], \quad n \neq 1$

# Integration Formulas   (Continued)

**Forms Involving $\sqrt{u^2 \pm a^2}$, a > 0**

**23.** $\displaystyle\int \sqrt{u^2 \pm a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \pm a^2 \ln\left|u + \sqrt{u^2 \pm a^2}\right|\right) + C$

**24.** $\displaystyle\int u^2 \sqrt{u^2 \pm a^2}\, du = \frac{1}{8}\left[u(2u^2 \pm a^2)\sqrt{u^2 \pm a^2} - a^4 \ln\left|u + \sqrt{u^2 \pm a^2}\right|\right] + C$

**25.** $\displaystyle\int \frac{\sqrt{u^2 + a^2}}{u}\, du = \sqrt{u^2 + a^2} - a\ln\left|\frac{a + \sqrt{u^2 + a^2}}{u}\right| + C$

**26.** $\displaystyle\int \frac{\sqrt{u^2 \pm a^2}}{u^2}\, du = \frac{-\sqrt{u^2 \pm a^2}}{u} + \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$

**27.** $\displaystyle\int \frac{1}{\sqrt{u^2 \pm a^2}}\, du = \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$

**28.** $\displaystyle\int \frac{1}{u\sqrt{u^2 + a^2}}\, du = \frac{-1}{a}\ln\left|\frac{a + \sqrt{u^2 + a^2}}{u}\right| + C$

**29.** $\displaystyle\int \frac{u^2}{\sqrt{u^2 \pm a^2}}\, du = \frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \mp a^2 \ln\left|u + \sqrt{u^2 \pm a^2}\right|\right) + C$

**30.** $\displaystyle\int \frac{1}{u^2\sqrt{u^2 \pm a^2}}\, du = \mp\frac{\sqrt{u^2 \pm a^2}}{a^2 u} + C$

**31.** $\displaystyle\int \frac{1}{(u^2 \pm a^2)^{3/2}}\, du = \frac{\pm u}{a^2\sqrt{u^2 \pm a^2}} + C$

**Forms Involving $\sqrt{a^2 - u^2}$, a > 0**

**32.** $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

**33.** $\displaystyle\int \frac{1}{u\sqrt{a^2 - u^2}}\, du = \frac{-1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

**34.** $\displaystyle\int \frac{1}{u^2\sqrt{a^2 - u^2}}\, du = \frac{-\sqrt{a^2 - u^2}}{a^2 u} + C$

**35.** $\displaystyle\int \frac{1}{(a^2 - u^2)^{3/2}}\, du = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$

**Forms Involving $e^u$**

**36.** $\displaystyle\int e^u\, du = e^u + C$

**37.** $\displaystyle\int u e^u\, du = (u - 1)e^u + C$

**38.** $\displaystyle\int u^n e^u\, du = u^n e^u - n\int u^{n-1} e^u\, du$

**39.** $\displaystyle\int \frac{1}{1 + e^u}\, du = u - \ln(1 + e^u) + C$

**40.** $\displaystyle\int \frac{1}{1 + e^{nu}}\, du = u - \frac{1}{n}\ln(1 + e^{nu}) + C$

**Forms Involving ln $u$**

**41.** $\displaystyle\int \ln u \, du = u(-1 + \ln u) + C$

**42.** $\displaystyle\int u \ln u \, du = \frac{u^2}{4}(-1 + 2 \ln u) + C$

**43.** $\displaystyle\int u^n \ln u \, du = \frac{u^{n+1}}{(n+1)^2}[-1 + (n+1)\ln u] + C, \quad n \neq -1$

**44.** $\displaystyle\int (\ln u)^2 \, du = u[2 - 2 \ln u + (\ln u)^2] + C$

**45.** $\displaystyle\int (\ln u)^n \, du = u(\ln u)^n - n \int (\ln u)^{n-1} \, du$

**Forms Involving $\sin u$ or $\cos u$**

**46.** $\displaystyle\int \sin u \, du = -\cos u + C$

**47.** $\displaystyle\int \cos u \, du = \sin u + C$

**48.** $\displaystyle\int \sin^2 u \, du = \frac{1}{2}(u - \sin u \cos u) + C$

**49.** $\displaystyle\int \cos^2 u \, du = \frac{1}{2}(u + \sin u \cos u) + C$

**50.** $\displaystyle\int \sin^n u \, du = -\frac{\sin^{n-1} u \cos u}{n} + \frac{n-1}{n} \int \sin^{n-2} u \, du$

**51.** $\displaystyle\int \cos^n u \, du = \frac{\cos^{n-1} u \sin u}{n} + \frac{n-1}{n} \int \cos^{n-2} u \, du$

**52.** $\displaystyle\int u \sin u \, du = \sin u - u \cos u + C$

**53.** $\displaystyle\int u \cos u \, du = \cos u + u \sin u + C$

**54.** $\displaystyle\int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$

**55.** $\displaystyle\int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$

**56.** $\displaystyle\int \frac{1}{1 \pm \sin u} \, du = \tan u \mp \sec u + C$

**57.** $\displaystyle\int \frac{1}{1 \pm \cos u} \, du = -\cot u \pm \csc u + C$

**58.** $\displaystyle\int \frac{1}{\sin u \cos u} \, du = \ln|\tan u| + C$

**Forms Involving $\tan u$, $\cot u$, $\sec u$, or $\csc u$**

**59.** $\displaystyle\int \tan u \, du = -\ln|\cos u| + C$

**60.** $\displaystyle\int \cot u \, du = \ln|\sin u| + C$

**61.** $\displaystyle\int \sec u \, du = \ln|\sec u + \tan u| + C$

**62.** $\displaystyle\int \csc u \, du = \ln|\csc u - \cot u| + C$

**63.** $\displaystyle\int \tan^2 u \, du = -u + \tan u + C$

**64.** $\displaystyle\int \cot^2 u \, du = -u - \cot u + C$

**65.** $\displaystyle\int \sec^2 u \, du = \tan u + C$

**66.** $\displaystyle\int \csc^2 u \, du = -\cot u + C$

**67.** $\displaystyle\int \tan^n u \, du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u \, du, \quad n \neq 1$

**68.** $\displaystyle\int \cot^n u \, du = -\frac{\cot^{n-1} u}{n-1} - \int \cot^{n-2} u \, du, \quad n \neq 1$

**69.** $\displaystyle\int \sec^n u \, du = \frac{\sec^{n-2} u \tan u}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} u \, du, \quad n \neq 1$

**70.** $\displaystyle \int \csc^n u \, du = -\frac{\csc^{n-2} u \cot u}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} u \, du, \quad n \neq 1$

**71.** $\displaystyle \int \frac{1}{1 \pm \tan u} \, du = \frac{1}{2}(u \pm \ln|\cos u \pm \sin u|) + C$

**72.** $\displaystyle \int \frac{1}{1 \pm \cot u} \, du = \frac{1}{2}(u \mp \ln|\sin u \pm \cos u|) + C$

**73.** $\displaystyle \int \frac{1}{1 \pm \sec u} \, du = u + \cot u \mp \csc u + C$

**74.** $\displaystyle \int \frac{1}{1 \pm \csc u} \, du = u - \tan u \pm \sec u + C$

# B.2   Formulas from Business and Finance

**Summary of business and finance formulas**

## Formulas from Business

### Basic Terms

$x$ = number of units produced (or sold)

$p$ = price per unit

$R$ = total revenue from selling $x$ units

$C$ = total cost of producing $x$ units
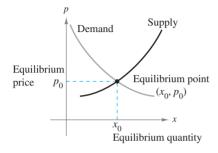
$\overline{C}$ = average cost per unit

$P$ = total profit from selling $x$ units

### Basic Equations

$$R = xp \qquad \overline{C} = \frac{C}{x} \qquad P = R - C$$

### Typical Graphs of Supply and Demand Curves



Supply curves increase as price increases and demand curves decrease as price increases. The equilibrium point occurs when the supply and demand curves intersect.
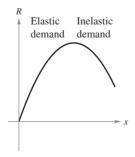
## Formulas from Business   (Continued)

Demand Function:  $p = f(x) =$ price required to sell $x$ units

$$\eta = \frac{p/x}{dp/dx} = \text{price elasticity of demand}$$

(If $|\eta| < 1$, the demand is inelastic. If $|\eta| > 1$, the demand is elastic.)
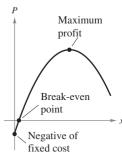
### Typical Graphs of Revenue, Cost, and Profit Functions

| Revenue Function | Cost Function | Profit Function |
|---|---|---|

The low prices required to sell more units eventually result in a decreasing revenue.

The total cost to produce $x$ units includes the fixed cost.

The break-even point occurs when $R = C$.

### Marginals

$$\frac{dR}{dx} = \text{marginal revenue}$$
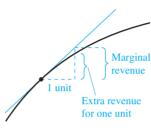
$\approx$ the *extra* revenue from selling one additional unit

$$\frac{dC}{dx} = \text{marginal cost}$$

$\approx$ the *extra* cost of producing one additional unit

$$\frac{dP}{dx} = \text{marginal profit}$$

$\approx$ the *extra* profit from selling one additional unit

Revenue Function

# Formulas from Finance

## Basic Terms

$P$ = amount of deposit

$r$ = interest rate

$n$ = number of times interest is compounded per year

$t$ = number of years

$A$ = balance after $t$ years

## Compound Interest Formulas

**1.** Balance when interest is compounded $n$ times per year

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

**2.** Balance when interest is compounded continuously

$$A = Pe^{rt}$$

## Effective Rate of Interest

$$r_{eff} = \left(1 + \frac{r}{n}\right)^{n} - 1$$

## Present Value of a Future Investment

$$P = \frac{A}{\left(1 + \dfrac{r}{n}\right)^{nt}}$$

## Balance of an Increasing Annuity After $n$ Deposits of $P$ per Year for $t$ Years

$$A = P\left[\left(1 + \frac{r}{n}\right)^{nt} - 1\right]\left(1 + \frac{n}{r}\right)$$

## Initial Deposit for a Decreasing Annuity with $n$ Withdrawals of $W$ per Year for $t$ Years

$$P = W\left(\frac{n}{r}\right)\left\{1 - \left[\frac{1}{1 + (r/n)}\right]^{nt}\right\}$$

## Monthly Installment $M$ for a Loan of $P$ Dollars over $t$ Years at $r\%$ Interest

$$M = P\left\{\frac{r/12}{1 - \left[\dfrac{1}{1 + (r/12)}\right]^{12t}}\right\}$$

## Amount of an Annuity

$$e^{rT}\int_{0}^{T} c(t)e^{-rt}\, dt$$

$c(t)$ is the continuous income function in dollars per year and $T$ is the term of the annuity in years.

# C Differential Equations

## C.1 Solutions of Differential Equations

**Find general solutions of differential equations. • Find particular solutions of differential equations.**

### General Solution of a Differential Equation

A **differential equation** is an equation involving a differentiable function and one or more of its derivatives. For instance,

$$y' + 2y = 0 \qquad \text{Differential equation}$$

is a differential equation. A function $y = f(x)$ is a **solution** of a differential equation if the equation is satisfied when $y$ and its derivatives are replaced by $f(x)$ and its derivatives. For instance,

$$y = e^{-2x} \qquad \text{Solution of differential equation}$$

is a solution of the differential equation shown above. To see this, substitute for $y$ and $y' = -2e^{-2x}$ in the original equation.

$$y' + 2y = -2e^{-2x} + 2(e^{-2x}) \qquad \text{Substitute for } y \text{ and } y'.$$
$$= 0$$

In the same way, you can show that $y = 2e^{-2x}$, $y = -3e^{-2x}$, and $y = \frac{1}{2}e^{-2x}$ are also solutions of the differential equation. In fact, each function given by

$$y = Ce^{-2x} \qquad \text{General solution}$$

where $C$ is a real number, is a solution of the equation. This family of solutions is called the **general solution** of the differential equation.

### Example 1 Checking Solutions

Show that

**a.** $y = Ce^x$ and **b.** $y = Ce^{-x}$

are solutions of the differential equation $y'' - y = 0$.

**SOLUTION**

**a.** Because $y' = Ce^x$ and $y'' = Ce^x$, it follows that

$$y'' - y = Ce^x - Ce^x$$
$$= 0.$$

So, $y = Ce^x$ is a solution.

**b.** Because $y' = -Ce^{-x}$ and $y'' = Ce^{-x}$, it follows that

$$y'' - y = Ce^{-x} - Ce^{-x}$$
$$= 0.$$

So, $y = Ce^{-x}$ is also a solution.
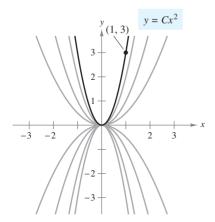
## Particular Solutions and Initial Conditions

A **particular solution** of a differential equation is any solution that is obtained by assigning specific values to the constants in the general equation.*

Geometrically, the general solution of a differential equation is a family of graphs called **solution curves.** For instance, the general solution of the differential equation $xy' - 2y = 0$ is

$$y = Cx^2. \qquad \text{General solution}$$

Figure A.7 shows several solution curves of this differential equation.

Particular solutions of a differential equation are obtained from **initial conditions** placed on the unknown function and its derivatives. For instance, in Figure A.7, suppose you want to find the particular solution whose graph passes through the point $(1, 3)$. This initial condition can be written as

$$y = 3 \quad \text{when} \quad x = 1. \qquad \text{Initial condition}$$

Substituting these values into the general solution produces $3 = C(1)^2$, which implies that $C = 3$. So, the particular solution is

$$y = 3x^2. \qquad \text{Particular solution}$$



**FIGURE A.7**

---

### Example 2  Finding a Particular Solution

Verify that

$$y = Cx^3 \qquad \text{General solution}$$

is a solution of the differential equation $xy' - 3y = 0$ for any value of $C$. Then find the particular solution determined by the initial condition

$$y = 2 \quad \text{when} \quad x = -3. \qquad \text{Initial condition}$$

**SOLUTION**    The derivative of $y = Cx^3$ is $y' = 3Cx^2$. Substituting into the differential equation produces

$$\begin{aligned} xy' - 3y &= x(3Cx^2) - 3(Cx^3) \\ &= 0. \end{aligned}$$

So, $y = Cx^3$ is a solution for any value of $C$. To find the particular solution, substitute $x = -3$ and $y = 2$ into the general solution to obtain

$$2 = C(-3)^3 \quad \text{or} \quad C = -\frac{2}{27}.$$

This implies that the particular solution is

$$y = -\frac{2}{27}x^3. \qquad \text{Particular solution}$$

---

*Some differential equations have solutions other than those given by their general solutions. These are called **singular solutions.** In this brief discussion of differential equations, singular solutions will not be discussed.

*Example 3* **Finding a Particular Solution** Ⓡ

You are working in the marketing department of a company that is producing a new cereal product to be sold nationally. You determine that a maximum of 10 million units of the product could be sold in a year. You hypothesize that the rate of growth of the sales $x$ (in millions of units) is proportional to the difference between the maximum sales and the current sales. As a differential equation, this hypothesis can be written as

$$\frac{dx}{dt} = k(10 - x), \quad 0 \le x \le 10.$$

Rate of change of $x$ | is proportional to | the difference between 10 and $x$.

The general solution of this differential equation is

$$x = 10 - Ce^{-kt} \qquad \text{General solution}$$

where $t$ is the time in years. After 1 year, 250,000 units have been sold. Sketch the graph of the sales function over a 10-year period.

**SOLUTION** Because the product is new, you can assume that $x = 0$ when $t = 0$. So, you have *two* initial conditions.

$$x = 0 \quad \text{when} \quad t = 0 \qquad \text{First initial condition}$$
$$x = 0.25 \quad \text{when} \quad t = 1 \qquad \text{Second initial condition}$$

Substituting the first initial condition into the general solution produces

$$0 = 10 - Ce^{-k(0)}$$

which implies that $C = 10$. Substituting the second initial condition into the general solution produces

$$0.25 = 10 - 10e^{-k(1)}$$

which implies that $k = \ln \frac{40}{39} \approx 0.0253$. So, the particular solution is

$$x = 10 - 10e^{-0.0253t}. \qquad \text{Particular solution}$$

The table shows the annual sales during the first 10 years, and the graph of the solution is shown in Figure A.8.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0.25 | 0.49 | 0.73 | 0.96 | 1.19 | 1.41 | 1.62 | 1.83 | 2.04 | 2.24 |



**Sales Projection**

$x = 10 - 10e^{-0.0253t}$

OATIES

Sales (in millions of units)

Time (in years)

**FIGURE A.8**

In the first three examples in this section, each solution was given in explicit form, such as $y = f(x)$. Sometimes you will encounter solutions for which it is more convenient to write the solution in implicit form, as shown in Example 4.
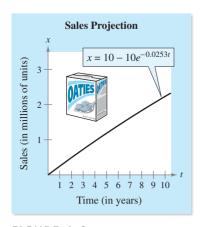
*Example 4*  **Sketching Graphs of Solutions**

Verify that

$$2y^2 - x^2 = C \qquad \text{General solution}$$

is a solution of the differential equation

$$2yy' - x = 0.$$

Then sketch the particular solutions represented by $C = 0$, $C = \pm 1$, and $C = \pm 4$.

**SOLUTION**   To verify the given solution, differentiate each side with respect to $x$.

$$2y^2 - x^2 = C \qquad \text{Given general solution}$$
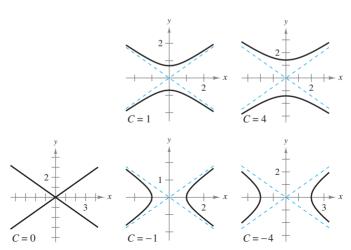
$$4yy' - 2x = 0 \qquad \text{Differentiate with respect to } x.$$

$$2yy' - x = 0 \qquad \text{Divide each side by 2.}$$

Because the third equation is the given differential equation, you can conclude that

$$2y^2 - x^2 = C$$

is a solution. The particular solutions represented by $C = 0$, $C = \pm 1$, and $C = \pm 4$ are shown in Figure A.9.



**FIGURE A.9**   Graphs of Five Particular Solutions

**CONCEPT CHECK**

1. Complete the following:  A _____ equation is an equation involving a differentiable function and one or more of its derivatives.

2. Complete the following:  Because each function given by $y = Ce^{-2x}$ is a solution of $y' + 2y = 0$, $y = Ce^{-2x}$ is the _____ solution of $y' + 2y = 0$.

3. Explain why $y' - 3y = 0$ is a differential equation.

4. In general, describe in words a particular solution of a differential equation.

In Exercises 1–4, find the first and second derivatives of the function.

**1.** $y = 3x^2 + 2x + 1$

**2.** $y = -2x^3 - 8x + 4$

**3.** $y = -3e^{2x}$

**4.** $y = -3e^{x^2}$

In Exercises 5–8, use implicit differentiation to find $dy/dx$.

**5.** $x^2 + y^2 = 2x$

**6.** $2x - y^3 = 4y$

**7.** $xy^2 = 3$

**8.** $3xy + x^2y^2 = 10$

In Exercises 9 and 10, solve for $k$.

**9.** $0.5 = 9 - 9e^{-k}$

**10.** $14.75 = 25 - 25e^{-2k}$

## Exercises C.1

In Exercises 1–10, verify that the function is a solution of the differential equation.

| *Solution* | *Differential Equation* |
|---|---|
| **1.** $y = x^3 + 5$ | $y' = 3x^2$ |
| **2.** $y = 2x^3 - x + 1$ | $y' = 6x^2 - 1$ |
| **3.** $y = e^{-2x}$ | $y' + 2y = 0$ |
| **4.** $y = 3e^{x^2}$ | $y' - 2xy = 0$ |
| **5.** $y = 2x^3$ | $y' - \dfrac{3}{x}y = 0$ |
| **6.** $y = 4x^2$ | $y' - \dfrac{2}{x}y = 0$ |
| **7.** $y = x^2$ | $x^2y'' - 2y = 0$ |
| **8.** $y = \dfrac{1}{x}$ | $xy'' + 2y' = 0$ |
| **9.** $y = 2e^{2x}$ | $y'' - y' - 2y = 0$ |
| **10.** $y = e^{x^3}$ | $y'' - 3x^2y' - 6xy = 0$ |

In Exercises 11–28, verify that the function is a solution of the differential equation for any value of $C$.

| *Solution* | *Differential Equation* |
|---|---|
| **11.** $y = \dfrac{1}{x} + C$ | $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$ |
| **12.** $y = \sqrt{4 - x^2} + C$ | $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{4 - x^2}}$ |
| **13.** $y = Ce^{4x}$ | $\dfrac{dy}{dx} = 4y$ |
| **14.** $y = Ce^{-4x}$ | $\dfrac{dy}{dx} = -4y$ |
| **15.** $y = Ce^{-t/3} + 7$ | $3\dfrac{dy}{dt} + y - 7 = 0$ |
| **16.** $y = Ce^{-t} + 10$ | $y' + y - 10 = 0$ |
| **17.** $y = Cx^2 - 3x$ | $xy' - 3x - 2y = 0$ |
| **18.** $y = x \ln x^2 + 2x^{3/2} + Cx$ | $y' - \dfrac{y}{x} = 2 + \sqrt{x}$ |
| **19.** $y = x^2 + 2x + \dfrac{C}{x}$ | $xy' + y = x(3x + 4)$ |
| **20.** $y = C_1 + C_2e^x$ | $y'' - y' = 0$ |

| Solution | Differential Equation |
|---|---|
| **21.** $y = C_1 e^{x/2} + C_2 e^{-2x}$ | $2y'' + 3y' - 2y = 0$ |
| **22.** $y = C_1 e^{4x} + C_2 e^{-x}$ | $y'' - 3y' - 4y = 0$ |
| **23.** $y = \dfrac{bx^4}{4-a} + Cx^a$ | $y' - \dfrac{ay}{x} = bx^3$ |
| **24.** $y = \dfrac{x^3}{5} - x + C\sqrt{x}$ | $2xy' - y = x^3 - x$ |
| **25.** $y = \dfrac{2}{1 + Ce^{x^2}}$ | $y' + 2xy = xy^2$ |
| **26.** $y = Ce^{x-x^2}$ | $y' + (2x - 1)y = 0$ |
| **27.** $y = x \ln x + Cx + 4$ | $x(y' - 1) - (y - 4) = 0$ |
| **28.** $y = x(\ln x + C)$ | $x + y - xy' = 0$ |

In Exercises 29–32, use implicit differentiation to verify that the function is a solution of the differential equation for any value of C.

| Solution | Differential Equation |
|---|---|
| **29.** $x^2 + y^2 = Cy$ | $y' = \dfrac{2xy}{x^2 - y^2}$ |
| **30.** $y^2 + 2xy - x^2 = C$ | $(x + y)y' - x + y = 0$ |
| **31.** $x^2 + xy = C$ | $x^2 y'' - 2(x + y) = 0$ |
| **32.** $x^2 - y^2 = C$ | $y^3 y'' + x^2 - y^2 = 0$ |

In Exercises 33–36, determine whether the function is a solution of the differential equation $y^{(4)} - 16y = 0$.

**33.** $y = e^{-2x}$

**34.** $y = 5 \ln x$

**35.** $y = \dfrac{4}{x}$

**36.** $y = 4e^{2x}$

In Exercises 37–40, determine whether the function is a solution of the differential equation $y''' - 3y' + 2y = 0$.

**37.** $y = \frac{2}{9}xe^{-2x}$

**38.** $y = 4e^x + \frac{2}{9}xe^{-2x}$

**39.** $y = xe^x$

**40.** $y = x \ln x$

In Exercises 41–48, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition.

**41.** General solution: $y = Ce^{-2x}$
   Differential equation: $y' + 2y = 0$
   Initial condition: $y = 3$ when $x = 0$

**42.** General solution: $2x^2 + 3y^2 = C$
   Differential equation: $2x + 3yy' = 0$
   Initial condition: $y = 2$ when $x = 1$

**43.** General solution: $y = C_1 + C_2 \ln x$
   Differential equation: $xy'' + y' = 0$
   Initial condition: $y = 5$ and $y' = 0.5$ when $x = 1$

**44.** General solution: $y = C_1 x + C_2 x^3$
   Differential equation: $x^2 y'' - 3xy' + 3y = 0$
   Initial condition: $y = 0$ and $y' = 4$ when $x = 2$

**45.** General solution: $y = C_1 e^{4x} + C_2 e^{-3x}$
   Differential equation: $y'' - y' - 12y = 0$
   Initial condition: $y = 5$ and $y' = 6$ when $x = 0$

**46.** General solution: $y = Ce^{x-x^2}$
   Differential equation: $y' + (2x - 1)y = 0$
   Initial condition: $y = 2$ when $x = 1$

**47.** General solution: $y = e^{2x/3}(C_1 + C_2 x)$
   Differential equation: $9y'' - 12y' + 4y = 0$
   Initial condition: $y = 4$ when $x = 0$
      $y = 0$ when $x = 3$

**48.** General solution: $y = \left(C_1 + C_2 x + \frac{1}{12}x^4\right)e^{2x}$
   Differential equation: $y'' - 4y' + 4y = x^2 e^{2x}$
   Initial condition: $y = 2$ and $y' = 1$ when $x = 0$

Ⓣ In Exercises 49–52, the general solution of the differential equation is given. Use a graphing utility to graph the particular solutions that correspond to the indicated values of C.

| General Solution | Differential Equation | C-Values |
|---|---|---|
| **49.** $y = Cx^2$ | $xy' - 2y = 0$ | $1, 2, 4$ |
| **50.** $4y^2 - x^2 = C$ | $4yy' - x = 0$ | $0, \pm 1, \pm 4$ |
| **51.** $y = C(x + 2)^2$ | $(x + 2)y' - 2y = 0$ | $0, \pm 1, \pm 2$ |
| **52.** $y = Ce^{-x}$ | $y' + y = 0$ | $0, \pm 1, \pm 2$ |

In Exercises 53–60, use integration to find the general solution of the differential equation.

**53.** $\dfrac{dy}{dx} = 3x^2$

**54.** $\dfrac{dy}{dx} = \dfrac{1}{1 + x}$

**55.** $\dfrac{dy}{dx} = \dfrac{x + 3}{x}$

**56.** $\dfrac{dy}{dx} = \dfrac{x - 2}{x}$

**57.** $\dfrac{dy}{dx} = \dfrac{1}{x^2 - 1}$

**58.** $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$
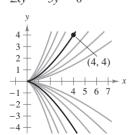
**59.** $\dfrac{dy}{dx} = x\sqrt{x - 3}$

**60.** $\dfrac{dy}{dx} = xe^x$

In Exercises 61–64, some of the curves corresponding to different values of $C$ in the general solution of the differential equation are given. Find the particular solution that passes through the point shown on the graph.

**61.** $y^2 = Cx^3$

$2xy' - 3y = 0$
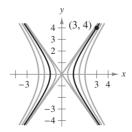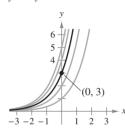


**62.** $2x^2 - y^2 = C$

$yy' - 2x = 0$
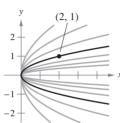


**63.** $y = Ce^x$

$y' - y = 0$



**64.** $y^2 = 2Cx$

$2xy' - y = 0$



**65. Biology** The limiting capacity of the habitat of a wildlife herd is 750. The growth rate $dN/dt$ of the herd is proportional to the unutilized opportunity for growth, as described by the differential equation

$$\frac{dN}{dt} = k(750 - N).$$

The general solution of this differential equation is

$$N = 750 - Ce^{-kt}.$$

When $t = 0$, the population of the herd is 100. After 2 years, the population has grown to 160.

(a) Write the population function $N$ as a function of $t$.

(T) (b) Use a graphing utility to graph the population function.

(c) What is the population of the herd after 4 years?

**66. Investment** The rate of growth of an investment is proportional to the amount in the investment at any time $t$. That is,

$$\frac{dA}{dt} = kA.$$

The initial investment is $1000, and after 10 years the balance is $3320.12. The general solution is

$$A = Ce^{kt}.$$

What is the particular solution?

**67. Marketing** You are working in the marketing department of a computer software company. Your marketing team determines that a maximum of 30,000 units of a new product can be sold in a year. You hypothesize that the rate of growth of the sales $x$ is proportional to the difference between the maximum sales and the current sales. That is,

$$\frac{dx}{dt} = k(30,000 - x).$$

The general solution of this differential equation is

$$x = 30,000 - Ce^{-kt}$$

where $t$ is the time in years. During the first year, 2000 units are sold. Complete the table showing the numbers of units sold in subsequent years.

| Year, $t$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| Units, $x$ | | | | | |

**68. Marketing** In Exercise 67, suppose that the maximum annual sales are 50,000 units. How does this change the sales shown in the table?

**69. Safety** Assume that the rate of change per hour in the number of miles $s$ of road cleared by a snowplow is inversely proportional to the depth $h$ of the snow. This rate of change is described by the differential equation

$$\frac{ds}{dh} = \frac{k}{h}.$$

Show that

$$s = 25 - \frac{13}{\ln 3} \ln \frac{h}{2}$$

is a solution of this differential equation.

**70.** Show that $y = a + Ce^{k(1-b)t}$ is a solution of the differential equation

$$y = a + b(y - a) + \left(\frac{1}{k}\right)\left(\frac{dy}{dt}\right)$$

where $k$ is a constant.

**71.** The function $y = Ce^{kx}$ is a solution of the differential equation

$$\frac{dy}{dx} = 0.07y.$$

Is it possible to determine $C$ or $k$ from the information given? If so, find its value.

**True or False?** In Exercises 72 and 73, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**72.** A differential equation can have more than one solution.

**73.** If $y = f(x)$ is a solution of a differential equation, then $y = f(x) + C$ is also a solution.

# C.2　Separation of Variables

**Use separation of variables to solve differential equations.** • **Use differential equations to model and solve real-life problems.**

## Separation of Variables

The simplest type of differential equation is one of the form $y' = f(x)$. You know that this type of equation can be solved by integration to obtain

$$y = \int f(x)\,dx.$$

In this section, you will learn how to use integration to solve another important family of differential equations—those in which the variables can be separated. This technique is called **separation of variables.**

---

**TECHNOLOGY**

(T) You can use a symbolic integration utility to solve a separable variables differential equation. Use a symbolic integration utility to solve the differential equation

$$y' = \frac{x}{y^2 + 1}.$$

---

> **Separation of Variables**
>
> If $f$ and $g$ are continuous functions, then the differential equation
>
> $$\frac{dy}{dx} = f(x)g(y)$$
>
> has a general solution of
>
> $$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C.$$

Essentially, the technique of separation of variables is just what its name implies. For a differential equation involving $x$ and $y$, you separate the $x$ variables to one side and the $y$ variables to the other. After separating variables, integrate each side to obtain the general solution. Here is an example.

### Example 1　Solving a Differential Equation

Find the general solution of

$$\frac{dy}{dx} = \frac{x}{y^2 + 1}.$$

**SOLUTION**　Begin by separating variables, then integrate each side.

$$\frac{dy}{dx} = \frac{x}{y^2 + 1}　　\text{Differential equation}$$

$$(y^2 + 1)\,dy = x\,dx　　\text{Separate variables.}$$

$$\int (y^2 + 1)\,dy = \int x\,dx　　\text{Integrate each side.}$$

$$\frac{y^3}{3} + y = \frac{x^2}{2} + C　　\text{General solution}$$

**Example 2**    Solving a Differential Equation

Find the general solution of

$$\frac{dy}{dx} = \frac{x}{y}.$$

**SOLUTION**    Begin by separating variables, then integrate each side.

| | |
|---|---|
| $\dfrac{dy}{dx} = \dfrac{x}{y}$ | Differential equation |
| $y \, dy = x \, dx$ | Separate variables. |
| $\displaystyle\int y \, dy = \int x \, dx$ | Integrate each side. |
| $\dfrac{y^2}{2} = \dfrac{x^2}{2} + C_1$ | Find antiderivatives. |
| $y^2 = x^2 + C$ | Multiply each side by 2. |

So, the general solution is $y^2 = x^2 + C$. Note that $C_1$ is used as a temporary constant of integration in anticipation of multiplying each side of the equation by 2 to produce the constant $C$.

---

**STUDY TIP**

After finding the general solution of a differential equation, you should use the techniques demonstrated in Section C.1 to check the solution. For instance, in Example 2 you can check the solution by differentiating the equation $y^2 = x^2 + C$ to obtain $2yy' = 2x$ or $y' = x/y$.

---

**Example 3**    Solving a Differential Equation    Ⓣ

Find the general solution of $e^y \dfrac{dy}{dx} = 2x$. Use a graphing utility to graph several solutions.

**SOLUTION**    Begin by separating variables, then integrate each side.

| | |
|---|---|
| $e^y \dfrac{dy}{dx} = 2x$ | Differential equation |
| $e^y \, dy = 2x \, dx$ | Separate variables. |
| $\displaystyle\int e^y \, dy = \int 2x \, dx$ | Integrate each side. |
| $e^y = x^2 + C$ | Find antiderivatives. |

By taking the natural logarithm of each side, you can write the general solution as

$$y = \ln(x^2 + C). \qquad \text{General solution}$$

The graphs of the particular solutions given by $C = 0$, $C = 5$, $C = 10$, and $C = 15$ are shown in Figure A.10.
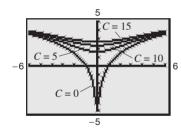


**FIGURE A.10**

> *Example 4*    **Finding a Particular Solution**

Solve the differential equation

$$xe^{x^2} + yy' = 0$$

subject to the initial condition $y = 1$ when $x = 0$.

**SOLUTION**

| | |
|---|---|
| $xe^{x^2} + yy' = 0$ | Differential equation |
| $y \dfrac{dy}{dx} = -xe^{x^2}$ | Subtract $xe^{x^2}$ from each side. |
| $y\,dy = -xe^{x^2}\,dx$ | Separate variables. |
| $\displaystyle\int y\,dy = \int -xe^{x^2}\,dx$ | Integrate each side. |
| $\dfrac{y^2}{2} = -\dfrac{1}{2}e^{x^2} + C_1$ | Find antiderivatives. |
| $y^2 = -e^{x^2} + C$ | Multiply each side by 2. |

To find the particular solution, substitute the initial condition values to obtain

$$(1)^2 = -e^{(0)^2} + C.$$

This implies that $1 = -1 + C$, or $C = 2$. So, the particular solution that satisfies the initial condition is

| | |
|---|---|
| $y^2 = -e^{x^2} + 2.$ | Particular solution |

> *Example 5*    **Solving a Differential Equation**    Ⓡ

Example 3 in Section C.1 uses the differential equation

$$\frac{dx}{dt} = k(10 - x)$$

to model the sales of a new product. Solve this differential equation.

**SOLUTION**

| | |
|---|---|
| $\dfrac{dx}{dt} = k(10 - x)$ | Differential equation |
| $\dfrac{1}{10 - x}\,dx = k\,dt$ | Separate variables. |
| $\displaystyle\int \dfrac{1}{10 - x}\,dx = \int k\,dt$ | Integrate each side. |
| $-\ln(10 - x) = kt + C_1$ | Find antiderivatives. |
| $\ln(10 - x) = -kt - C_1$ | Multiply each side by $-1$. |
| $10 - x = e^{-kt - C_1}$ | Exponentiate each side. |
| $x = 10 - Ce^{-kt}$ | Solve for $x$. |

**STUDY TIP**

In Example 5, the context of the original model indicates that $(10 - x)$ is positive. So, when you integrate $1/(10 - x)$, you can write $-\ln(10 - x)$, rather than $-\ln|10 - x|$.

   Also note in Example 5 that the solution agrees with the one that was given in Example 3 in Section C.1.

## Applications

Example 6    **Modeling National Income**    ®

Let $y$ represent the national income, let $a$ represent the income spent on necessities, and let $b$ represent the percent of the remaining income spent on luxuries. A commonly used economic model that relates these three quantities is

$$\frac{dy}{dt} = k(1 - b)(y - a)$$

where $t$ is the time in years. Assume that $b$ is 75%, and solve the resulting differential equation.

**SOLUTION**    Because $b$ is 75%, it follows that $(1 - b)$ is 0.25. So, you can solve the differential equation as shown.

$$\frac{dy}{dt} = k(0.25)(y - a) \qquad \text{Differential equation}$$

$$\frac{1}{y - a} \, dy = 0.25k \, dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y - a} \, dy = \int 0.25k \, dt \qquad \text{Integrate each side.}$$

$$\ln(y - a) = 0.25kt + C_1 \qquad \text{Find antiderivatives, given } y - a > 0.$$

$$y - a = Ce^{0.25kt} \qquad \text{Exponentiate each side.}$$

$$y = a + Ce^{0.25kt} \qquad \text{Add } a \text{ to each side.}$$

The graph of this solution is shown in Figure A.11. In the figure, note that the national income is spent in three ways.

$$(\text{National income}) = (\text{necessities}) + (\text{luxuries}) + (\text{capital investment})$$
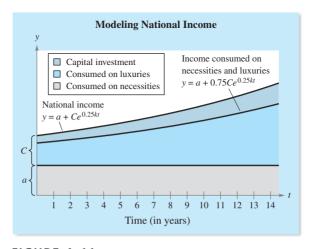
Corporate profits in the United States are closely monitored by New York City's Wall Street executives. Corporate profits, however, represent only about 12.4% of the national income. In 2005, the national income was more than $10.9 trillion. Of this, about 65.3% was employee compensation.



**Modeling National Income**

- Capital investment
- Consumed on luxuries
- Consumed on necessities

National income
$y = a + Ce^{0.25kt}$

Income consumed on necessities and luxuries
$y = a + 0.75Ce^{0.25kt}$

Time (in years)

**FIGURE A.11**

*Example 7*  **Using Graphical Information**

Find the equation of the graph that has the characteristics listed below.

**1.** At each point $(x, y)$ on the graph, the slope is $-x/2y$.

**2.** The graph passes through the point $(2, 1)$.

**SOLUTION**    Using the information about the slope of the graph, you can write the differential equation

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

Using the point on the graph, you can determine the initial condition $y = 1$ when $x = 2$.

| | |
|---|---|
| $\dfrac{dy}{dx} = -\dfrac{x}{2y}$ | Differential equation |
| $2y\,dy = -x\,dx$ | Separate variables. |
| $\displaystyle\int 2y\,dy = \int -x\,dx$ | Integrate each side. |
| $y^2 = -\dfrac{x^2}{2} + C_1$ | Find antiderivatives. |
| $2y^2 = -x^2 + C$ | Multiply each side by 2. |
| $x^2 + 2y^2 = C$ | Simplify. |

Applying the initial condition yields

$$(2)^2 + 2(1)^2 = C$$

which implies that $C = 6$. So, the equation that satisfies the two given conditions is

$$x^2 + 2y^2 = 6. \qquad \text{Particular solution}$$

As shown in Figure A.12, the graph of this equation is an ellipse.
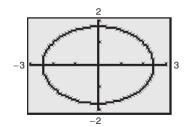


**FIGURE A.12**

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**CONCEPT CHECK**

1. Complete the following: If $f$ and $g$ are continuous functions, then the differential equation $dy/dx = f(x)g(y)$ has a general solution of

   $$\int \frac{1}{g(y)}dy = \underline{\hspace{2cm}} + C.$$

2. True or false: The differential equation $\dfrac{dy}{dx} = \dfrac{3x}{y}$ can be written in separated variables form.

3. True or false: The differential equation $\dfrac{dy}{dx} = \dfrac{3x}{y} + 1$ can be written in separated variables form.

4. In your own words, describe how to solve differential equations that can be solved by separation of variables.

**Skills Review C.2**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 4.4, 5.2, and 5.3.

In Exercises 1–6, find the indefinite integral and check your result by differentiating.

**1.** $\int x^{3/2}\, dx$

**2.** $\int (t^3 - t^{1/3})\, dt$

**3.** $\int \dfrac{2}{x - 5}\, dx$

**4.** $\int \dfrac{y}{2y^2 + 1}\, dy$

**5.** $\int e^{2y}\, dy$

**6.** $\int x e^{1 - x^2}\, dx$

In Exercises 7–10, solve the equation for $C$ or $k$.

**7.** $(3)^2 - 6(3) = 1 + C$

**8.** $(-1)^2 + (-2)^2 = C$

**9.** $10 = 2e^{2k}$

**10.** $(6)^2 - 3(6) = e^{-k}$

## Exercises C.2

In Exercises 1–6, decide whether the variables in the differential equation can be separated.

**1.** $\dfrac{dy}{dx} = \dfrac{x}{y + 3}$

**2.** $\dfrac{dy}{dx} = \dfrac{x + 1}{x}$

**3.** $\dfrac{dy}{dx} = \dfrac{1}{x} + 1$

**4.** $\dfrac{dy}{dx} = \dfrac{x}{x + y}$

**5.** $\dfrac{dy}{dx} = x - y$

**6.** $x\dfrac{dy}{dx} = \dfrac{1}{y}$

In Exercises 7–26, use separation of variables to find the general solution of the differential equation.

**7.** $\dfrac{dy}{dx} = 2x$

**8.** $\dfrac{dy}{dx} = \dfrac{1}{x}$

**9.** $3y^2 \dfrac{dy}{dx} = 1$

**10.** $\dfrac{dy}{dx} = x^2 y$

**11.** $(y + 1)\dfrac{dy}{dx} = 2x$

**12.** $(1 + y)\dfrac{dy}{dx} - 4x = 0$

**13.** $y' - xy = 0$

**14.** $y' - y = 5$

**15.** $\dfrac{dy}{dt} = \dfrac{e^t}{4y}$

**16.** $e^y \dfrac{dy}{dt} = 3t^2 + 1$

**17.** $\dfrac{dy}{dx} = \sqrt{1 - y}$

**18.** $\dfrac{dy}{dx} = \sqrt{\dfrac{x}{y}}$

**19.** $(2 + x)y' = 2y$

**20.** $y' = (2x - 1)(y + 3)$

**21.** $xy' = y$

**22.** $y' - y(x + 1) = 0$

**23.** $y' = \dfrac{x}{y} - \dfrac{x}{1 + y}$

**24.** $\dfrac{dy}{dx} = \dfrac{x^2 + 2}{3y^2}$

**25.** $e^x(y' + 1) = 1$

**26.** $yy' - 2xe^x = 0$

In Exercises 27–32, use the initial condition to find the particular solution of the differential equation.

| *Differential Equation* | *Initial Condition* |
|---|---|
| **27.** $yy' - e^x = 0$ | $y = 4$ when $x = 0$ |
| **28.** $\sqrt{x} + \sqrt{y}\, y' = 0$ | $y = 4$ when $x = 1$ |
| **29.** $x(y + 4) + y' = 0$ | $y = -5$ when $x = 0$ |
| **30.** $\dfrac{dy}{dx} = x^2(1 + y)$ | $y = 3$ when $x = 0$ |
| **31.** $dP - 6P\, dt = 0$ | $P = 5$ when $t = 0$ |
| **32.** $dT + k(T - 70)\, dt = 0$ | $T = 140$ when $t = 0$ |

In Exercises 33 and 34, find an equation for the graph that passes through the point and has the specified slope. Then graph the equation.

**33.** Point: $(-1, 1)$

Slope: $y' = \dfrac{6x}{5y}$

**34.** Point: $(8, 2)$

Slope: $y' = \dfrac{2y}{3x}$

**Velocity** In Exercises 35 and 36, solve the differential equation to find velocity $v$ as a function of time $t$ if $v = 0$ when $t = 0$. The differential equation models the motion of two people on a toboggan after consideration of the force of gravity, friction, and air resistance.

**35.** $12.5\dfrac{dv}{dt} = 43.2 - 1.25v$

**36.** $12.5\dfrac{dv}{dt} = 43.2 - 1.75v$

**Chemistry: Newton's Law of Cooling** In Exercises 37–39, use Newton's Law of Cooling, which states that the rate of change in the temperature $T$ of an object is proportional to the difference between the temperature $T$ of the object and the temperature $T_0$ of the surrounding environment. This is described by the differential equation $dT/dt = k(T - T_0)$.

**37.** A steel ingot whose temperature is 1500°F is placed in a room whose temperature is a constant 90°F. One hour later, the temperature of the ingot is 1120°F. What is the ingot's temperature 5 hours after it is placed in the room?

**38.** A room is kept at a constant temperature of 70°F. An object placed in the room cools from 350°F to 150°F in 45 minutes. How long will it take for the object to cool to a temperature of 80°F?

**39.** Food at a temperature of 70°F is placed in a freezer that is set at 0°F. After 1 hour, the temperature of the food is 48°F.

 (a) Find the temperature of the food after it has been in the freezer 6 hours.

 (b) How long will it take the food to cool to a temperature of 10°F?

**40.** **Biology: Cell Growth** The rate of growth of a spherical cell with volume $V$ is proportional to its surface area $S$. For a sphere, the surface area and volume are related by $S = kV^{2/3}$. So, a model for the cell's growth is

$$\dfrac{dV}{dt} = kV^{2/3}.$$

Solve this differential equation.

**41.** **Learning Theory** The management of a factory has found that a worker can produce at most 30 units per day. The number of units $N$ per day produced by a new employee will increase at a rate proportional to the difference between 30 and $N$. This is described by the differential equation

$$\dfrac{dN}{dt} = k(30 - N)$$

where $t$ is the time in days. Solve this differential equation.

**42.** **Sales** The rate of increase in sales $S$ (in thousands of units) of a product is proportional to the current level of sales and inversely proportional to the square of the time $t$. This is described by the differential equation

$$\dfrac{dS}{dt} = \dfrac{kS}{t^2}$$
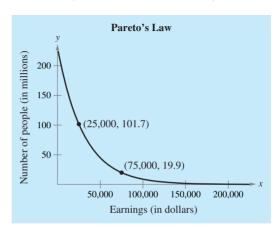
where $t$ is the time in years. The saturation point for the market is 50,000 units. That is, the limit of $S$ as $t \to \infty$ is 50. After 1 year, 10,000 units have been sold. Find $S$ as a function of the time $t$.

**43.** **Economics: Pareto's Law** According to the economist Vilfredo Pareto (1848–1923), the rate of decrease of the number of people $y$ in a stable economy having an income of at least $x$ dollars is directly proportional to the number of such people and inversely proportional to their income $x$. This is modeled by the differential equation

$$\dfrac{dy}{dx} = -k\dfrac{y}{x}.$$

Solve this differential equation.

**44.** **Economics: Pareto's Law** In 2005, 19.9 million people in the United States earned at least $75,000 and 101.7 million people earned at least $25,000 (see figure). Assume that Pareto's Law holds and use the result of Exercise 43 to determine the number of people (in millions) who earned (a) at least $20,000 and (b) at least $100,000. *(Source: U.S. Census Bureau)*

**Pareto's Law**

# C.3  First-Order Linear Differential Equations

**Solve first-order linear differential equations.** • **Use first-order linear differential equations to model and solve real-life problems.**

## First-Order Linear Differential Equations

**Definition of a First-Order Linear Differential Equation**

A **first-order linear differential equation** is an equation of the form

$$y' + P(x)y = Q(x)$$

where $P$ and $Q$ are functions of $x$. An equation that is written in this form is said to be in **standard form.**

**STUDY TIP**

The term "first-order" refers to the fact that the highest-order derivative of $y$ in the equation is the first derivative.

To solve a linear differential equation, write it in standard form to identify the functions $P(x)$ and $Q(x)$. Then integrate $P(x)$ and form the expression

$$u(x) = e^{\int P(x)\, dx} \qquad \text{Integrating factor}$$

which is called an **integrating factor.** The general solution of the equation is

$$y = \frac{1}{u(x)} \int Q(x)u(x)\, dx. \qquad \text{General solution}$$

---

*Example 1*  **Solving a Linear Differential Equation**

Find the general solution of

$$y' + y = e^x.$$

**SOLUTION**   For this equation, $P(x) = 1$ and $Q(x) = e^x$. So, the integrating factor is

$$u(x) = e^{\int dx} \qquad \text{Integrating factor}$$
$$= e^x.$$

This implies that the general solution is

$$y = \frac{1}{e^x} \int e^x (e^x)\, dx$$
$$= e^{-x}\left(\frac{1}{2}e^{2x} + C\right)$$
$$= \frac{1}{2}e^x + Ce^{-x}. \qquad \text{General solution}$$

In Example 1, the differential equation was given in standard form. For equations that are not written in standard form, you should first convert to standard form so that you can identify the functions $P(x)$ and $Q(x)$.

*Example 2*  **Solving a Linear Differential Equation**

Find the general solution of

$$xy' - 2y = x^2.$$

Assume $x > 0$.

**SOLUTION**    Begin by writing the equation in standard form.

$$y' - \left(\frac{2}{x}\right)y = x \qquad \text{Standard form, } y' + P(x)y = Q(x)$$

In this form, you can see that $P(x) = -2/x$ and $Q(x) = x$. So,

$$\int P(x)\,dx = -\int \frac{2}{x}\,dx$$
$$= -2 \ln x$$
$$= -\ln x^2$$

which implies that the integrating factor is

$$u(x) = e^{\int P(x)\,dx}$$
$$= e^{-\ln x^2}$$
$$= \frac{1}{x^2}. \qquad \text{Integrating factor}$$

This implies that the general solution is

$$y = \frac{1}{u(x)} \int Q(x)u(x)\,dx \qquad \text{Form of general solution}$$
$$= \frac{1}{1/x^2} \int x\left(\frac{1}{x^2}\right) dx \qquad \text{Substitute.}$$
$$= x^2 \int \frac{1}{x}\,dx \qquad \text{Simplify.}$$
$$= x^2(\ln x + C). \qquad \text{General solution}$$

**TECHNOLOGY**

(T) From Example 2, you can see that it can be difficult to solve a linear differential equation. Fortunately, the task is greatly simplified by symbolic integration utilities. Use a symbolic integration utility to find the particular solution of the differential equation in Example 2, given the initial condition $y = 1$ when $x = 1$.

**Guidelines for Solving a Linear Differential Equation**

1. Write the equation in standard form

$$y' + P(x)y = Q(x).$$

2. Find the integrating factor

$$u(x) = e^{\int P(x)\,dx}.$$

3. Evaluate the integral below to find the general solution.

$$y = \frac{1}{u(x)} \int Q(x)u(x)\,dx$$

## Application

*Example 3* Finding a Balance ®

You are setting up a "continuous annuity" trust fund. For 20 years, money is continuously transferred from your checking account to the trust fund at the rate of $1000 per year (about $2.74 per day). The account earns 8% interest, compounded continuously. What is the balance in the account after 20 years?

**SOLUTION** Let $A$ represent the balance after $t$ years. The balance increases in two ways: with interest *and* with additional deposits. The rate at which the balance is changing can be modeled by

$$\frac{dA}{dt} = \underbrace{0.08A}_{\text{Interest}} + \underbrace{1000}_{\text{Deposits}}.$$

In standard form, this linear differential equation is

$$\frac{dA}{dt} - 0.08A = 1000 \qquad \text{Standard form}$$

which implies that $P(t) = -0.08$ and $Q(t) = 1000$. The general solution is

$$A = -12{,}500 + Ce^{0.08t}. \qquad \text{General solution}$$

Because $A = 0$ when $t = 0$, you can determine that $C = 12{,}500$. So, the revenue after 20 years is

$$A = -12{,}500 + 12{,}500e^{0.08(20)}$$
$$\approx -12{,}500 + 61{,}912.91$$
$$= \$49{,}412.91.$$

---

**CONCEPT CHECK**

1. Given a first-order linear differential equation, what does the term "first-order" refer to?

2. True or false: $y' + \dfrac{1}{x}y = x + 1$ is a first-order linear differential equation.

3. Give the standard form of a first-order linear differential equation. What is its integrating factor?

4. Give the guidelines for solving a first-order linear differential equation.

**Skills Review C.3**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 4.1, 4.4, 5.2, and 5.3.

In Exercises 1–4, simplify the expression.

**1.** $e^{-x}(e^{2x} + e^x)$

**2.** $\dfrac{1}{e^{-x}}(e^{-x} + e^{2x})$

**3.** $e^{-\ln x^3}$

**4.** $e^{2\ln x + x}$

In Exercises 5–10, find the indefinite integral.

**5.** $\displaystyle\int 4e^{2x}\, dx$

**6.** $\displaystyle\int xe^{3x^2}\, dx$

**7.** $\displaystyle\int \dfrac{1}{2x + 5}\, dx$

**8.** $\displaystyle\int \dfrac{x + 1}{x^2 + 2x + 3}\, dx$

**9.** $\displaystyle\int (4x - 3)^2\, dx$

**10.** $\displaystyle\int x(1 - x^2)^2\, dx$

# Exercises C.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, write the linear differential equation in standard form.

**1.** $x^3 - 2x^2 y' + 3y = 0$

**2.** $y' - 5(2x - y) = 0$

**3.** $xy' + y = xe^x$

**4.** $xy' + y = x^3 y$

**5.** $y + 1 = (x - 1)y'$

**6.** $x = x^2(y' + y)$

In Exercises 7–18, solve the differential equation.

**7.** $\dfrac{dy}{dx} + 3y = 6$

**8.** $\dfrac{dy}{dx} + 5y = 15$

**9.** $\dfrac{dy}{dx} + y = e^{-x}$

**10.** $\dfrac{dy}{dx} + 3y = e^{-3x}$

**11.** $\dfrac{dy}{dx} + \dfrac{y}{x} = 3x + 4$

**12.** $\dfrac{dy}{dx} + \dfrac{2y}{x} = 3x + 1$

**13.** $y' + 5xy = x$

**14.** $y' + 5y = e^{5x}$

**15.** $(x - 1)y' + y = x^2 - 1$

**16.** $xy' + y = x^2 + 1$

**17.** $x^3 y' + 2y = e^{1/x^2}$

**18.** $xy' + y = x^2 \ln x$

In Exercises 19–22, solve for y in two ways.

**19.** $y' + y = 4$

**20.** $y' + 10y = 5$

**21.** $y' - 2xy = 2x$

**22.** $y' + 4xy = x$

In Exercises 23–26, match the differential equation with its solution.

| Differential Equation | Solution |
|---|---|
| **23.** $y' - 2x = 0$ | (a) $y = Ce^{x^2}$ |
| **24.** $y' - 2y = 0$ | (b) $y = -\frac{1}{2} + Ce^{x^2}$ |
| **25.** $y' - 2xy = 0$ | (c) $y = x^2 + C$ |
| **26.** $y' - 2xy = x$ | (d) $y = Ce^{2x}$ |

In Exercises 27–34, find the particular solution that satisfies the initial condition.

| Differential Equation | Initial Condition |
|---|---|
| **27.** $y' + y = 6e^x$ | $y = 3$ when $x = 0$ |
| **28.** $y' + 2y = e^{-2x}$ | $y = 4$ when $x = 1$ |
| **29.** $xy' + y = 0$ | $y = 2$ when $x = 2$ |
| **30.** $y' + y = x$ | $y = 4$ when $x = 0$ |
| **31.** $y' + 3x^2 y = 3x^2$ | $y = 6$ when $x = 0$ |
| **32.** $y' + (2x - 1)y = 0$ | $y = 2$ when $x = 1$ |
| **33.** $xy' - 2y = -x^2$ | $y = 5$ when $x = 1$ |
| **34.** $x^2 y' - 4xy = 10$ | $y = 10$ when $x = 1$ |

**35. Sales**   The rate of change (in thousands of units) in sales $S$ is modeled by

$$\frac{dS}{dt} = 0.2(100 - S) + 0.2t$$

where $t$ is the time in years. Solve this differential equation and use the result to complete the table.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S$ | | | | | | | | | | | |

**36. Sales**   The rate of change in sales $S$ is modeled by

$$\frac{dS}{dt} = k_1(L - S) + k_2 t$$

where $t$ is the time in years and $S = 0$ when $t = 0$. Solve this differential equation for $S$ as a function of $t$.

**Elasticity of Demand**   In Exercises 37 and 38, find the demand function $p = f(x)$. Recall from Section 3.5 that the price elasticity of demand was defined as $\eta = (p/x)/(dp/dx)$.

**37.** $\eta = 1 - \dfrac{400}{3x}, \quad p = 340$ when $x = 20$

**38.** $\eta = 1 - \dfrac{500}{3x}, \quad p = 2$ when $x = 100$

**Supply and Demand**   In Exercises 39 and 40, use the demand and supply functions to find the price $p$ as a function of time $t$. Begin by setting $D(t)$ equal to $S(t)$ and solving the resulting differential equation. Find the general solution, and then use the initial condition to find the particular solution.

**39.** $D(t) = 480 + 5p(t) - 2p'(t)$    Demand function
    $S(t) = 300 + 8p(t) + p'(t)$    Supply function
    $p(0) = \$75.00$    Initial condition

**40.** $D(t) = 4000 + 5p(t) - 4p'(t)$    Demand function
    $S(t) = 2800 + 7p(t) + 2p'(t)$    Supply function
    $p(0) = \$1000.00$    Initial condition

**41. Investment**   A brokerage firm opens a new real estate investment plan for which the earnings are equivalent to continuous compounding at the rate of $r$. The firm estimates that deposits from investors will create a net cash flow of $Pt$ dollars, where $t$ is the time in years. The rate of change in the total investment $A$ is modeled by

$$\frac{dA}{dt} = rA + Pt.$$

(a) Solve the differential equation and find the total investment $A$ as a function of $t$. Assume that $A = 0$ when $t = 0$.

(b) Find the total investment $A$ after 10 years given that $P = \$500,000$ and $r = 9\%$.

**42. Investment**   Let $A(t)$ be the amount in a fund earning interest at the annual rate of $r$, compounded continuously. If a continuous cash flow of $P$ dollars per year is withdrawn from the fund, then the rate of decrease of $A$ is given by the differential equation

$$\frac{dA}{dt} = rA - P$$

where $A = A_0$ when $t = 0$.

(a) Solve this equation for $A$ as a function of $t$.

(b) Use the result of part (a) to find $A$ when $A_0 = \$2,000,000$, $r = 7\%$, $P = \$250,000$, and $t = 5$ years.

(c) Find $A_0$ if a retired person wants a continuous cash flow of \$40,000 per year for 20 years. Assume that the person's investment will earn 8\%, compounded continuously.

**43. Velocity**   A booster rocket carrying an observation satellite is launched into space. The rocket and satellite have mass $m$ and are subject to air resistance proportional to the velocity $v$ at any time $t$. A differential equation that models the velocity of the rocket and satellite is

$$m\frac{dv}{dt} = -mg - kv$$

where $g$ is the acceleration due to gravity. Solve the differential equation for $v$ as a function of $t$.

**44. Health**   An infectious disease spreads through a large population according to the model

$$\frac{dy}{dt} = \frac{1 - y}{4}$$

where $y$ is the percent of the population exposed to the disease, and $t$ is the time in years.

(a) Solve this differential equation, assuming $y(0) = 0$.

(b) Find the number of years it takes for half of the population to have been exposed to the disease.

(c) Find the percent of the population that has been exposed to the disease after 4 years.

**45. Research Project**   Use your school's library, the Internet, or some other reference source to find an article in a scientific or business journal that uses a differential equation to model a real-life situation. Write a short paper describing the situation. If possible, describe the solution of the differential equation.

# C.4   Applications of Differential Equations

**Use differential equations to model and solve real-life problems.**

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**Example 1**   **Modeling Advertising Awareness**   Ⓡ

The new cereal product from Example 3 in Section C.1 is introduced through an advertising campaign to a population of 1 million potential customers. The rate at which the population hears about the product is assumed to be proportional to the number of people who are not yet aware of the product. By the end of 1 year, half of the population has heard of the product. How many will have heard of it by the end of 2 years?

**SOLUTION**   Let $y$ be the number (in millions) of people at time $t$ who have heard of the product. This means that $(1 - y)$ is the number of people who have not heard of it, and $dy/dt$ is the rate at which the population hears about the product. From the given assumption, you can write the differential equation as shown.

$$\frac{dy}{dt} = k(1 - y)$$

Rate of change of $y$ — is proportional to — the difference between 1 and $y$.

Using separation of variables *or* a symbolic integration utility, you can find the general solution to be

$$y = 1 - Ce^{-kt}. \qquad \text{General solution}$$

To solve for the constants $C$ and $k$, use the initial conditions. That is, because $y = 0$ when $t = 0$, you can determine that $C = 1$. Similarly, because $y = 0.5$ when $t = 1$, it follows that $0.5 = 1 - e^{-k}$, which implies that

$$k = \ln 2 \approx 0.693.$$

So, the particular solution is

$$y = 1 - e^{-0.693t}. \qquad \text{Particular solution}$$

This model is shown graphically in Figure A.13. Using the model, you can determine that the number of people who have heard of the product after 2 years is

$$y = 1 - e^{-0.693(2)}$$
$$\approx 0.75 \text{ or } 750,000 \text{ people.}$$



**Advertising Awareness**

$y = 1 - e^{-0.693t}$

(2, 0.75)

(1, 0.50)

(0, 0)

Potential customers (in millions)

Time (in years)
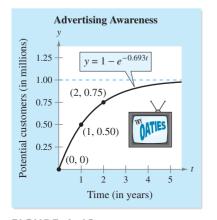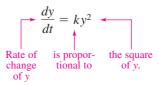
**FIGURE A.13**

**Example 2** Modeling a Chemical Reaction ®

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When $t = 0$, 60 grams of A is present, and after 1 hour ($t = 1$), only 10 grams of A remains unconverted. How much of A is present after 2 hours?

**SOLUTION** Let $y$ be the amount of unconverted substance A at any time $t$. From the given assumption about the conversion rate, you can write the differential equation as shown.

$$\frac{dy}{dt} = ky^2$$

Rate of change of $y$ — is proportional to — the square of $y$.

Using separation of variables *or* a symbolic integration utility, you can find the general solution to be

$$y = \frac{-1}{kt + C}.$$  General solution

To solve for the constants $C$ and $k$, use the initial conditions. That is, because $y = 60$ when $t = 0$, you can determine that $C = -\frac{1}{60}$. Similarly, because $y = 10$ when $t = 1$, it follows that

$$10 = \frac{-1}{k - (1/60)}$$

which implies that $k = -\frac{1}{12}$. So, the particular solution is

$$y = \frac{-1}{(-1/12)t - (1/60)}$$  Substitute for $k$ and $C$.

$$= \frac{60}{5t + 1}.$$  Particular solution

Using the model, you can determine that the amount of unconverted substance A after 2 hours is

$$y = \frac{60}{5(2) + 1}$$

$$\approx 5.45 \text{ grams}.$$

In Figure A.14, note that the chemical conversion is occurring rapidly during the first hour. Then, as more and more of substance A is converted, the conversion rate slows down.



**Chemical Reaction**

$y = \frac{60}{5t + 1}$

(0, 60)

(1, 10)

(2, 5.45)

Amount (in grams)

Time (in hours)
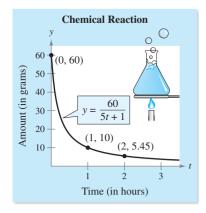
**FIGURE A.14**

**STUDY TIP**

In Example 2, the rate of conversion was assumed to be proportional to the *square* of the amount of unconverted substance A. How would the result change if the rate of conversion were assumed to be proportional to the amount of unconverted substance A?

Earlier in the text, you studied two models for population growth: *exponential growth*, which assumes that the rate of change of $y$ is proportional to $y$, and *logistic growth*, which assumes that the rate of change of $y$ is proportional to $y$ and $1 - y/L$, where $L$ is the population limit.
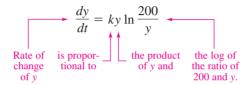
The next example describes a third type of growth model called a **Gompertz growth model.** This model assumes that the rate of change of $y$ is proportional to $y$ and the natural log of $L/y$, where $L$ is the population limit.

**Example 3** **Modeling Population Growth** ®

A population of 20 wolves has been introduced into a national park. The forest service estimates that the maximum population the park can sustain is 200 wolves. After 3 years, the population is estimated to be 40 wolves. If the population follows a Gompertz growth model, how many wolves will there be 10 years after their introduction?

**SOLUTION**   Let $y$ be the number of wolves at any time $t$. From the given assumption about the rate of growth of the population, you can write the differential equation as shown.

$$\frac{dy}{dt} = ky \ln \frac{200}{y}$$

Rate of change of $y$ — is proportional to — the product of $y$ and — the log of the ratio of 200 and $y$.

Using separation of variables *or* a symbolic integration utility, you can find the general solution to be

$$y = 200e^{-Ce^{-kt}}. \qquad \text{General solution}$$

To solve for the constants $C$ and $k$, use the initial conditions. That is, because $y = 20$ when $t = 0$, you can determine that

$$C = \ln 10$$
$$\approx 2.3026.$$

Similarly, because $y = 40$ when $t = 3$, it follows that

$$40 = 200e^{-2.3026e^{-3k}}$$

which implies that $k \approx 0.1194$. So, the particular solution is

$$y = 200e^{-2.3026e^{-0.1194t}}. \qquad \text{Particular solution}$$

Using the model, you can estimate the wolf population after 10 years to be

$$y = 200e^{-2.3026e^{-0.1194(10)}}$$
$$\approx 100 \text{ wolves.}$$

In Figure A.15, note that after 10 years the population has reached about half of the estimated maximum population. Try checking the growth model to see that it yields $y = 20$ when $t = 0$ and $y = 40$ when $t = 3$.

**Population Growth**
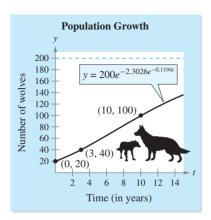


**FIGURE A.15**

In genetics, a commonly used hybrid selection model is based on the differential equation

$$\frac{dy}{dt} = ky(1 - y)(a - by).$$

In this model, $y$ represents the portion of the population that has a certain characteristic and $t$ represents the time (measured in generations). The numbers $a$, $b$, and $k$ are constants that depend on the genetic characteristic that is being studied.

### *Example 4*   Modeling Hybrid Selection ®

You are studying a population of beetles to determine how quickly characteristic D will pass from one generation to the next. At the beginning of your study ($t = 0$), you find that half the population has characteristic D. After four generations ($t = 4$), you find that 80% of the population has characteristic D. Use the hybrid selection model above with $a = 2$ and $b = 1$ to find the percent of the population that will have characteristic D after 10 generations.

**SOLUTION**   Using $a = 2$ and $b = 1$, the differential equation for the hybrid selection model is

$$\frac{dy}{dt} = ky(1 - y)(2 - y).$$

Using separation of variables *or* a symbolic integration utility, you can find the general solution to be

$$\frac{y(2 - y)}{(1 - y)^2} = Ce^{2kt}. \qquad \text{General solution}$$

To solve for the constants $C$ and $k$, use the initial conditions. That is, because $y = 0.5$ when $t = 0$, you can determine that $C = 3$. Similarly, because $y = 0.8$ when $t = 4$, it follows that

$$\frac{0.8(1.2)}{(0.2)^2} = 3e^{8k}$$

which implies that

$$k = \frac{1}{8} \ln 8 \approx 0.2599.$$

So, the particular solution is

$$\frac{y(2 - y)}{(1 - y)^2} = 3e^{0.5199t}. \qquad \text{Particular solution}$$

Using the model, you can estimate the percent of the population that will have characteristic D after 10 generations to be given by

$$\frac{y(2 - y)}{(1 - y)^2} = 3e^{0.5199(10)}.$$

Using a symbolic algebra utility, you can solve this equation for $y$ to obtain $y \approx 0.96$. The graph of the model is shown in Figure A.16.



**Hybrid Selection**

**FIGURE A.16**

4 gal/min



4 gal/min
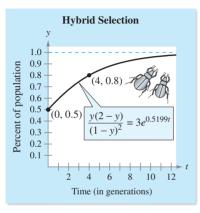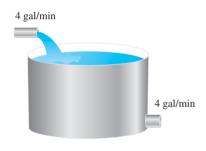
**FIGURE A.17**

*Example 5*  **Modeling a Chemical Mixture**  (R)

A tank contains 40 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing half water and half alcohol is added to the tank at the rate of 4 gallons per minute. At the same time, the tank is being drained at the rate of 4 gallons per minute, as shown in Figure A.17. Assuming that the solution is stirred constantly, how much alcohol will be in the tank after 10 minutes?

**SOLUTION**   Let $y$ be the number of gallons of alcohol in the tank at any time $t$. The *percent* of alcohol in the 40-gallon tank at any time is $y/40$. Moreover, because 4 gallons of solution is being drained each minute, the rate of change of $y$ is

$$\frac{dy}{dt} = -4\left(\frac{y}{40}\right) + 2$$

| Rate of change of $y$ | is equal to the amount of alcohol draining out | plus the amount of alcohol entering. |

where 2 represents the number of gallons of alcohol entering each minute in the 50% solution. In standard form, this linear differential equation is

$$y' + \frac{1}{10}y = 2. \qquad \text{Standard form}$$

Using an integrating factor *or* a symbolic integration utility, you can find the general solution to be

$$y = 20 + Ce^{-t/10}. \qquad \text{General solution}$$

Because $y = 4$ when $t = 0$, you can conclude that $C = -16$. So, the particular solution is

$$y = 20 - 16e^{-t/10}. \qquad \text{Particular solution}$$

Using this model, you can determine that the amount of alcohol in the tank when $t = 10$ is

$$y = 20 - 16e^{-(10)/10}$$
$$\approx 14.1 \text{ gallons.}$$

**CONCEPT CHECK**

1. What does the exponential growth model assume about the rate of change of $y$?
2. What does the logistic growth model assume about the rate of change of $y$?
3. What does the Gompertz growth model assume about the rate of change of $y$?
4. In the logistic and Gompertz growth models, what does $L$ represent?

**Skills Review C.4**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 2.3, C.2, and C.3.

In Exercises 1–4, use separation of variables to find the general solution of the differential equation.

**1.** $\dfrac{dy}{dx} = 3x$

**2.** $2y\dfrac{dy}{dx} = 3$

**3.** $\dfrac{dy}{dx} = 2xy$

**4.** $\dfrac{dy}{dx} = \dfrac{x - 4}{4y^3}$

In Exercises 5–8, use an integrating factor to solve the first-order linear differential equation.

**5.** $y' + 2y = 4$

**6.** $y' + 2y = e^{-2x}$

**7.** $y' + xy = x$

**8.** $xy' + 2y = x^2$

In Exercises 9 and 10, write the equation that models the statement.

**9.** The rate of change of $y$ with respect to $x$ is proportional to the square of $x$.

**10.** The rate of change of $x$ with respect to $t$ is proportional to the difference of $x$ and $t$.

## Exercises C.4

In Exercises 1–6, assume that the rate of change of $y$ is proportional to $y$. Solve the resulting differential equation $dy/dx = ky$ and find the particular solution that passes through the points.

**1.** $(0, 1), (3, 2)$

**2.** $(0, 4), (1, 6)$

**3.** $(0, 4), (4, 1)$

**4.** $(0, 60), (5, 30)$

**5.** $(2, 2), (3, 4)$

**6.** $(1, 4), (2, 1)$

**7. Investment** The rate of growth of an investment is proportional to the amount $A$ of the investment at any time $t$. An investment of \$2000 increases to a value of \$2983.65 in 5 years. Find its value after 10 years.

**8. Population Growth** The rate of change of the population of a city is proportional to the population $P$ at any time $t$. In 1998, the population was 400,000, and the constant of proportionality was 0.015. Estimate the population of the city in the year 2005.

**9. Sales Growth** The rate of change in sales $S$ (in thousands of units) of a new product is proportional to the difference between $L$ and $S$ (in thousands of units) at any time $t$. When $t = 0$, $S = 0$. Write and solve the differential equation for this sales model.

**10. Sales Growth** Use the result of Exercise 9 to write $S$ as a function of $t$ if (a) $L = 100$, $S = 25$ when $t = 2$, and (b) $L = 500$, $S = 50$ when $t = 1$.

In Exercises 11–14, the rate of change of $y$ is proportional to the product of $y$ and the difference of $L$ and $y$. Solve the resulting differential equation $dy/dx = ky(L - y)$ and find the particular solution that passes through the points for the given value of $L$.

**11.** $L = 20; (0, 1), (5, 10)$

**12.** $L = 100; (0, 10), (5, 30)$

**13.** $L = 5000; (0, 250), (25, 2000)$

**14.** $L = 1000; (0, 100), (4, 750)$

**15. Biology** At any time $t$, the rate of growth of the population $N$ of deer in a state park is proportional to the product of $N$ and $L - N$, where $L = 500$ is the maximum number of deer the park can maintain. When $t = 0$, $N = 100$, and when $t = 4$, $N = 200$. Write $N$ as a function of $t$.

**16. Sales Growth** The rate of change in sales $S$ (in thousands of units) of a new product is proportional to the product of $S$ and $L - S$. $L$ (in thousands of units) is the estimated maximum level of sales, and $S = 10$ when $t = 0$. Write and solve the differential equation for this sales model.

**Learning Theory** In Exercises 17 and 18, assume that the rate of change in the proportion $P$ of correct responses after $n$ trials is proportional to the product of $P$ and $L - P$, where $L$ is the limiting proportion of correct responses.

**17.** Write and solve the differential equation for this learning theory model.

(T) **18.** Use the solution of Exercise 17 to write $P$ as a function of $n$, and then use a graphing utility to graph the solution.

    (a) $L = 1.00$

        $P = 0.50$ when $n = 0$

        $P = 0.85$ when $n = 4$

    (b) $L = 0.80$

        $P = 0.25$ when $n = 0$

        $P = 0.60$ when $n = 10$

(T) **Chemical Reaction** In Exercises 19 and 20, use the chemical reaction model in Example 2 to find the amount $y$ as a function of $t$, and use a graphing utility to graph the function.

**19.** $y = 45$ grams when $t = 0$; $y = 4$ grams when $t = 2$

**20.** $y = 75$ grams when $t = 0$; $y = 12$ grams when $t = 1$

In Exercises 21 and 22, use the Gompertz growth model described in Example 3 to find the growth function, and sketch its graph.

**21.** $L = 500$; $y = 100$ when $t = 0$; $y = 150$ when $t = 2$

**22.** $L = 5000$; $y = 500$ when $t = 0$; $y = 625$ when $t = 1$

**23. Biology** A population of eight beavers has been introduced into a new wetlands area. Biologists estimate that the maximum population the wetlands can sustain is 60 beavers. After 3 years, the population is 15 beavers. If the population follows a Gompertz growth model, how many beavers will be in the wetlands after 10 years?

**24. Biology** A population of 30 rabbits has been introduced into a new region. It is estimated that the maximum population the region can sustain is 400 rabbits. After 1 year, the population is estimated to be 90 rabbits. If the population follows a Gompertz growth model, how many rabbits will be present after 3 years?

**Biology** In Exercises 25 and 26, use the hybrid selection model in Example 4 to find the percent of the population that has the indicated characteristic.

**25.** You are studying a population of mayflies to determine how quickly characteristic A will pass from one generation to the next. At the start of the study, half the population has characteristic A. After four generations, 75% of the population has characteristic A. Find the percent of the population that will have characteristic A after 10 generations. (Assume $a = 2$ and $b = 1$.)

**26.** A research team is studying a population of snails to determine how quickly characteristic B will pass from one generation to the next. At the start of the study, 40% of the snails have characteristic B. After five generations, 80% of the population has characteristic B. Find the percent of the population that will have characteristic B after eight generations. (Assume $a = 2$ and $b = 1$.)

**27. Chemical Reaction** In a chemical reaction, a compound changes into another compound at a rate proportional to the unchanged amount, according to the model

$$\frac{dy}{dt} = ky.$$

    (a) Solve the differential equation.

    (b) If the initial amount of the original compound is 20 grams, and the amount remaining after 1 hour is 16 grams, when will 75% of the compound have been changed?

**28. Chemical Mixture** A 100-gallon tank is full of a solution containing 25 pounds of a concentrate. Starting at time $t = 0$, distilled water is admitted to the tank at the rate of 5 gallons per minute, and the well-stirred solution is withdrawn at the same rate.

    (a) Find the amount $Q$ of the concentrate in the solution as a function of $t$. (*Hint:* $Q' + Q/20 = 0$)

    (b) Find the time when the amount of concentrate in the tank reaches 15 pounds.

**29. Chemical Mixture** A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the same rate. Find the amount $Q$ of concentrate in the tank after 30 minutes. $\left(\textit{Hint:}\ Q' + Q/20 = \frac{5}{2}\right)$

**30. Safety** Assume that the rate of change per hour in the number of miles $s$ of road cleared by a snowplow is inversely proportional to the depth $h$ of snow. That is,

$$\frac{ds}{dh} = \frac{k}{h}.$$

Find $s$ as a function of $h$ if $s = 25$ miles when $h = 2$ inches and $s = 12$ miles when $h = 6$ inches ($2 \le h \le 15$).

**31. Chemistry** A wet towel hung from a clothesline to dry loses moisture through evaporation at a rate proportional to its moisture content. If after 1 hour the towel has lost 40% of its original moisture content, after how long will it have lost 80%?

**32. Biology** Let $x$ and $y$ be the sizes of two internal organs of a particular mammal at time $t$. Empirical data indicate that the relative growth rates of these two organs are equal, and can be modeled by

$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{y}\frac{dy}{dt}.$$

Use this differential equation to write $y$ as a function of $x$.

**33. Population Growth** When predicting population growth, demographers must consider birth and death rates as well as the net change caused by the difference between the rates of immigration and emigration. Let $P$ be the population at time $t$ and let $N$ be the net increase per unit time due to the difference between immigration and emigration. So, the rate of growth of the population is given by

$$\frac{dP}{dt} = kP + N, \quad N \text{ is constant.}$$

Solve this differential equation to find $P$ as a function of time.

**34. Meteorology** The barometric pressure $y$ (in inches of mercury) at an altitude of $x$ miles above sea level decreases at a rate proportional to the current pressure according to the model

$$\frac{dy}{dx} = -0.2y$$

where $y = 29.92$ inches when $x = 0$. Find the barometric pressure (a) at the top of Mt. St. Helens (8364 feet) and (b) at the top of Mt. McKinley (20,320 feet).

**35. Investment** A large corporation starts at time $t = 0$ to invest part of its receipts at a rate of $P$ dollars per year in a fund for future corporate expansion. Assume that the fund earns $r$ percent interest per year compounded continuously. So, the rate of growth of the amount $A$ in the fund is given by

$$\frac{dA}{dt} = rA + P$$

where $A = 0$ when $t = 0$. Solve this differential equation for $A$ as a function of $t$.

**Investment** In Exercises 36–38, use the result of Exercise 35.

**36.** Find $A$ for each situation.

(a) $P = \$100,000$, $r = 12\%$, and $t = 5$ years

(b) $P = \$250,000$, $r = 15\%$, and $t = 10$ years

**37.** Find $P$ if the corporation needs $\$120,000,000$ in 8 years and the fund earns $16\frac{1}{4}\%$ interest compounded continuously.

**38.** Find $t$ if the corporation needs $\$800,000$ and it can invest $\$75,000$ per year in a fund earning 13% interest compounded continuously.

**Medical Science** In Exercises 39–41, a medical researcher wants to determine the concentration $C$ (in moles per liter) of a tracer drug injected into a moving fluid. Solve this problem by considering a single-compartment dilution model (see figure). Assume that the fluid is continuously mixed and that the volume of fluid in the compartment is constant.
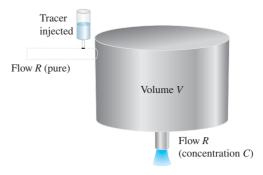


Figure for 39–41

**39.** If the tracer is injected instantaneously at time $t = 0$, then the concentration of the fluid in the compartment begins diluting according to the differential equation

$$\frac{dC}{dt} = \left(-\frac{R}{V}\right)C, \quad C = C_0 \text{ when } t = 0.$$

(a) Solve this differential equation to find the concentration as a function of time.

(b) Find the limit of $C$ as $t \to \infty$.

**40.** Use the solution of the differential equation in Exercise 39 to find the concentration as a function of time, and use a graphing utility to graph the function.
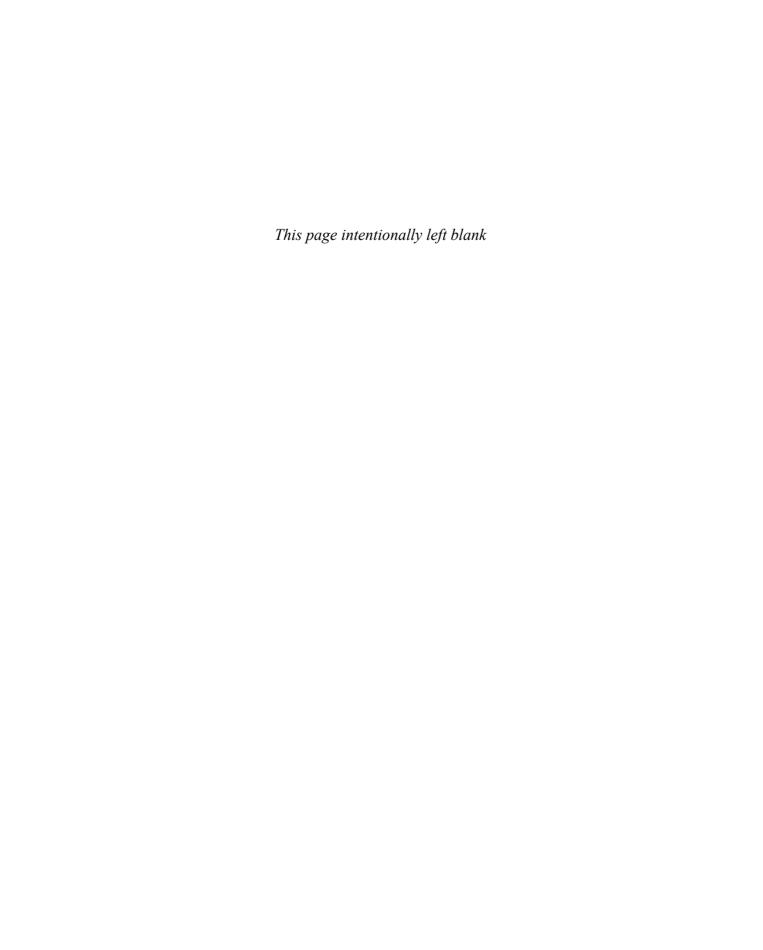
(a) $V = 2$ liters, $R = 0.5$ L/min, and $C_0 = 0.6$ mol/L

(b) $V = 2$ liters, $R = 1.5$ L/min, and $C_0 = 0.6$ mol/L

**41.** In Exercises 39 and 40, it was assumed that there was a single initial injection of the tracer drug into the compartment. Now consider the case in which the tracer is continuously injected (beginning at $t = 0$) at the rate of $Q$ mol/min. Considering $Q$ to be negligible compared with $R$, use the differential equation

$$\frac{dC}{dt} = \frac{Q}{V} - \left(\frac{R}{V}\right)C, \quad C = 0 \text{ when } t = 0.$$

(a) Solve this differential equation to find the concentration as a function of time.

(b) Find the limit of $C$ as $t \to \infty$.

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# Answers to Selected Exercises
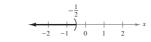
## CHAPTER 0

### SECTION 0.1   (page 7)

**1.** Rational   **3.** Irrational   **5.** Rational

**7.** Rational   **9.** Irrational

**11.** (a) Yes   (b) No   (c) Yes

**13.** (a) Yes   (b) No   (c) No

**15.** $x \geq 12$



**17.** $x < -\frac{1}{2}$



**19.** $x > 1$
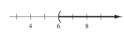


**21.** $-\frac{1}{2} < x < \frac{7}{2}$



**23.** $-\frac{3}{4} < x < -\frac{1}{4}$



**25.** $x > 6$



**27.** $-\frac{3}{2} < x < 2$



**29.** $4.1 \leq E \leq 4.25$

**31.** $p \leq 0.4$   **33.** $[120, 180]$   **35.** $(160, 280)$

**37.** (a) False   (b) True   (c) True   (d) False

### SECTION 0.2   (page 12)

**1.** (a) $-51$   (b) $51$   (c) $51$

**3.** (a) $-14.99$   (b) $14.99$   (c) $14.99$

**5.** (a) $-\frac{128}{75}$   (b) $\frac{128}{75}$   (c) $\frac{128}{75}$   **7.** $|x| \leq 2$

**9.** $|x| > 2$   **11.** $|x - 5| \leq 3$   **13.** $|x - 2| > 2$

**15.** $|x - 5| < 3$   **17.** $|y - a| \leq 2$

**19.** $-4 < x < 4$   **21.** $x < -6$ or $x > 6$

   

**23.** $3 < x < 7$   **25.** $x \leq -7$ or $x \geq 13$

   

**27.** $x < 6$ or $x > 14$   **29.** $4 < x < 5$

   

**31.** $a - b \leq x \leq a + b$   **33.** $\dfrac{a - 8b}{3} < x < \dfrac{a + 8b}{3}$

   

**35.** 16   **37.** 1.25   **39.** $\frac{1}{8}$   **41.** $|M - 1083.4| < 0.2$

**43.** $65.8 \leq h \leq 71.2$   **45.** $175,000 \leq x \leq 225,000$

**47.** (a) $|4750 - E| \leq 500$, $|4750 - E| \leq 237.50$

(b) At variance

**49.** (a) $|20,000 - E| \leq 500$, $|20,000 - E| \leq 1000$

(b) At variance

**51.** $\$11,759.40 \leq C \leq \$17,639.10$

### SECTION 0.3   (page 18)

**1.** $-54$   **3.** $\frac{1}{2}$   **5.** 4   **7.** 44   **9.** 5   **11.** 9

**13.** $\frac{1}{2}$   **15.** $\frac{1}{4}$   **17.** 908.3483   **19.** $-5.3601$

**21.** $\dfrac{3}{4y^{14}}$   **23.** $10x^4$   **25.** $7x^5$

**27.** $\frac{5}{2}(x + y)^5$, $x \neq -y$   **29.** $3x$, $x > 0$

**31.** $2\sqrt{2}$   **33.** $3x\sqrt[3]{2x^2}$   **35.** $\dfrac{2x^3z}{y}\sqrt[3]{\dfrac{18z^2}{y}}$

**37.** $3x(x + 2)(x - 2)$   **39.** $(2x^3 + 1)/x^{1/2}$

**41.** $3(x + 1)^{1/2}(x + 2)(x - 1)$   **43.** $\dfrac{2(x - 1)^2}{(x + 1)^2}$

**45.** $x \geq 4$   **47.** $(-\infty, \infty)$   **49.** $(-\infty, 4) \cup (4, \infty)$

**51.** $x \neq 1$, $x \geq -2$   **53.** $\$19,121.84$

**55.** $\$11,345.46$   **57.** $\dfrac{\sqrt{2}}{2}\pi$ sec or about 2.22 sec

**59.** Answers will vary.

### SECTION 0.4   (page 24)

**1.** $\dfrac{1}{6}, 1$   **3.** $\dfrac{3}{2}$   **5.** $-2 \pm \sqrt{3}$   **7.** $\dfrac{-3 \pm \sqrt{41}}{4}$

**9.** $(x - 2)^2$   **11.** $(2x + 1)^2$   **13.** $(3x - 1)(x - 1)$

**15.** $(3x - 2)(x - 1)$   **17.** $(x - 2y)^2$

**19.** $(3 + y)(3 - y)(9 + y^2)$   **21.** $(x - 2)(x^2 + 2x + 4)$

**23.** $(y + 4)(y^2 - 4y + 16)$   **25.** $(x - y)(x^2 + xy + y^2)$

**27.** $(x - 4)(x - 1)(x + 1)$   **29.** $(2x - 3)(x^2 + 2)$

**31.** $(x - 2)(2x^2 - 1)$     **33.** $(x + 4)(x - 4)(x^2 + 1)$

**35.** $0, 5$    **37.** $\pm 3$    **39.** $\pm\sqrt{3}$    **41.** $0, 6$

**43.** $-2, 1$    **45.** $-1, 6$    **47.** $-1, -\frac{2}{3}$    **49.** $-4$

**51.** $\pm 2$    **53.** $1, \pm 2$    **55.** $(-\infty, -2] \cup [2, \infty)$

**57.** $(-\infty, 3] \cup [4, \infty)$    **59.** $(-\infty, -1] \cup \left[-\frac{1}{5}, \infty\right)$

**61.** $(x + 1)(x^2 - 4x - 2)$    **63.** $(x + 1)(2x^2 - 3x + 1)$

**65.** $-2, -1, 4$    **67.** $1, 2, 3$    **69.** $-\frac{2}{3}, -\frac{1}{2}, 3$

**71.** $4$    **73.** $-2, -1, \frac{1}{4}$

**75.** Two solutions; The solutions of the equation are $\pm 2000$, but the minimum average cost occurs at the positive value, $2000$; $2000$ units

**77.** $3.4 \times 10^{-5}$

## SECTION 0.5    *(page 32)*

**1.** $\dfrac{x + 3}{x - 2}$    **3.** $\dfrac{5x - 1}{x^2 + 2}$    **5.** $-\dfrac{x}{x^2 - 4}$    **7.** $\dfrac{2}{x - 3}$

**9.** $\dfrac{(A + C)x^2 + (A + B - 2C)x - (2A - 2B - C)}{(x - 1)^2(x + 2)}$

**11.** $\dfrac{(A + B)x^2 - (6B - C)x + 3(A - 2C)}{(x - 6)(x^2 + 3)}$

**13.** $\dfrac{-2x^2 + x - 4}{x(x^2 + 2)}$    **15.** $-\dfrac{x^2 + 3}{(x + 1)(x - 2)(x - 3)}$

**17.** $\dfrac{x + 2}{(x + 1)^{3/2}}$    **19.** $-\dfrac{3t}{2\sqrt{1 + t}}$    **21.** $\dfrac{x(x^2 + 2)}{(x^2 + 1)^{3/2}}$

**23.** $\dfrac{2}{x^2\sqrt{x^2 + 2}}$    **25.** $\dfrac{1}{2\sqrt{x}(x + 1)^{3/2}}$    **27.** $\dfrac{3x(x + 2)}{(2x + 3)^{3/2}}$

**29.** $\dfrac{\sqrt{10}}{5}$    **31.** $\dfrac{4x\sqrt{x - 1}}{x - 1}$    **33.** $\dfrac{49\sqrt{x^2 - 9}}{x + 3}$

**35.** $\dfrac{\sqrt{14} + 2}{2}$    **37.** $\dfrac{x(5 + \sqrt{3})}{11}$    **39.** $\sqrt{6} - \sqrt{5}$

**41.** $\sqrt{x} - \sqrt{x - 2}$    **43.** $\dfrac{1}{\sqrt{x + 2} + \sqrt{2}}$

**45.** $\dfrac{4 - 3x^2}{x^4(4 - x^2)^{3/2}}$    **47.** $\$200.38$

# CHAPTER 1
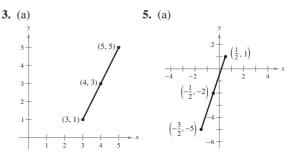
## SECTION 1.1    *(page 40)*

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**Skills Review**    *(page 40)*

**1.** $3\sqrt{5}$    **2.** $2\sqrt{5}$    **3.** $\frac{1}{2}$    **4.** $-2$    **5.** $5\sqrt{3}$
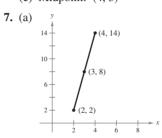
**6.** $-\sqrt{2}$    **7.** $x = -3, x = 9$

**8.** $y = -8, y = 4$    **9.** $x = 19$    **10.** $y = 1$

---

**1.**



**3. (a)**



**(b)** $d = 2\sqrt{5}$

**(c)** Midpoint: $(4, 3)$

**5. (a)**
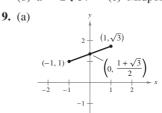


**(b)** $d = 2\sqrt{10}$

**(c)** Midpoint: $\left(-\frac{1}{2}, -2\right)$

**7. (a)**
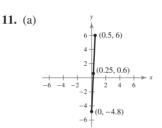


**(b)** $d = 2\sqrt{37}$    **(c)** Midpoint: $(3, 8)$

**9. (a)**



**(b)** $d = \sqrt{8 - 2\sqrt{3}}$    **(c)** Midpoint: $\left(0, \dfrac{1 + \sqrt{3}}{2}\right)$

**11. (a)**
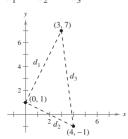


**(b)** $d = \sqrt{116.89}$    **(c)** Midpoint: $(0.25, 0.6)$
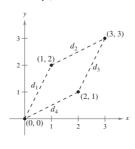
**13.** (a) $a = 4$, $b = 3$, $c = 5$

(b) $4^2 + 3^2 = 5^2$

**15.** (a) $a = 10$, $b = 3$, $c = \sqrt{109}$

(b) $10^2 + 3^2 = \left(\sqrt{109}\right)^2$

**17.** $d_1 = \sqrt{45}$, $d_2 = \sqrt{20}$,        **19.** $d_1 = d_2 = d_3 = d_4$

$d_3 = \sqrt{65}$                    $= \sqrt{5}$

$d_1^2 + d_2^2 = d_3^2$
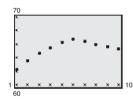
**21.** $x = 4, -2$        **23.** $y = \pm\sqrt{55}$

**25.** (a) 16.76 ft        (b) 1341.04 ft$^2$

**27.** Answers will vary. Sample answer:

The number of subscribers appears to be increasing from 1996 to 2001 and decreasing from 2001 to 2005.

**29.** (a) 10,700        (b) 10,900        (c) 11,400        (d) 12,500

**31.** (a) $92 thousand        (b) $100 thousand

(c) $122 thousand        (d) $207 thousand

**33.** (a) Revenue: $28,606.5 million

Profit: $2393.5 million

(b) Actual 2003 revenue: $27,061 million

Actual 2003 profit: $1354 million

(c) No, the increase in revenue from 2003 to 2005 is greater than the increase in revenue from 2001 to 2003.
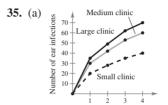
No, the profit decreased from 2001 to 2003 and then increased from 2003 to 2005.

(d) Expenses for 2001: $23,211 million
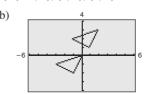
Expenses for 2003: $25,707 million

Expenses for 2005: $29,215 million

(e) Answers will vary.

**35.** (a)

(b) The larger the clinic, the more patients a doctor can treat.

**37.** (a) $(-1, 2)$, $(1, 1)$, $(2, 3)$

(b)

**39.** $\left(\dfrac{3x_1 + x_2}{4}, \dfrac{3y_1 + y_2}{4}\right)$, $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$,

$\left(\dfrac{x_1 + 3x_2}{4}, \dfrac{y_1 + 3y_2}{4}\right)$

**41.** (a) $\left(\frac{7}{4}, -\frac{7}{4}\right)$, $\left(\frac{5}{2}, -\frac{3}{2}\right)$, $\left(\frac{13}{4}, -\frac{5}{4}\right)$

(b) $\left(-\frac{3}{2}, -\frac{9}{4}\right)$, $\left(-1, -\frac{3}{2}\right)$, $\left(-\frac{1}{2}, -\frac{3}{4}\right)$

## SECTION 1.2   *(page 53)*

**Skills Review**   *(page 53)*
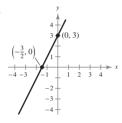
**1.** $y = \frac{1}{5}(x + 12)$        **2.** $y = x - 15$

**3.** $y = \dfrac{1}{x^3 + 2}$

**4.** $y = \pm\sqrt{x^2 + x - 6} = \pm\sqrt{(x + 3)(x - 2)}$

**5.** $y = -1 \pm \sqrt{9 - (x - 2)^2}$

**6.** $y = 5 \pm \sqrt{81 - (x + 6)^2}$        **7.** $x^2 - 4x + 4$

**8.** $x^2 + 6x + 9$        **9.** $x^2 - 5x + \frac{25}{4}$

**10.** $x^2 + 3x + \frac{9}{4}$        **11.** $(x - 2)(x - 1)$

**12.** $(x + 3)(x + 2)$        **13.** $\left(y - \frac{3}{2}\right)^2$        **14.** $\left(y - \frac{7}{2}\right)^2$

**1.** (a) Not a solution point        (b) Solution point

(c) Solution point

**3.** (a) Solution point        (b) Not a solution point

(c) Not a solution point

**5.** e        **6.** b        **7.** c        **8.** f        **9.** a        **10.** d
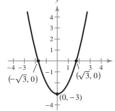
**11.** $(0, -3)$, $\left(\frac{3}{2}, 0\right)$        **13.** $(0, -2)$, $(-2, 0)$, $(1, 0)$

**15.** $(-2, 0)$, $(0, 2)$, $(2, 0)$        **17.** $(-2, 0)$, $(0, 2)$

**19.** $(0, 0)$

**21.**



**23.**



**49.** $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = 2$



**25.**



**27.**



**51.** $(x + 1.5)^2 + (y - 3)^2 = 1$



**29.**



**31.**



**53.** $(1, 1)$     **55.** $(3, 4), (5, 0)$

**57.** $(0, 0), \left(\sqrt{2}, 2\sqrt{2}\right), \left(-\sqrt{2}, -2\sqrt{2}\right)$

**59.** $(-1, 0), (0, 1), (1, 0)$

**61.** (a) $C = 11.8x + 15,000; R = 19.3x$

(b) 2000 units

(c) 2134 units

**63.** 50,000 units

**65.** 193 units
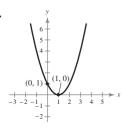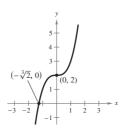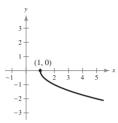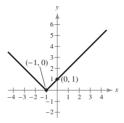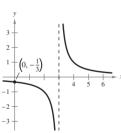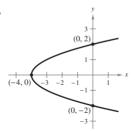
**33.**



**35.**
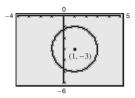






**37.** $x^2 + y^2 - 16 = 0$

**39.** $x^2 + y^2 - 4x + 2y - 4 = 0$

**67.** $(15, 120)$

**69.** (a) The model fits the data well. Explanations will vary.

(b) \$8622.7 million
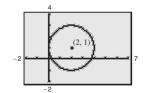
**41.** $x^2 + y^2 + 2x - 4y = 0$    **43.** $x^2 + y^2 - 6x - 8y = 0$

**45.** $(x - 1)^2 + (y + 3)^2 = 4$

**71.** (a)

| Year | 2000 | 2001 | 2002 |
|------|------|------|------|
| Salary | 587 | 613.53 | 638.52 |

| Year | 2003 | 2004 | 2007 |
|------|------|------|------|
| Salary | 661.97 | 683.88 | 740.37 |
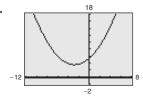


**47.** $(x - 2)^2 + (y - 1)^2 = 4$

(b) Answers will vary.

(c) \$770.33; answers will vary.

**73.**
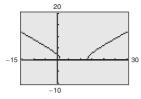
The greater the value of $c$, the steeper the line.

**75.**
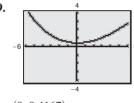
$(0, 5.36)$

**77.**

$(1.4780, 0), (12.8553, 0), (0, 2.3875)$

**79.**

$(0, 0.4167)$

**81.** Answers will vary.

## SECTION 1.3  *(page 65)*

**Skills Review**  *(page 65)*

**1.** $-1$   **2.** $1$   **3.** $\frac{1}{3}$   **4.** $-\frac{7}{6}$

**5.** $y = 4x + 7$   **6.** $y = 3x - 7$
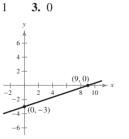
**7.** $y = 3x - 10$   **8.** $y = -x - 7$

**9.** $y = 7x - 17$   **10.** $y = \frac{2}{3}x + \frac{5}{3}$

**1.** $1$   **3.** $0$

**5.**                                     $m = \frac{1}{3}$

**7.**

$m = 3$

**9.**

$m = 0$

**11.**

$m$ is undefined.

**13.**

$m = -\frac{2}{3}$

**15.**

$m = -\frac{24}{5}$

**17.**
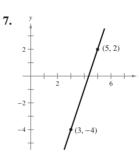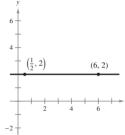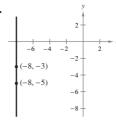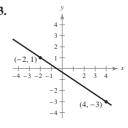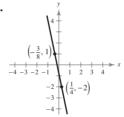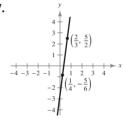
$m = 8$

**19.** $(0, 1), (1, 1), (3, 1)$   **21.** $(3, -6), (9, -2), (12, 0)$

**23.** $(0, 10), (2, 4), (3, 1)$   **25.** $(-8, 0), (-8, 2), (-8, 3)$

**27.** $m = -\frac{1}{5}, (0, 4)$   **29.** $m = -\frac{7}{6}, (0, 5)$
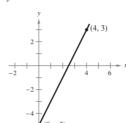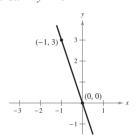
**31.** $m = 3, (0, -15)$   **33.** $m$ is undefined; no $y$-intercept.

**35.** $m = 0, (0, 4)$
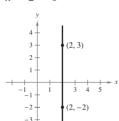
**37.** $y = 2x - 5$   **39.** $3x + y = 0$

**41.** $x - 2 = 0$



**43.** $y + 1 = 0$



**67.** (a) $x + y + 1 = 0$    (b) $x - y + 5 = 0$



**45.** $3x - 6y + 7 = 0$



**47.** $4x - y + 6 = 0$



**69.** (a) $6x + 8y - 3 = 0$    (b) $96x - 72y + 127 = 0$



**49.** $3x - 4y + 12 = 0$



**51.** $x + 1 = 0$



**71.** (a) $y = 0$    (b) $x + 1 = 0$



**53.** $y - 7 = 0$



**55.** $4x + y + 2 = 0$



**73.** (a) $x - 1 = 0$    (b) $y - 1 = 0$



**57.** $9x - 12y + 8 = 0$



**75.**



**77.**



**79.**



**81.**
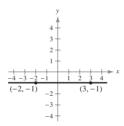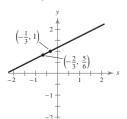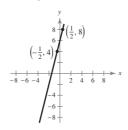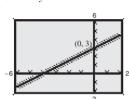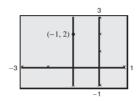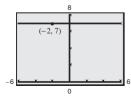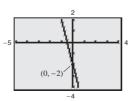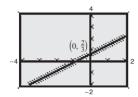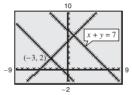


**59.** The points are not collinear.
Explanations will vary.

**61.** The points are collinear. Explanations will vary.
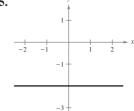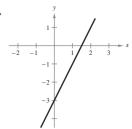
**63.** $x - 3 = 0$    **65.** $y + 10 = 0$
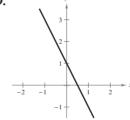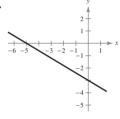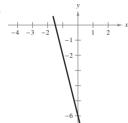
**83.**



**85.** $F = \frac{9}{5}C + 32$   or   $C = \frac{5}{9}F - \frac{160}{9}$

**87.** (a) $y = 46.2t + 4024$; The slope $m = 46.2$ tells you that the population increases by 46.2 thousand each year.

(b) 4116.4 thousand (4,116,400)

(c) 4208.8 thousand (4,208,800)

(d) Answers will vary. Sample answer:
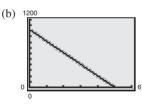
2002:  4103 thousand (4,103,000)

2004:  4198 thousand (4,198,000)

The estimates were close to the actual populations.

(e) The model could possibly be used to predict the population in 2009 if the population continues to grow at the same linear rate.

**89.** (a) $y = 1025 - 205t, 0 \le t \le 5$

(b)
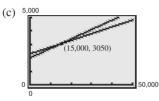


(c) \$410   (d) $t = 2.07$ yr

**91.** (a) $Y = 406\frac{1}{6}t + 4146.5$     (b) \$8614.3 billion

(c) \$11,051.3 billion     (d) Answers will vary.

**93.** (a) $C = 50x + 350,000$     (b) $R = 120x$

(c) $P = 70x - 350,000$     (d) \$560,000 profit

(e) 5000 units
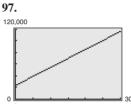
**95.** (a) $W = 0.07S + 2000$     (b) $W = 0.05S + 2300$

(c)
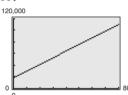


(15,000, 3050); The point of intersection tells you that your monthly wage will be \$3050 at either job when your sales are \$15,000.

(d) No, you will earn a higher monthly wage if you stay at your current job.

**97.**



$x \le 24$ units

**99.**



$x \le 70$ units

**101.**



$x \le 275$ units

**103.**
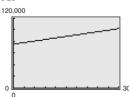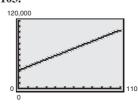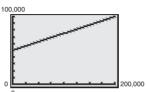


$x \le 104$ units
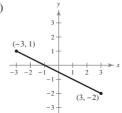
**105.**



$x \le 200,000$ units
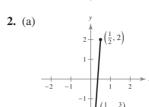
## MID-CHAPTER QUIZ   *(page 68)*

**1.** (a)
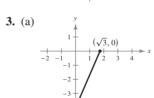


(b) $d = 3\sqrt{5}$

(c) Midpoint: $(0, -0.5)$

**2.** (a)



(b) $d = \sqrt{12.3125}$

(c) Midpoint: $\left(\frac{3}{8}, \frac{1}{4}\right)$

**3.** (a)
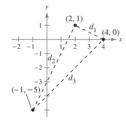


(b) $d = \sqrt{19}$

(c) Midpoint: $\left(\frac{\sqrt{3}}{2}, -2\right)$

**4.** $d_1 = \sqrt{5}$

$d_2 = \sqrt{45}$
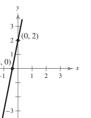
$d_3 = \sqrt{50}$

$d_1{}^2 + d_2{}^2 = d_3{}^2$



**5.** 5759.5 thousand

**6.**



**7.**



**8.**
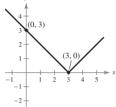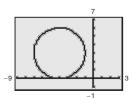


**9.** $x^2 + 2x + y^2 - 36 = 0$
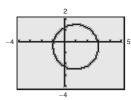
**10.** $x^2 - 4x + y^2 + 4y - 17 = 0$

**11.** $(x + 4)^2 + (y - 3)^2 = 9$



**12.** $(x - 1)^2 + (y + 0.5)^2 = 4$     **13.** 4735 units
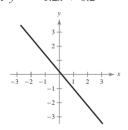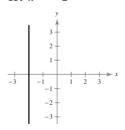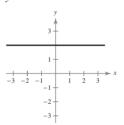


**14.** $y = -1.2x + 0.2$     **15.** $x = -2$





**16.** $y = 2$
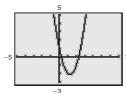


**17.** (a) $y = -0.25x - 4.25$     **18.** 2006: \$1,565,000

(b) $y = 4x - 17$         2009: \$2,270,000

**19.** $C = 0.42x + 175$

**20.** (a) $y = 1700t + 21,500$     (b) \$38,500

## SECTION 1.4   (page 78)

**Skills Review**   (page 78)
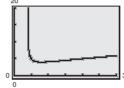
**1.** 20     **2.** 10     **3.** $x^2 + x - 6$

**4.** $x^3 + 9x^2 + 26x + 30$     **5.** $\dfrac{1}{x}$     **6.** $\dfrac{2x - 1}{x}$

**7.** $y = -2x + 17$     **8.** $y = \frac{6}{5}x^2 + \frac{1}{5}$

**9.** $y = 3 \pm \sqrt{5 + (x + 1)^2}$     **10.** $y = \pm\sqrt{4x^2 + 2}$

**11.** $y = 2x + \dfrac{1}{2}$     **12.** $y = \dfrac{x^3}{2} + \dfrac{1}{2}$

**1.** $y$ is not a function of $x$.     **3.** $y$ is a function of $x$.

**5.** $y$ is a function of $x$.     **7.** $y$ is a function of $x$.

**9.**
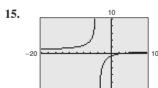


**11.**



Domain: $(-\infty, \infty)$

Range: $[-2.125, \infty)$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $y = -1$ or $y = 1$

**13.**



Domain: $(4, \infty)$

Range: $[4, \infty)$

**15.**



Domain:

$(-\infty, -4) \cup (-4, \infty)$

Range: $(-\infty, 1) \cup (1, \infty)$

**17.** Domain: $(-\infty, \infty)$      **19.** Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$          Range: $(-\infty, 4]$

**21.** (a) $-2$    (b) $3x - 5$    (c) $3x + 3\Delta x - 2$
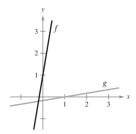
**23.** (a) $4$    (b) $\dfrac{1}{x + 4}$    (c) $-\dfrac{\Delta x}{x(x + \Delta x)}$

**25.** $\Delta x + 2x - 5,\ \Delta x \neq 0$

**27.** $\dfrac{1}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}},\ \Delta x \neq 0$

**29.** $-\dfrac{1}{(x + \Delta x - 2)(x - 2)},\ \Delta x \neq 0$

**31.** $y$ is not a function of $x$.    **33.** $y$ is a function of $x$.

**35.** (a) $2x$    (b) $10x - 25$    (c) $\dfrac{2x - 5}{5}$    (d) $5$    (e) $5$
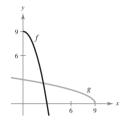
**37.** (a) $x^2 + x$    (b) $(x^2 + 1)(x - 1) = x^3 - x^2 + x - 1$

(c) $\dfrac{x^2 + 1}{x - 1}$    (d) $x^2 - 2x + 2$    (e) $x^2$

**39.** (a) $0$    (b) $0$    (c) $-1$    (d) $\sqrt{15}$
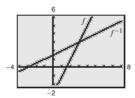
(e) $\sqrt{x^2 - 1}$    (f) $x - 1, x \geq 0$

**41.** $f(g(x)) = 5\left(\dfrac{x - 1}{5}\right) + 1 = x$

$g(f(x)) = \dfrac{5x + 1 - 1}{5} = x$
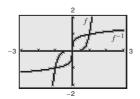


**43.** $f(g(x)) = 9 - \left(\sqrt{9 - x}\right)^2 = 9 - (9 - x) = x$

$g(f(x)) = \sqrt{9 - (9 - x^2)} = \sqrt{x^2} = x$
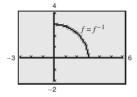


**45.** $f(x) = 2x - 3,\ f^{-1}(x) = \dfrac{x + 3}{2}$
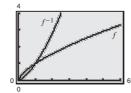


**47.** $f(x) = x^5,\ f^{-1}(x) = \sqrt[5]{x}$
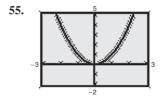


**49.** $f(x) = \sqrt{9 - x^2}, 0 \leq x \leq 3$

$f^{-1}(x) = \sqrt{9 - x^2}, 0 \leq x \leq 3$
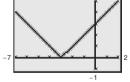


**51.** $f(x) = x^{2/3}, x \geq 0$

$f^{-1}(x) = x^{3/2}, x \geq 0$



**53.**



$f(x)$ is one-to-one. $f^{-1}(x) = \dfrac{3 - x}{7}$

**55.**



$f(x)$ is not one-to-one.
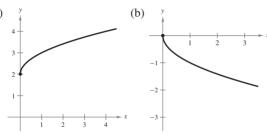
**57.**



$f(x)$ is not one-to-one.

**59.** (a)



(b)



(c)



(d)



(e)



(f)



**61.** (a) $y = (x + 3)^2$     (b) $y = -(x + 6)^2 - 3$

**63.** (a)
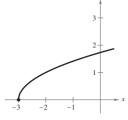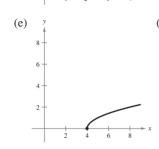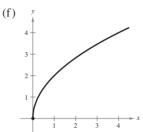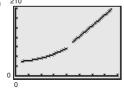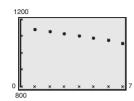


(b) 1997: \$76.22 billion

2000: \$122 billion

2004: \$188.8 billion

**65.** $R_T = R_1 + R_2 = -0.8t^2 - 7.22t + 1148,$

$t = 1, 2, \ldots, 7$



**67.** (a) $x = \dfrac{1475}{p} - 100$     (b) About 48 units

**69.** $C(x(t)) = 2800t + 375$

$C$ is the weekly cost in terms of $t$ hours of manufacturing.

**71.** (a) $p = \begin{cases} 90, & 0 \le x \le 100 \\ 91 - 0.01x, & 100 < x \le 1600 \\ 75, & x > 1600 \end{cases}$

(b) $P = \begin{cases} 30x, & 0 \le x \le 100 \\ 31x - 0.01x^2, & 100 < x \le 1600 \\ 15x, & x > 1600 \end{cases}$
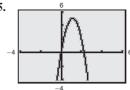
**73.** (a) $R = rn = \left[15 - 0.05(n - 80)\right]n$
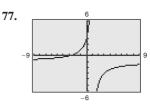
(b)

| $n$ | 100 | 125 | 150 | 175 |
|-----|-----|-----|-----|-----|
| $R$ | 1400 | 1593.75 | 1725 | 1793.75 |

| $n$ | 200 | 225 | 250 |
|-----|-----|-----|-----|
| $R$ | 1800 | 1743.75 | 1625 |

(c) Answers will vary.

**75.**



Zeros: $x = 0, \frac{9}{4}$

$f(x)$ is not one-to-one.

**77.**
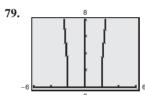


Zero: $t = -3$

$g(t)$ is one-to-one.

**79.**



Zero: $\pm 2$

$g(x)$ is not one-to-one.

**81.** Answers will vary.

## SECTION 1.5   *(page 91)*

**Skills Review**   *(page 91)*
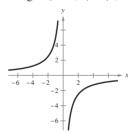
**1.** (a) 7    (b) $c^2 - 3c + 3$

(c) $x^2 + 2xh + h^2 - 3x - 3h + 3$
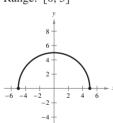
**2.** (a) $-4$    (b) 10    (c) $3t^2 + 4$     **3.** $h$     **4.** 4

**5.** Domain: $(-\infty, 0) \cup (0, \infty)$
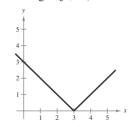
Range: $(-\infty, 0) \cup (0, \infty)$

**6.** Domain: $[-5, 5]$
Range: $[0, 5]$
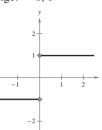


**7.** Domain: $(-\infty, \infty)$
Range: $[0, \infty)$



**8.** Domain:
$(-\infty, 0) \cup (0, \infty)$
Range: $-1, 1$



**9.** $y$ is not a function of $x$.
**10.** $y$ is a function of $x$.

**7.**

| $x$ | $-0.5$ | $-0.1$ | $-0.01$ | $-0.001$ | $0$ |
|---|---|---|---|---|---|
| $f(x)$ | $-0.0714$ | $-0.0641$ | $-0.0627$ | $-0.0625$ | ? |

$$\lim_{x \to 0^-} \frac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x} = -\frac{1}{16}$$

**9.** (a) 1    (b) 3    **11.** (a) 1    (b) 3

**13.** (a) 12    (b) 27    (c) $\frac{1}{3}$

**15.** (a) 4    (b) 48    (c) 256

**17.** (a) 1    (b) 1    (c) 1

**19.** (a) 0    (b) 0    (c) 0

**21.** (a) 3    (b) $-3$    (c) Limit does not exist.

**23.** 4    **25.** $-1$    **27.** 0    **29.** 3    **31.** $-2$

**33.** $-\frac{3}{4}$    **35.** $\frac{35}{9}$    **37.** $\frac{1}{3}$    **39.** $-\frac{1}{20}$    **41.** 2

**43.** Limit does not exist.    **45.** Limit does not exist.

**47.** 12    **49.** Limit does not exist.    **51.** 2

**53.** $-1$    **55.** 2    **57.** $\dfrac{1}{2\sqrt{x+2}}$    **59.** $2t - 5$

**1.**

| $x$ | 1.9 | 1.99 | 1.999 | 2 |
|---|---|---|---|---|
| $f(x)$ | 8.8 | 8.98 | 8.998 | ? |

| $x$ | 2.001 | 2.01 | 2.1 |
|---|---|---|---|
| $f(x)$ | 9.002 | 9.02 | 9.2 |

$$\lim_{x \to 2}(2x + 5) = 9$$

**3.**

| $x$ | 1.9 | 1.99 | 1.999 | 2 |
|---|---|---|---|---|
| $f(x)$ | 0.2564 | 0.2506 | 0.2501 | ? |

| $x$ | 2.001 | 2.01 | 2.1 |
|---|---|---|---|
| $f(x)$ | 0.2499 | 0.2494 | 0.2439 |

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \frac{1}{4}$$

**5.**

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0$ |
|---|---|---|---|---|
| $f(x)$ | 0.5132 | 0.5013 | 0.5001 | ? |

| $x$ | 0.001 | 0.01 | 0.1 |
|---|---|---|---|
| $f(x)$ | 0.4999 | 0.4988 | 0.4881 |

$$\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} = 0.5$$

**61.**



| $x$ | 0 | 0.5 | 0.9 | 0.99 |
|---|---|---|---|---|
| $f(x)$ | $-2$ | $-2.67$ | $-10.53$ | $-100.5$ |

| $x$ | 0.999 | 0.9999 | 1 |
|---|---|---|---|
| $f(x)$ | $-1000.5$ | $-10,000.5$ | Undefined |
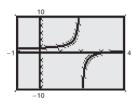
$-\infty$

**63.**
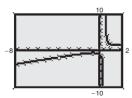


| $x$ | $-3$ | $-2.5$ | $-2.1$ | $-2.01$ |
|---|---|---|---|---|
| $f(x)$ | $-1$ | $-2$ | $-10$ | $-100$ |

| $x$ | $-2.001$ | $-2.0001$ | $-2$ |
|---|---|---|---|
| $f(x)$ | $-1000$ | $-10,000$ | Undefined |

$-\infty$

**65.**



Limit does not exist.

**67.**
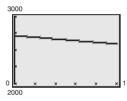


$-\frac{17}{9} \approx -1.8889$

**69.** (a) \$25,000    (b) 80%

(c) $\infty$; The cost function increases without bound as $x$ approaches 100 from the left. Therefore, according to the model, it is not possible to remove 100% of the pollutants.

**71.** (a)



(b) For $x = 0.25$, $A \approx \$2685.06$.

For $x = \frac{1}{365}$, $A \approx \$2717.91$.

(c) $\lim\limits_{x \to 0^+} 1000(1 + 0.1x)^{10/x} = 1000e \approx \$2718.28$;

continuous compounding

**73.** (a)
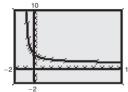
| $x$ | $-0.01$ | $-0.001$ | $-0.0001$ | $0$ |
|---|---|---|---|---|
| $f(x)$ | 2.732 | 2.720 | 2.718 | Undefined |

| $x$ | 0.0001 | 0.001 | 0.01 |
|---|---|---|---|
| $f(x)$ | 2.718 | 2.717 | 2.705 |

$\lim\limits_{x \to 0} (1 + x)^{1/x} \approx 2.718$

(b)



(c) Domain: $(-1, 0) \cup (0, \infty)$

Range: $(1, e) \cup (e, \infty)$

## SECTION 1.6    (page 102)

**1.** Continuous; The function is a polynomial.

**3.** Not continuous $(x \neq \pm 2)$

**5.** Continuous; The rational function's domain is the set of real numbers.

**7.** Not continuous $(x \neq 3 \text{ and } x \neq 5)$

**9.** Not continuous $(x \neq \pm 2)$

**11.** $(-\infty, 0)$ and $(0, \infty)$; Explanations will vary. There is a discontinuity at $x = 0$, because $f(0)$ is not defined.

**13.** $(-\infty, -1)$ and $(-1, \infty)$; Explanations will vary. There is a discontinuity at $x = -1$, because $f(-1)$ is not defined.

**15.** $(-\infty, \infty)$; Explanations will vary.

**17.** $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$; Explanations will vary. There are discontinuities at $x = \pm 1$, because $f(\pm 1)$ is not defined.

**19.** $(-\infty, \infty)$; Explanations will vary.

**21.** $(-\infty, 4)$, $(4, 5)$, and $(5, \infty)$; Explanations will vary. There are discontinuities at $x = 4$ and $x = 5$, because $f(4)$ and $f(5)$ are not defined.

**23.** Continuous on all intervals $\left(\dfrac{c}{2}, \dfrac{c}{2} + \dfrac{1}{2}\right)$, where $c$ is an integer. Explanations will vary. There are discontinuities at $x = \dfrac{c}{2}$ where $c$ is an integer, because $\lim\limits_{x \to c} f\left(\dfrac{c}{2}\right)$ does not exist.

**25.** $(-\infty, \infty)$; Explanations will vary.
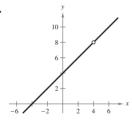
**27.** $(-\infty, 2]$ and $(2, \infty)$; Explanations will vary. There is a discontinuity at $x = 2$, because $\lim\limits_{x \to 2} f(2)$ does not exist.

**29.** $(-\infty, -1)$ and $(-1, \infty)$; Explanations will vary. There is a discontinuity at $x = -1$, because $f(-1)$ is not defined.

**31.** Continuous on all intervals $(c, c + 1)$, where $c$ is an integer. Explanations will vary. There are discontinuities at $x = c$ where $c$ is an integer, because $\lim\limits_{x \to c} f(c)$ does not exist.
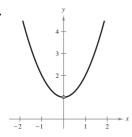
**33.** $(1, \infty)$; Explanations will vary.    **35.** Continuous

**37.** Nonremovable discontinuity at $x = 2$

**39.**
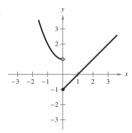


Continuous on $(-\infty, 4)$ and $(4, \infty)$

**41.**



Continuous on $(-\infty, 0)$ and $(0, \infty)$
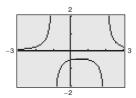
**43.**



Continuous on $(-\infty, 0)$ and $(0, \infty)$
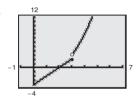
**45.** $a = 2$

**47.**
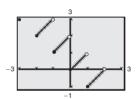


Not continuous at $x = 2$ and $x = -1$, because $f(-1)$ and $f(2)$ are not defined.

**49.**



Not continuous at $x = 3$, because $\lim\limits_{x \to 3} f(3)$ does not exist.

**51.**
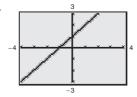


Not continuous at all integers $c$, because $\lim\limits_{x \to c} f(c)$ does not exist.
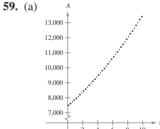
**53.** $(-\infty, \infty)$

**55.** Continuous on all intervals $\left(\dfrac{c}{2}, \dfrac{c+1}{2}\right)$, where $c$ is an integer.

**57.**
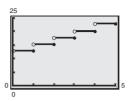


The graph of $f(x) = \dfrac{x^2 + x}{x}$ appears to be continuous on $[-4, 4]$, but $f$ is not continuous at $x = 0$.

**59.** (a)



The graph has nonremovable discontinuities at $t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \ldots$ .
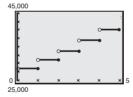
(b) \$11,379.17

**61.** $C = 12.80 - 2.50 [\![ 1 - x ]\!]$



$C$ is not continuous at $x = 1, 2, 3, \ldots$ .

**63.** (a)



(b) \$43,850.78
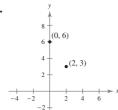
$S$ is not continuous at $t = 1, 2, \ldots, 5$.

**65.** The model is continuous. The actual revenue probably would not be continuous, because the revenue is usually recorded over larger units of time (hourly, daily, or monthly). In these cases, the revenue may jump between different units of time.

**67.** The function is continuous at $x = 100$, because the function is defined at $P(100)$, $\lim\limits_{x \to 100} P(x)$ exists, and
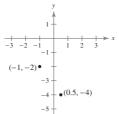
$$\lim\limits_{x \to 100} P(x) = P(100) = 3000.$$

## REVIEW EXERCISES FOR CHAPTER 1
*(page 109)*

**1.**



**3.**
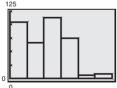


**5.** a    **6.** c    **7.** b    **8.** d    **9.** $\sqrt{29}$    **11.** $3\sqrt{2}$

**13.** $(7, 4)$    **15.** $(-8, 6)$

**17.** The tallest bars in the graph represent revenues. The middle bars represent costs. The bars on the left in each group represent profits, because $P = R - C$.

**19.** $(4, 7), (5, 8), (8, 10)$
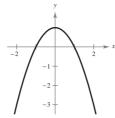
**21.**
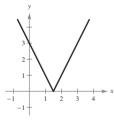


**23.**
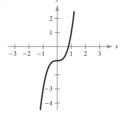


**25.**
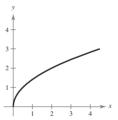


**27.**



**29.**
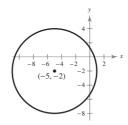


**31.**



**33.** $(0, 1), (1, 0), (-1, 0)$    **35.** $(x - 2)^2 + (y + 1)^2 = 73$

**37.** $(x + 5)^2 + (y + 2)^2 = 36$

Center: $(-5, -2)$

Radius: 6
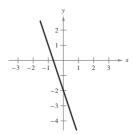


**39.** $(2, 1), (-1, -2)$

**41.** $\left(-1 + \sqrt{6}, -3 + 2\sqrt{6}\right), \ \left(-1 - \sqrt{6}, -3 - 2\sqrt{6}\right)$

**43.** (a) $C = 6000 + 6.50x$

$R = 13.90x$

(b) $\approx 811$ units

**45.** Slope: $-3$

$y$-intercept: $(0, -2)$
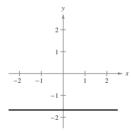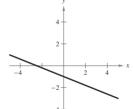


**47.** Slope: 0 (horizontal line)

$y$-intercept: $\left(0, -\frac{5}{3}\right)$

**49.** Slope: $-\frac{2}{5}$
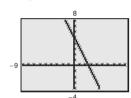
$y$-intercept: $(0, -1)$
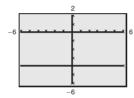




**51.** $\frac{6}{7}$    **53.** $\frac{20}{21}$

**55.** $y = -2x + 5$



**57.** $y = -4$



**59.** (a) $7x - 8y + 69 = 0$    (b) $2x + y = 0$

(c) $2x + y = 0$    (d) $2x + 3y - 12 = 0$

**61.** (a) $x = -10p + 1070$    (b) 725 units    (c) 650 units
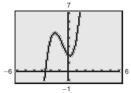
**63.** $y$ is a function of $x$.    **65.** $y$ is not a function of $x$.

**67.** (a) 7    (b) $3x + 7$    (c) $10 + 3\Delta x$

**69.**
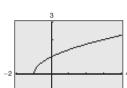


Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$
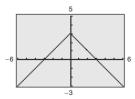
**71.**



Domain: $[-1, \infty)$

Range: $[0, \infty)$

**73.**



Domain: $(-\infty, \infty)$

Range: $(-\infty, 3]$

**75.** (a) $x^2 + 2x$    (b) $x^2 - 2x + 2$

(c) $2x^3 - x^2 + 2x - 1$    (d) $\dfrac{1 + x^2}{2x - 1}$

(e) $4x^2 - 4x + 2$    (f) $2x^2 + 1$

**77.** $f^{-1}(x) = \frac{2}{3}x$

**79.** $f(x)$ does not have an inverse function.
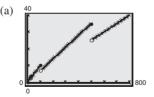
**81.** 7    **83.** 49    **85.** $\frac{10}{3}$    **87.** $-2$

**89.** $-\frac{1}{4}$    **91.** $-\infty$    **93.** Limit does not exist.

**95.** $-\frac{1}{16}$    **97.** $3x^2 - 1$    **99.** 0.5774

**101.** False, limit does not exist.
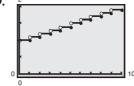
**103.** False, limit does not exist.

**105.** False, limit does not exist.

**107.** $(-\infty, -4)$ and $(-4, \infty)$; $f(-4)$ is undefined.

**109.** $(-\infty, -1)$ and $(-1, \infty)$; $f(-1)$ is undefined.

**111.** Continuous on all intervals $(c, c + 1)$, where $c$ is an integer; $\lim\limits_{x \to c} f(c)$ does not exist.

**113.** $(-\infty, 0)$ and $(0, \infty)$; $\lim\limits_{x \to 0} f(x)$ does not exist.

**115.** $a = 2$

**117.** (a)
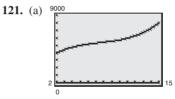


$C$ is not continuous at $x = 25, 100,$ and 500.

(b) \$10

**119.**



$C$ is not continuous at $t = 1, 2, 3, \ldots$.

**121.** (a)



(b)

| $t$ | 2 | 3 | 4 | 5 | 6 |
|-------|--------|--------|--------|--------|--------|
| $D$ | 4001.8 | 4351.0 | 4643.3 | 4920.6 | 5181.5 |
| Model | 3937.0 | 4391.3 | 4727.4 | 4971.0 | 5147.8 |

| $t$ | 7 | 8 | 9 | 10 | 11 |
|-------|--------|--------|--------|--------|--------|
| $D$ | 5369.2 | 5478.2 | 5605.5 | 5628.7 | 5769.9 |
| Model | 5283.6 | 5404.0 | 5534.7 | 5701.5 | 5930.1 |

| $t$ | 12 | 13 | 14 | 15 |
|-------|--------|--------|--------|--------|
| $D$ | 6198.4 | 6760.0 | 7354.7 | 7905.3 |
| Model | 6246.1 | 6675.3 | 7243.3 | 7976.0 |
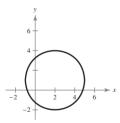
(c) \$15,007.9 billion

## CHAPTER TEST    *(page 113)*

**1.** (a) $d = 5\sqrt{2}$    (b) Midpoint: $(-1.5, 1.5)$

(c) $m = -1$

**2.** (a) $d = 2.5$    (b) Midpoint: $(1.25, 2)$    (c) $m = 0$
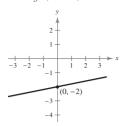
**3.** (a) $d = 3$    (b) Midpoint: $\left(2\sqrt{2}, 1.5\right)$    (c) $m = \dfrac{\sqrt{2}}{4}$

**4.**
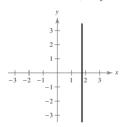


**5.** $(5.5, 53.45)$
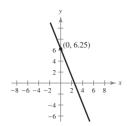
**6.** $m = \frac{1}{5}; (0, -2)$
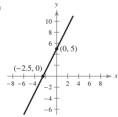


**7.** $m$ is undefined; no $y$-intercept



**8.** $m = -2.5; (0, 6.25)$



**9.** (a)
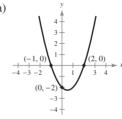


(b) Domain: $(-\infty, \infty)$

   Range: $(-\infty, \infty)$

(c) $f(-3) = -1; f(-2) = 1; f(3) = 11$

(d) The function is one-to-one.

**10.** (a)
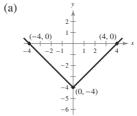


(b) Domain: $(-\infty, \infty)$

   Range: $(-2.25, \infty)$

(c) $f(-3) = 10; f(-2) = 4; f(3) = 4$

(d) The function is not one-to-one.

**11.** (a)



(b) Domain: $(-\infty, \infty)$

   Range: $(-4, \infty)$

(c) $f(-3) = -1; f(-2) = -2; f(3) = -1$

(d) The function is not one-to-one.

**12.** $f^{-1}(x) = \frac{1}{4}x - \frac{3}{2}$

$f(f^{-1}(x)) = 4\left(\frac{1}{4}x - \frac{3}{2}\right) + 6 = x - 6 + 6 = x$

$f^{-1}(f(x)) = \frac{1}{4}(4x + 6) - \frac{3}{2} = x + \frac{3}{2} - \frac{3}{2} = x$

**13.** $f^{-1}(x) = -\frac{1}{3}x^3 + \frac{8}{3}$

$f(f^{-1}(x)) = \sqrt[3]{8 - 3\left(-\frac{1}{3}x^3 + \frac{8}{3}\right)}$

$\qquad\quad = \sqrt[3]{8 + x^3 - 8}$

$\qquad\quad = \sqrt[3]{x^3} = x$

$f^{-1}(f(x)) = -\frac{1}{3}\left(\sqrt[3]{8 - 3x}\right)^3 + \frac{8}{3}$

$\qquad\quad = -\frac{1}{3}(8 - 3x) + \frac{8}{3}$

$\qquad\quad = -\frac{8}{3} + x + \frac{8}{3} = x$

**14.** $-1$    **15.** Limit does not exist.    **16.** 2    **17.** $\frac{1}{6}$

**18.** $(-\infty, 4)$ and $(4, \infty)$; Explanations will vary. There is a discontinuity at $x = 4$, because $f(4)$ is not defined.

**19.** $(-\infty, 5]$; Explanations will vary.

**20.** $(-\infty, \infty)$; Explanations will vary.

**21.** (a) The model fits the data well. Explanations will vary.

(b) 2071.14 thousand (2,071,140)

# CHAPTER 2

## SECTION 2.1   *(page 123)*

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**Skills Review**   *(page 123)*

**1.** $x = 2$     **2.** $y = 2$     **3.** $y = -x + 2$
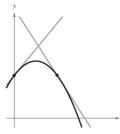
**4.** $2x$     **5.** $3x^2$     **6.** $\dfrac{1}{x^2}$     **7.** $2x$

**8.** $(-\infty, 1) \cup (1, \infty)$     **9.** $(-\infty, \infty)$
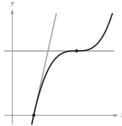
**10.** $(-\infty, 0) \cup (0, \infty)$

---

**1.**



**3.**



**5.** $m = 1$     **7.** $m = 0$     **9.** $m = -\frac{1}{3}$

**11.** 2002: $m \approx 200$

    2004: $m \approx 500$

**13.** $t = 1$: $m \approx 65$

    $t = 8$: $m \approx 0$

    $t = 12$: $m \approx -1000$

**15.** $f'(x) = -2$

    $f'(2) = -2$

**17.** $f'(x) = 0$

    $f'(0) = 0$

**19.** $f'(x) = 2x$

    $f'(2) = 4$

**21.** $f'(x) = 3x^2 - 1$

    $f'(2) = 11$

**23.** $f'(x) = \dfrac{1}{\sqrt{x}}$

    $f'(4) = \dfrac{1}{2}$

**25.** $f(x) = 3$

    $f(x + \Delta x) = 3$

    $f(x + \Delta x) - f(x) = 0$

    $\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = 0$

    $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = 0$

**27.** $f(x) = -5x$

    $f(x + \Delta x) = -5x - 5\Delta x$

    $f(x + \Delta x) - f(x) = -5\Delta x$

    $\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = -5$

    $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = -5$

**29.** $g(s) = \dfrac{1}{3}s + 2$

    $g(s + \Delta s) = \dfrac{1}{3}s + \dfrac{1}{3}\Delta s + 2$

    $g(s + \Delta s) - g(s) = \dfrac{1}{3}\Delta s$

    $\dfrac{g(s + \Delta s) - g(s)}{\Delta s} = \dfrac{1}{3}$

    $\lim\limits_{\Delta s \to 0} \dfrac{g(s + \Delta s) - g(s)}{\Delta s} = \dfrac{1}{3}$

**31.** $f(x) = x^2 - 4$

    $f(x + \Delta x) = x^2 + 2x\Delta x + (\Delta x)^2 - 4$

    $f(x + \Delta x) - f(x) = 2x\Delta x + (\Delta x)^2$

    $\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = 2x + \Delta x$

    $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = 2x$

**33.** $h(t) = \sqrt{t - 1}$

    $h(t + \Delta t) = \sqrt{t + \Delta t - 1}$

    $h(t + \Delta t) - h(t) = \sqrt{t + \Delta t - 1} - \sqrt{t - 1}$

    $\dfrac{h(t + \Delta t) - h(t)}{\Delta t} = \dfrac{1}{\sqrt{t + \Delta t - 1} + \sqrt{t - 1}}$

    $\lim\limits_{\Delta t \to 0} \dfrac{h(t + \Delta t) - h(t)}{\Delta t} = \dfrac{1}{2\sqrt{t - 1}}$

**35.** $f(t) = t^3 - 12t$

    $f(t + \Delta t) = t^3 + 3t^2\Delta t + 3t(\Delta t)^2$
    $\qquad\qquad\quad + (\Delta t)^3 - 12t - 12\Delta t$

    $f(t + \Delta t) - f(t) = 3t^2\Delta t + 3t(\Delta t)^2 + (\Delta t)^3 - 12\Delta t$

    $\dfrac{f(t + \Delta t) - f(t)}{\Delta t} = 3t^2 + 3t\Delta t + (\Delta t)^2 - 12$

    $\lim\limits_{\Delta t \to 0} \dfrac{f(t + \Delta t) - f(t)}{\Delta t} = 3t^2 - 12$

**37.** $f(x) = \dfrac{1}{x + 2}$

$f(x + \Delta x) = \dfrac{1}{x + \Delta x + 2}$
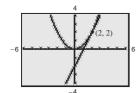
$f(x + \Delta x) - f(x) = \dfrac{-\Delta x}{(x + \Delta x + 2)(x + 2)}$
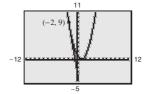
$\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{-1}{(x + \Delta x + 2)(x + 2)}$

$\displaystyle\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = -\dfrac{1}{(x + 2)^2}$

**39.** $y = 2x - 2$          **41.** $y = -6x - 3$
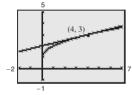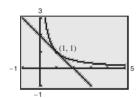




**43.** $y = \dfrac{x}{4} + 2$          **45.** $y = -x + 2$





**47.** $y = -x + 1$

**49.** $y = -6x + 8$ and $y = -6x - 8$

**51.** $x \neq -3$ (node)        **53.** $x \neq 3$ (cusp)        **55.** $x > 1$
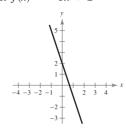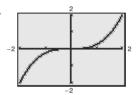
**57.** $x \neq 0$ (nonremovable discontinuity)

**59.** $x \neq 1$

**61.** $f(x) = -3x + 2$



**63.**
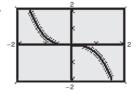


$f'(x) = \frac{3}{4}x^2$

| $x$ | $-2$ | $-\frac{3}{2}$ | $-1$ | $-\frac{1}{2}$ |
|---|---|---|---|---|
| $f(x)$ | $-2$ | $-0.8438$ | $-0.25$ | $-0.0313$ |
| $f'(x)$ | $3$ | $1.6875$ | $0.75$ | $0.1875$ |

| $x$ | $0$ | $\frac{1}{2}$ | $1$ | $\frac{3}{2}$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $0$ | $0.0313$ | $0.25$ | $0.8438$ | $2$ |
| $f'(x)$ | $0$ | $0.1875$ | $0.75$ | $1.6875$ | $3$ |

**65.**
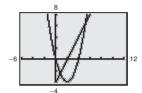


$f'(x) = -\frac{3}{2}x^2$

| $x$ | $-2$ | $-\frac{3}{2}$ | $-1$ | $-\frac{1}{2}$ |
|---|---|---|---|---|
| $f(x)$ | $4$ | $1.6875$ | $0.5$ | $0.0625$ |
| $f'(x)$ | $-6$ | $-3.375$ | $-1.5$ | $-0.375$ |

| $x$ | $0$ | $\frac{1}{2}$ | $1$ | $\frac{3}{2}$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $0$ | $-0.0625$ | $-0.5$ | $-1.6875$ | $-4$ |
| $f'(x)$ | $0$ | $-0.375$ | $-1.5$ | $-3.375$ | $-6$ |
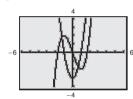
**67.** $f'(x) = 2x - 4$
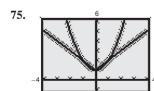


The $x$-intercept of the derivative indicates a point of horizontal tangency for $f$.

**69.** $f'(x) = 3x^2 - 3$



The $x$-intercepts of the derivative indicate points of horizontal tangency for $f$.

**71.** True     **73.** True

**75.**



The graph of $f$ is smooth at $(0, 1)$, but the graph of $g$ has a sharp point at $(0, 1)$. The function $g$ is not differentiable at $x = 0$.

## SECTION 2.2   *(page 135)*

### Skills Review   *(page 135)*

**1.** (a) 8     (b) 16     (c) $\frac{1}{2}$

**2.** (a) $\frac{1}{36}$     (b) $\frac{1}{32}$     (c) $\frac{1}{64}$

**3.** $4x(3x^2 + 1)$     **4.** $\frac{3}{2}x^{1/2}(x^{3/2} - 1)$     **5.** $\frac{1}{4x^{3/4}}$

**6.** $x^2 - \frac{1}{x^{1/2}} + \frac{1}{3x^{2/3}}$     **7.** $0, -\frac{2}{3}$

**8.** $0, \pm 1$     **9.** $-10, 2$     **10.** $-2, 12$

**1.** (a) 2     (b) $\frac{1}{2}$     **3.** (a) $-1$     (b) $-\frac{1}{3}$     **5.** 0

**7.** $4x^3$     **9.** 4     **11.** $2x + 5$     **13.** $-6t + 2$

**15.** $3t^2 - 2$     **17.** $\frac{16}{3}t^{1/3}$     **19.** $\frac{2}{\sqrt{x}}$     **21.** $-\frac{8}{x^3} + 4x$

**23.** Function: $y = \dfrac{1}{x^3}$

Rewrite: $y = x^{-3}$

Differentiate: $y' = -3x^{-4}$

Simplify: $y' = -\dfrac{3}{x^4}$

**25.** Function: $y = \dfrac{1}{(4x)^3}$

Rewrite: $y = \dfrac{1}{64}x^{-3}$

Differentiate: $y' = -\dfrac{3}{64}x^{-4}$

Simplify: $y' = -\dfrac{3}{64x^4}$

**27.** Function: $y = \dfrac{\sqrt{x}}{x}$

Rewrite: $y = x^{-1/2}$
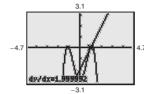
Differentiate: $y' = -\dfrac{1}{2}x^{-3/2}$
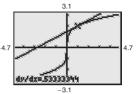
Simplify: $y' = -\dfrac{1}{2x^{3/2}}$

**29.** $-1$     **31.** $-2$     **33.** 4     **35.** $2x + \dfrac{4}{x^2} + \dfrac{6}{x^3}$

**37.** $2x - 2 + \dfrac{8}{x^5}$     **39.** $3x^2 + 1$     **41.** $6x^2 + 16x - 1$

**43.** $\dfrac{2x^3 - 6}{x^3}$     **45.** $\dfrac{4x^3 - 2x - 10}{x^3}$     **47.** $\dfrac{4}{5x^{1/5}} + 1$

**49.** (a) $y = 2x - 2$          **51.** (a) $y = \frac{8}{15}x + \frac{22}{15}$

(b) and (c)               (b) and (c)
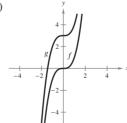
   

**53.** $(0, -1), \left(-\dfrac{\sqrt{6}}{2}, \dfrac{5}{4}\right), \left(\dfrac{\sqrt{6}}{2}, \dfrac{5}{4}\right)$     **55.** $(-5, -12.5)$

**57.** (a)                          (b) $f'(1) = g'(1) = 3$



(c)                          (d) $f' = g' = 3x^2$ for
                            every value of $x$.



**59.** (a) 3     (b) 6     (c) $-3$     (d) 6

**61.** (a) 2001: 2.03

2004: 249.01

(b) The results are similar.

(c) Millions of dollars/yr/yr

**63.** $P = 350x - 7000$

$P' = 350$

**65.**



$(0.11, 0.14), (1.84, -10.49)$

**67.** False. Let $f(x) = x$ and $g(x) = x + 1$.

## SECTION 2.3   *(page 149)*

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### Skills Review   *(page 149)*

**1.** 3    **2.** $-7$    **3.** $y' = 8x - 2$

**4.** $y' = -9t^2 + 4t$    **5.** $s' = -32t + 24$

**6.** $y' = -32x + 54$    **7.** $A' = -\frac{3}{5}r^2 + \frac{3}{5}r + \frac{1}{2}$
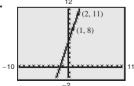
**8.** $y' = 2x^2 - 4x + 7$    **9.** $y' = 12 - \dfrac{x}{2500}$
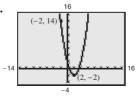
**10.** $y' = 74 - \dfrac{3x^2}{10,000}$

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**1.** (a) $10.4 billion/yr    (b) $7.4 billion/yr

(c) $6.4 billion/yr    (d) $16.6 billion/yr

(e) $10.4 billion/yr    (f) $11.4 billion/yr

**3.**



Average rate: 3

Instantaneous rates:

$f'(1) = f'(2) = 3$

**5.**
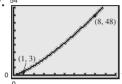


Average rate: $-4$

Instantaneous rates:

$h'(-2) = -8, h'(2) = 0$

**7.**



Average rate: $\frac{45}{7}$

Instantaneous rates:

$f'(1) = 4, f'(8) = 8$

**9.**
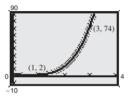


Average rate: $-\frac{1}{4}$

Instantaneous rates:

$f'(1) = -1, f'(4) = -\frac{1}{16}$

**11.**



Average rate: 36

Instantaneous rates: $g'(1) = 2, g'(3) = 102$

**13.** (a) $-500$

The number of visitors to the park is decreasing at an average rate of 500 hundred thousand people per month from September to December.

(b) Answers will vary. The instantaneous rate of change at $t = 8$ is approximately 0.

**15.** (a) Average rate: $\frac{11}{27}$

Instantaneous rates: $E'(0) = \frac{1}{3}, E'(1) = \frac{4}{9}$

(b) Average rate: $\frac{11}{27}$

Instantaneous rates: $E'(1) = \frac{4}{9}, E'(2) = \frac{1}{3}$

(c) Average rate: $\frac{5}{27}$

Instantaneous rates: $E'(2) = \frac{1}{3}, E'(3) = 0$

(d) Average rate: $-\frac{7}{27}$

Instantaneous rates: $E'(3) = 0, E'(4) = -\frac{5}{9}$
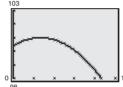
**17.** (a) $-80$ ft/sec

(b) $s'(2) = -64$ ft/sec,
$s'(3) = -96$ ft/sec

(c) $\dfrac{\sqrt{555}}{4} \approx 5.89$ sec

(d) $-8\sqrt{555} \approx -188.5$ ft/sec

**19.** 1.47 dollars    **21.** $470 - 0.5x$ dollars, $0 \le x \le 940$

**23.** $50 - x$ dollars    **25.** $-18x^2 + 16x + 200$ dollars

**27.** $-4x + 72$ dollars    **29.** $-0.0005x + 12.2$ dollars

**31.** (a) $0.58    (b) $0.60

(c) The results are nearly the same.

**33.** (a) $4.95    (b) $5.00

(c) The results are nearly the same.

**35.** (a)



(b) For $t < 4$, positive; for $t > 4$, negative; shows when fever is going up and down.
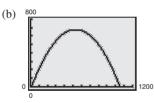
(c)  $T(0) = 100.4°F$

$T(4) = 101°F$

$T(8) = 100.4°F$

$T(12) = 98.6°F$

(d)  $T'(t) = -0.075t + 0.3$

The rate of change of temperature

(e)  $T'(0) = 0.3°F/hr$

$T'(4) = 0°F/hr$

$T'(8) = -0.3°F/hr$

$T'(12) = -0.6°F/hr$

**37.** (a)  $R = 5x - 0.001x^2$

(b)  $P = -0.001x^2 + 3.5x - 35$

(c)

| $x$ | 600 | 1200 | 1800 | 2400 | 3000 |
|---|---|---|---|---|---|
| $dR/dx$ | 3.8 | 2.6 | 1.4 | 0.2 | $-1$ |
| $dP/dx$ | 2.3 | 1.1 | $-0.1$ | $-1.3$ | $-2.5$ |
| $P$ | 1705 | 2725 | 3025 | 2605 | 1465 |
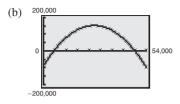
**39.** (a)  $P = -0.0025x^2 + 2.65x - 25$

(b)



When $x = 300$, slope is positive.

When $x = 700$, slope is negative.

(c)  $P'(300) = 1.15$

$P'(700) = -0.85$

**41.** (a)  $P = -\frac{1}{3000}x^2 + 17.8x - 85,000$

(b)



When $x = 18,000$, slope is positive.

When $x = 36,000$, slope is negative.

(c)  $P'(18,000) = 5.8$

$P'(36,000) = -6.2$

**43.** (a)  $0.33/\text{unit}$     (b)  $0.13/\text{unit}$

(c)  $0/\text{unit}$     (d)  $-\$0.08/\text{unit}$

$p'(2500) = 0$ indicates that $x = 2500$ is the optimal value

of $x$. So, $p = \dfrac{50}{\sqrt{x}} = \dfrac{50}{\sqrt{2500}} = \$1.00$.

**45.**  $C = \dfrac{44,250}{x}$;

| $x$ | 10 | 15 | 20 | 25 |
|---|---|---|---|---|
| $C$ | 4425.00 | 2950.00 | 2212.50 | 1770.00 |
| $dC/dx$ | $-442.5$ | $-196.67$ | $-110.63$ | $-70.80$ |

| $x$ | 30 | 35 | 40 |
|---|---|---|---|
| $C$ | 1475.00 | 1264.29 | 1106.25 |
| $dC/dx$ | $-49.17$ | $-36.12$ | $-27.66$ |

15 mi/gal; Explanations will vary.

**47.** (a)  $654.43     (b)  $1084.65     (c)  $1794.44

(d)  Answers will vary.

## SECTION 2.4   *(page 161)*

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**Skills Review**   *(page 161)*

**1.** $2(3x^2 + 7x + 1)$     **2.** $4x^2(6 - 5x^2)$

**3.** $8x^2(x^2 + 2)^3 + (x^2 + 4)$

**4.** $(2x)(2x + 1)[2x + (2x + 1)^3]$

**5.** $\dfrac{23}{(2x + 7)^2}$     **6.** $-\dfrac{x^2 + 8x + 4}{(x^2 - 4)^2}$

**7.** $-\dfrac{2(x^2 + x - 1)}{(x^2 + 1)^2}$     **8.** $\dfrac{4(3x^4 - x^3 + 1)}{(1 - x^4)^2}$

**9.** $\dfrac{4x^3 - 3x^2 + 3}{x^2}$     **10.** $\dfrac{x^2 - 2x + 4}{(x - 1)^2}$

**11.** $11$     **12.** $0$     **13.** $-\frac{1}{4}$     **14.** $\frac{17}{4}$

---

**1.** $f'(2) = 15$; Product Rule

**3.** $f'(1) = 13$; Product Rule

**5.** $f'(0) = 0$; Constant Multiple Rule

**7.** $g'(4) = 11$; Product Rule

**9.** $h'(6) = -5$; Quotient Rule

**11.** $f'(3) = \frac{3}{4}$; Quotient Rule

**13.** $g'(6) = -11$; Quotient Rule

**15.** $f'(1) = \frac{2}{5}$; Quotient Rule

**17.** Function: $y = \dfrac{x^2 + 2x}{x}$

Rewrite: $y = x + 2, \ x \neq 0$

Differentiate: $y' = 1, \ x \neq 0$

Simplify: $y' = 1, \ x \neq 0$

**19.** Function: $y = \dfrac{7}{3x^3}$

Rewrite: $y = \dfrac{7}{3}x^{-3}$

Differentiate: $y' = -7x^{-4}$

Simplify: $y' = -\dfrac{7}{x^4}$

**21.** Function: $y = \dfrac{4x^2 - 3x}{8\sqrt{x}}$

Rewrite: $y = \dfrac{1}{2}x^{3/2} - \dfrac{3}{8}x^{1/2}, \ x \neq 0$

Differentiate: $y' = \dfrac{3}{4}x^{1/2} - \dfrac{3}{16}x^{-1/2}$

Simplify: $y' = \dfrac{3}{4}\sqrt{x} - \dfrac{3}{16\sqrt{x}}$

**23.** Function: $y = \dfrac{x^2 - 4x + 3}{x - 1}$

Rewrite: $y = x - 3, x \neq 1$

Differentiate: $y' = 1, x \neq 1$

Simplify: $y' = 1, x \neq 1$

**25.** $10x^4 + 12x^3 - 3x^2 - 18x - 15$; Product Rule

**27.** $12t^2(2t^3 - 1)$; Product Rule

**29.** $\dfrac{5}{6x^{1/6}} + \dfrac{1}{x^{2/3}}$; Product Rule

**31.** $-\dfrac{5}{(2x - 3)^2}$; Quotient Rule

**33.** $\dfrac{2}{(x + 1)^2}, \ x \neq 1$; Quotient Rule
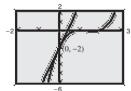
**35.** $\dfrac{x^2 + 2x - 1}{(x + 1)^2}$; Quotient Rule
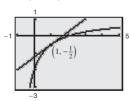
**37.** $\dfrac{3s^2 - 2s - 5}{2s^{3/2}}$; Quotient Rule

**39.** $\dfrac{2x^3 + 11x^2 - 8x - 17}{(x + 4)^2}$; Quotient Rule
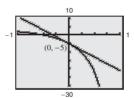
**41.** $y = 5x - 2$       **43.** $y = \frac{3}{4}x - \frac{5}{4}$
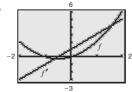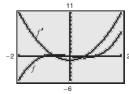


**45.** $y = -16x - 5$



**47.** $(0, 0), (2, 4)$       **49.** $(0, 0), (\sqrt[3]{-4}, -2.117)$

**51.**        **53.** 
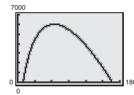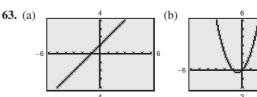
**55.** $-\$1.87/\text{unit}$

**57.** (a) $-0.480/\text{wk}$     (b) $0.120/\text{wk}$     (c) $0.015/\text{wk}$

**59.** 31.55 bacteria/hr
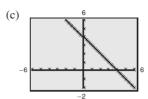
**61.** (a) $p = \dfrac{4000}{\sqrt{x}}$     (b) $C = 250x + 10{,}000$

(c) $P = 4000\sqrt{x} - 250x - 10{,}000$

     $\$500/\text{unit}$

**63.** (a)      (b) 

(c) 

The graph of (c) would most likely represent a demand function. As the number of units increases, demand is likely to decrease, not increase as in (a) and (b).

**65.** (a) $-38.125$     (b) $-10.37$     (c) $-3.80$

Increasing the order size reduces the cost per item; Choices and explanations will vary.

**67.** $\dfrac{dP}{dt} = \dfrac{17{,}091 - 1773.4t + 39.5t^2}{(1000 - 128.2t + 4.34t^2)^2}$

$P'(8) = 0.0854$

$P'(10) = 0.1431$

$P'(12) = 0.2000$

$P'(14) = 0.0017$

The rate of change in price at year $t$

**69.** $f'(2) = 0$     **71.** $f'(2) = 14$     **73.** Answers will vary.

## MID-CHAPTER QUIZ    *(page 164)*

**1.** $f(x) = -x + 2$

$f(x + \Delta x) = -x - \Delta x + 2$

$f(x + \Delta x) - f(x) = -\Delta x$

$\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = -1$

$\displaystyle\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = -1$

$f'(x) = -1$

$f'(2) = -1$

**2.** $f(x) = \sqrt{x + 3}$

$f(x + \Delta x) = \sqrt{x + \Delta x + 3}$

$f(x + \Delta x) - f(x) = \sqrt{x + \Delta x + 3} - \sqrt{x + 3}$

$\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{1}{\sqrt{x + \Delta x + 3} + \sqrt{x + 3}}$

$\displaystyle\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{1}{2\sqrt{x + 3}}$

$f'(x) = \dfrac{1}{2\sqrt{x + 3}}$

$f'(1) = \dfrac{1}{4}$

**3.** $f(x) = \dfrac{4}{x}$

$f(x + \Delta x) = \dfrac{4}{x + \Delta x}$

$f(x + \Delta x) - f(x) = -\dfrac{4\Delta x}{x(x + \Delta x)}$

$\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = -\dfrac{4}{x(x + \Delta x)}$

$\displaystyle\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = -\dfrac{4}{x^2}$

$f'(x) = -\dfrac{4}{x^2}$

$f'(1) = -4$

**4.** $f'(x) = 0$     **5.** $f'(x) = 19$     **6.** $f'(x) = -6x$

**7.** $f'(x) = \dfrac{3}{x^{3/4}}$     **8.** $f'(x) = -\dfrac{8}{x^3}$     **9.** $f'(x) = \dfrac{1}{\sqrt{x}}$
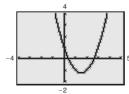
**10.** $f'(x) = -\dfrac{5}{(3x + 2)^2}$     **11.** $f'(x) = -6x^2 + 8x - 2$

**12.** $f'(x) = -\dfrac{9}{(x + 5)^2}$
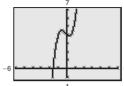
**13.**



Average rate: 0

Instantaneous rates: $f'(0) = -3, f'(3) = 3$
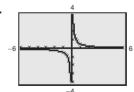
**14.**



Average rate: 1

Instantaneous rates: $f'(-1) = 3, f'(1) = 7$
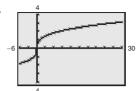
**15.**



Average rate: $-\dfrac{1}{20}$

Instantaneous rates: $f'(2) = -\dfrac{1}{8}, f'(5) = -\dfrac{1}{50}$

**16.**
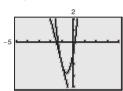


Average rate: $\dfrac{1}{19}$

Instantaneous rates: $f'(8) = \dfrac{1}{12}, f'(27) = \dfrac{1}{27}$
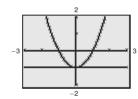
**17.** (a) \$11.61     (b) \$11.63

(c) The results are approximately equal.

**18.** $y = -4x - 6$          **19.** $y = -1$

          

**20.** (a) $\dfrac{dS}{dt} = 0.5517t^2 - 1.6484t + 3.492$

   (b) 2001: $\$2.3953/\text{yr}$

   2004: $\$5.7256/\text{yr}$

   2005: $\$9.0425/\text{yr}$

## SECTION 2.5   (page 172)

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**Skills Review**   (page 172)

**1.** $(1 - 5x)^{2/5}$     **2.** $(2x - 1)^{3/4}$

**3.** $(4x^2 + 1)^{-1/2}$     **4.** $(x - 6)^{-1/3}$

**5.** $x^{1/2}(1 - 2x)^{-1/3}$     **6.** $(2x)^{-1}(3 - 7x)^{3/2}$

**7.** $(x - 2)(3x^2 + 5)$     **8.** $(x - 1)\left(5\sqrt{x} - 1\right)$

**9.** $(x^2 + 1)^2(4 - x - x^3)$

**10.** $(3 - x^2)(x - 1)(x^2 + x + 1)$

---

| $y = f(g(x))$ | $u = g(x)$ | $y = f(u)$ |
|---|---|---|
| **1.** $y = (6x - 5)^4$ | $u = 6x - 5$ | $y = u^4$ |
| **3.** $y = (4 - x^2)^{-1}$ | $u = 4 - x^2$ | $y = u^{-1}$ |
| **5.** $y = \sqrt{5x - 2}$ | $u = 5x - 2$ | $y = \sqrt{u}$ |
| **7.** $y = (3x + 1)^{-1}$ | $u = 3x + 1$ | $y = u^{-1}$ |

**9.** $\dfrac{dy}{du} = 2u$          **11.** $\dfrac{dy}{du} = \dfrac{1}{2\sqrt{u}}$

$\dfrac{du}{dx} = 4$               $\dfrac{du}{dx} = -2x$

$\dfrac{dy}{dx} = 32x + 56$        $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{3 - x^2}}$

**13.** $\dfrac{dy}{du} = \dfrac{2}{3u^{1/3}}$

$\dfrac{du}{dx} = 20x^3 - 2$

$\dfrac{dy}{dx} = \dfrac{40x^3 - 4}{3\sqrt[3]{5x^4 - 2x}}$

**15.** c     **17.** b     **19.** a     **21.** c     **23.** $6(2x - 7)^2$

**25.** $-6(4 - 2x)^2$     **27.** $6x(6 - x^2)(2 - x^2)$

**29.** $\dfrac{4x}{3(x^2 - 9)^{1/3}}$     **31.** $\dfrac{1}{2\sqrt{t + 1}}$     **33.** $\dfrac{4t + 5}{2\sqrt{2t^2 + 5t + 2}}$
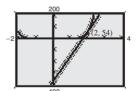
**35.** $\dfrac{6x}{(9x^2 + 4)^{2/3}}$     **37.** $\dfrac{27}{4(2 - 9x)^{3/4}}$     **39.** $\dfrac{4x^2}{(4 - x^3)^{7/3}}$
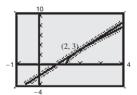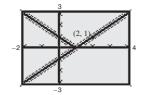
**41.** $y = 216x - 378$          **43.** $y = \frac{8}{3}x - \frac{7}{3}$
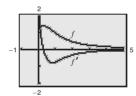
          

**45.** $y = x - 1$
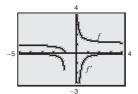


**47.** $f'(x) = \dfrac{1 - 3x^2 - 4x^{3/2}}{2\sqrt{x}(x^2 + 1)^2}$



The zero of $f'(x)$ corresponds to the point on the graph of $f(x)$ where the tangent line is horizontal.

**49.** $f'(x) = -\dfrac{\sqrt{(x + 1)/x}}{2x(x + 1)}$



$f'(x)$ has no zeros.

In Exercises 51–65, the differentiation rule(s) used may vary. A sample answer is provided.

**51.** $-\dfrac{1}{(x - 2)^2}$; Chain Rule     **53.** $\dfrac{8}{(t + 2)^3}$; Chain Rule
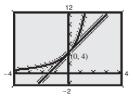
**55.** $-\dfrac{2(2x - 3)}{(x^2 - 3x)^3}$; Chain Rule     **57.** $-\dfrac{2t}{(t^2 - 2)^2}$; Chain Rule

**59.** $27(x - 3)^2(4x - 3)$; Product Rule and Chain Rule
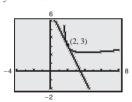
**61.** $\dfrac{3(x + 1)}{\sqrt{2x + 3}}$; Product Rule and Chain Rule

**63.** $\dfrac{t(5t - 8)}{2\sqrt{t - 2}}$; Product Rule and Chain Rule

**65.** $\dfrac{2(6 - 5x)(5x^2 - 12x + 5)}{(x^2 - 1)^3}$; Chain Rule and Quotient Rule

**67.** $y = \frac{8}{3}t + 4$



**69.** $y = -6t - 14$



**71.** $y = -2x + 7$



**73.** (a) \$74.00 per 1%

(b) \$81.59 per 1%

(c) \$89.94 per 1%

**75.**

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\dfrac{dN}{dt}$ | 0 | 177.78 | 44.44 | 10.82 | 3.29 |

The rate of growth of $N$ is decreasing.

**77.** (a) $V = \dfrac{10,000}{\sqrt[3]{t + 1}}$

(b) $-\$1322.83/\text{yr}$

(c) $-\$524.97/\text{yr}$

**79.** False. $y' = \frac{1}{2}(1 - x)^{-1/2}(-1) = -\frac{1}{2}(1 - x)^{-1/2}$

**81.** (a) 15    (b) $-10$

## SECTION 2.6   *(page 179)*

**Skills Review**   *(page 179)*

**1.** $t = 0, \frac{3}{2}$    **2.** $t = -2, 7$    **3.** $t = -2, 10$

**4.** $t = \dfrac{9 \pm 3\sqrt{10,249}}{32}$    **5.** $\dfrac{dy}{dx} = 6x^2 + 14x$

**6.** $\dfrac{dy}{dx} = 8x^3 + 18x^2 - 10x - 15$

**7.** $\dfrac{dy}{dx} = \dfrac{2x(x + 7)}{(2x + 7)^2}$    **8.** $\dfrac{dy}{dx} = -\dfrac{6x^2 + 10x + 15}{(2x^2 - 5)^2}$

**9.** Domain: $(-\infty, \infty)$    **10.** Domain: $[7, \infty)$

Range: $[-4, \infty)$    Range: $[0, \infty)$

**1.** 0    **3.** 2    **5.** $2t - 8$    **7.** $\dfrac{9}{2t^4}$

**9.** $18(2 - x^2)(5x^2 - 2)$

**11.** $12(x^3 - 2x)^2(11x^4 - 16x^2 + 4)$    **13.** $\dfrac{4}{(x - 1)^3}$

**15.** $12x^2 + 24x + 16$    **17.** $60x^2 - 72x$

**19.** $120x + 360$    **21.** $-\dfrac{9}{2x^5}$    **23.** 260    **25.** $-\dfrac{1}{648}$

**27.** $-126$    **29.** $4x$    **31.** $\dfrac{1}{x^2}$    **33.** $12x^2 + 4$

**35.** $f''(x) = 6(x - 3) = 0$ when $x = 3$.

**37.** $f''(x) = 2(3x + 4) = 0$ when $x = -\frac{4}{3}$.

**39.** $f''(x) = \dfrac{x(2x^2 - 3)}{(x^2 - 1)^{3/2}} = 0$ when $x = \pm\dfrac{\sqrt{6}}{2}$.

**41.** $f''(x) = \dfrac{2x(x + 3)(x - 3)}{(x^2 + 3)^3}$

$= 0$ when $x = 0$ or $x = \pm3$.

**43.** (a) $s(t) = -16t^2 + 144t$

$v(t) = -32t + 144$

$a(t) = -32$

(b) 4.5 sec; 324 ft

(c) $v(9) = -144$ ft/sec, which is the same speed as the initial velocity
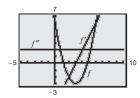
**45.**

| $t$ | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| $\dfrac{ds}{dt}$ | 0 | 45 | 60 | 67.5 | 72 | 75 | 77.1 |
| $\dfrac{d^2s}{dt^2}$ | 9 | 2.25 | 1 | 0.56 | 0.36 | 0.25 | 0.18 |

As time increases, velocity increases and acceleration decreases.

**47.** $f(x) = x^2 - 6x + 6$
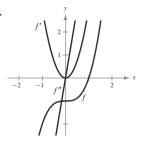
$f'(x) = 2x - 6$

$f''(x) = 2$
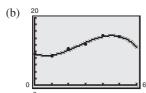


The degrees of the successive derivatives decrease by 1.

**49.**



We know that the degrees of the successive derivatives decrease by 1.

**51.** (a) $y(t) = -0.2093t^3 + 1.637t^2 - 1.95t + 9.4$

(b)



The model fits the data well.

(c) $y'(t) = -0.6279t^2 + 3.274t - 1.95$

$y''(t) = -1.2558t + 3.274$

(d) $y'(t) > 0$ on $[1, 4]$

(e) 2002 $(t = 2.607)$

(f) The first derivative is used to show that the retail value of motor homes is increasing in (d), and the retail value increased at the greatest rate at the zero of the second derivative as shown in (e).

**53.** False. The product rule is

$[f(x)g(x)]' = f(x)g'(x) + g(x)f'(x).$

**55.** True    **57.** $[xf(x)]^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x)$

## SECTION 2.7   *(page 186)*

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### Skills Review   *(page 186)*

**1.** $y = x^2 - 2x$    **2.** $y = \dfrac{x - 3}{4}$

**3.** $y = 1, x \neq -6$    **4.** $y = -4, x \neq \pm\sqrt{3}$

**5.** $y = \pm\sqrt{5 - x^2}$    **6.** $y = \pm\sqrt{6 - x^2}$    **7.** $\frac{8}{3}$

**8.** $-\frac{1}{2}$    **9.** $\frac{5}{7}$    **10.** 1

---

**1.** $-\dfrac{y}{x}$    **3.** $-\dfrac{x}{y}$    **5.** $\dfrac{1 - xy^2}{x^2y}$    **7.** $\dfrac{y}{8y - x}$

**9.** $-\dfrac{1}{10y - 2}$    **11.** $\dfrac{1}{2}$    **13.** $-\dfrac{x}{y}, 0$

**15.** $-\dfrac{y}{x + 1}, -\dfrac{1}{4}$    **17.** $\dfrac{y - 3x^2}{2y - x}, \dfrac{1}{2}$    **19.** $\dfrac{1 - 3x^2y^3}{3x^3y^2 - 1}, -1$

**21.** $-\sqrt{\dfrac{y}{x}}, -\dfrac{5}{4}$    **23.** $-\sqrt[3]{\dfrac{y}{x}}, -\dfrac{1}{2}$    **25.** 3
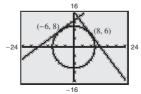
**27.** 0    **29.** $-\dfrac{\sqrt{5}}{3}$    **31.** $-\dfrac{x}{y}, \dfrac{4}{3}$    **33.** $\dfrac{1}{2y}, -\dfrac{1}{2}$
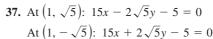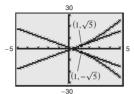
**35.** At $(8, 6)$: $y = -\frac{4}{3}x + \frac{50}{3}$

At $(-6, 8)$: $y = \frac{3}{4}x + \frac{25}{2}$



**37.** At $\left(1, \sqrt{5}\right)$: $15x - 2\sqrt{5}y - 5 = 0$

At $\left(1, -\sqrt{5}\right)$: $15x + 2\sqrt{5}y - 5 = 0$
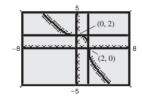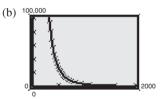


**39.** At $(0, 2)$: $y = 2$

At $(2, 0)$: $x = 2$

**41.** $-\dfrac{2}{p^2(0.00003x^2 + 0.1)}$    **43.** $-\dfrac{4xp}{2p^2 + 1}$

**45.** (a) $-2$

(b)
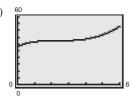

As more labor is used, less capital is available.

As more capital is used, less labor is available.

**47.** (a)


The numbers of cases of HIV/AIDS increases from 2001 to 2005.

(b) 2005

(c)

| $t$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 37.90 | 38.91 | 39.05 | 40.23 | 44.08 |
| $y'$ | 2.130 | 0.251 | 0.347 | 2.288 | 5.565 |

2005

## SECTION 2.8    (page 194)

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### Skills Review    (page 194)

**1.** $A = \pi r^2$    **2.** $V = \frac{4}{3}\pi r^3$    **3.** $S = 6s^2$

**4.** $V = s^3$    **5.** $V = \frac{1}{3}\pi r^2 h$    **6.** $A = \frac{1}{2}bh$

**7.** $-\dfrac{x}{y}$    **8.** $\dfrac{2x - 3y}{3x}$    **9.** $-\dfrac{2x + y}{x + 2}$

**10.** $-\dfrac{y^2 - y + 1}{2xy - 2y - x}$

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**1.** (a) $\frac{3}{4}$    (b) 20    **3.** (a) $-\frac{5}{8}$    (b) $\frac{3}{2}$

**5.** (a) $36\pi$ in.$^2$/min

(b) $144\pi$ in.$^2$/min

**7.** If $\dfrac{dr}{dt}$ is constant, $\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$ and so is proportional to $r$.

**9.** (a) $\dfrac{5}{2\pi}$ ft/min    (b) $\dfrac{5}{8\pi}$ ft/min

**11.** (a) 112.5 dollars/wk

(b) 7500 dollars/wk

(c) 7387.5 dollars/wk

**13.** (a) 9 cm$^3$/sec

(b) 900 cm$^3$/sec

**15.** (a) $-12$ cm/min

(b) 0 cm/min

(c) 4 cm/min

(d) 12 cm/min

**17.** (a) $-\frac{7}{12}$ ft/sec    (b) $-\frac{3}{2}$ ft/sec

(c) $-\frac{48}{7}$ ft/sec

**19.** (a) $-750$ mi/hr    (b) 20 min

**21.** $-8.33$ ft/sec    **23.** About 37.7 ft$^3$/min

**25.** 4 units/wk

## REVIEW EXERCISES FOR CHAPTER 2
*(page 200)*

**1.** $-2$    **3.** 0

**5.** Answers will vary. Sample answer:

$t = 10$: slope $\approx$ \$7025 million/yr; Sales were increasing by about \$7025 million/yr in 2000.

$t = 13$: slope $\approx$ \$6750 million/yr; Sales were increasing by about \$6750 million/yr in 2003.

$t = 15$: slope $\approx$ \$10,250 million/yr; Sales were increasing by about \$10,250 million/yr in 2005.

**7.** $t = 0$: slope $\approx 180$
$t = 4$: slope $\approx -70$
$t = 6$: slope $\approx -900$
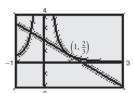
**9.** $-3; -3$    **11.** $2x - 4; -2$
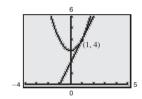
**13.** $\dfrac{1}{2\sqrt{x + 9}}; \dfrac{1}{4}$    **15.** $-\dfrac{1}{(x - 5)^2}; -1$

**17.** $-3$    **19.** 0    **21.** $\frac{1}{6}$    **23.** $-5$    **25.** 1    **27.** 0

**29.** $y = -\frac{4}{3}t + 2$    **31.** $y = 2x + 2$
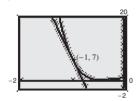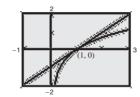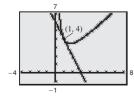
**33.** $y = -34x - 27$       **35.** $y = x - 1$





**37.** $y = -2x + 6$



**39.** Average rate of change: 4

Instantaneous rate of change when $x = 0$: 3

Instantaneous rate of change when $x = 1$: 5

**41.** (a) About \$7219 million/yr/yr
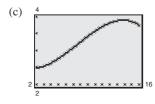
(b) 1999: about \$8618 million/yr/yr

2005: about \$10,279 million/yr/yr

(c) Sales were increasing in 1999 and 2005, and grew at a rate of about \$7219 million over the period 1999–2005.

**43.** (a) $P'(t) = -0.00447t^2 + 0.068t - 0.086$

(b) 1997: \$0.17/half gallon

2003: \$0.04/half gallon

2005: $-$\$0.07/half gallon

(c)



The price is increasing from 1992 to 2004, and decreasing from 2004 to 2006.

(d) Positive slope: $2 < t < 14$

Negative slope: $14 < t < 16$

(e) When the price increases, the slope is positive.

When the price decreases, the slope is negative.

**45.** (a) $s(t) = -16t^2 + 276$    (b) $-32$ ft/sec

(c) $t = 2$: $-64$ ft/sec

$t = 3$: $-96$ ft/sec

(d) About 4.15 sec    (e) About $-132.8$ ft/sec

**47.** $R = 27.50x$

$C = 15x + 2500$

$P = 12.50x - 2500$

**49.** $\dfrac{dC}{dx} = 320$    **51.** $\dfrac{dC}{dx} = \dfrac{1.275}{\sqrt{x}}$

**53.** $\dfrac{dR}{dx} = 200 - \dfrac{2}{5}x$    **55.** $\dfrac{dR}{dx} = \dfrac{35(x-4)}{2(x-2)^{3/2}}$

**57.** $\dfrac{dP}{dx} = -0.0006x^2 + 12x - 1$

In Exercises 59–77, the differentiation rule(s) used may vary. A sample answer is provided.

**59.** $15x^2(1 - x^2)$; Power Rule

**61.** $16x^3 - 33x^2 + 12x$; Product Rule

**63.** $\dfrac{2(3 + 5x - 3x^2)}{(x^2 + 1)^2}$; Quotient Rule

**65.** $30x(5x^2 + 2)^2$; Chain Rule

**67.** $-\dfrac{1}{(x+1)^{3/2}}$; Quotient Rule

**69.** $\dfrac{2x^2 + 1}{\sqrt{x^2 + 1}}$; Product Rule

**71.** $80x^4 - 24x^2 + 1$; Product Rule
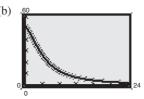
**73.** $18x^5(x + 1)(2x + 3)^2$; Chain Rule

**75.** $x(x - 1)^4(7x - 2)$; Product Rule

**77.** $\dfrac{3(9t + 5)}{2\sqrt{3t + 1}(1 - 3t)^3}$; Quotient Rule

**79.** (a) $t = 1$: $-6.63$        $t = 3$: $-6.5$

$t = 5$: $-4.33$        $t = 10$: $-1.36$

(b)



The rate of decrease is approaching zero.

**81.** 6    **83.** $-\dfrac{120}{x^6}$    **85.** $\dfrac{35x^{3/2}}{2}$    **87.** $\dfrac{2}{x^{2/3}}$

**89.** (a) $s(t) = -16t^2 + 5t + 30$    (b) About 1.534 sec

(c) About $-44.09$ ft/sec    (d) $-32$ ft/sec$^2$

**91.** $-\dfrac{2x + 3y}{3(x + y^2)}$    **93.** $\dfrac{2x - 8}{2y - 9}$    **95.** $y = \dfrac{1}{3}x + \dfrac{1}{3}$

**97.** $y = \frac{4}{3}x + \frac{2}{3}$    **99.** $\frac{1}{64}$ ft/min

## CHAPTER TEST     *(page 204)*

**1.** $f(x) = x^2 + 1$

$f(x + \Delta x) = x^2 + 2x\Delta x + \Delta x^2 + 1$

$f(x + \Delta x) - f(x) = 2x\Delta x + \Delta x^2$

$\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = 2x + \Delta x$

$\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = 2x$

$f'(x) = 2x$

$f'(2) = 4$

**2.** $f(x) = \sqrt{x} - 2$

$f(x + \Delta x) = \sqrt{x + \Delta x} - 2$

$f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x}$

$\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$

$\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{1}{2\sqrt{x}}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f'(4) = \dfrac{1}{4}$

**3.** $f'(t) = 3t^2 + 2$     **4.** $f'(x) = 8x - 8$

**5.** $f'(x) = \dfrac{3\sqrt{x}}{2}$     **6.** $f'(x) = 2x$     **7.** $f'(x) = \dfrac{9}{x^4}$
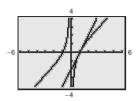
**8.** $f'(x) = \dfrac{5 + x}{2\sqrt{x}} + \sqrt{x}$     **9.** $f'(x) = 36x^3 + 48x$

**10.** $f'(x) = -\dfrac{1}{\sqrt{1 - 2x}}$

**11.** $f'(x) = \dfrac{(10x + 1)(5x - 1)^2}{x^2} = 250x - 75 + \dfrac{1}{x^2}$

**12.** $y = 2x - 2$



**13.** (a) \$169.80 million/yr

   (b) 2001: \$68.84 million/yr

      2005: \$223.30 million/yr

   (c) The annual sales of Bausch & Lomb from 2001 to 2005 increased on average by about \$169.80 million/yr, and the instantaneous rates of change for 2001 and 2005 are \$68.84 million/yr and \$223.30 million/yr, respectively.

**14.** $P = -0.016x^2 + 1460x - 715,000$     **15.** 0

**16.** $-\dfrac{3}{8(3 - x)^{5/2}}$     **17.** $-\dfrac{96}{(2x - 1)^4}$     **18.** $\dfrac{dy}{dx} = -\dfrac{1 + y}{x}$

**19.** $\dfrac{dy}{dx} = -\dfrac{1}{y - 1}$     **20.** $\dfrac{dy}{dx} = \dfrac{x}{2y}$

**21.** (a) $3.75\pi \text{ cm}^3/\text{min}$     (b) $15\pi \text{ cm}^3/\text{min}$

# *CHAPTER 3*

## SECTION 3.1     *(page 213)*

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### Skills Review     *(page 213)*

**1.** $x = 0, x = 8$     **2.** $x = 0, x = 24$     **3.** $x = \pm 5$

**4.** $x = 0$     **5.** $(-\infty, 3) \cup (3, \infty)$     **6.** $(-\infty, 1)$

**7.** $(-\infty, -2) \cup (-2, 5) \cup (5, \infty)$     **8.** $\left(-\sqrt{3}, \sqrt{3}\right)$

**9.** $x = -2: -6$     **10.** $x = -2: 60$

   $x = 0: 2$          $x = 0: -4$

   $x = 2: -6$          $x = 2: 60$

**11.** $x = -2: -\frac{1}{3}$     **12.** $x = -2: \frac{1}{18}$

   $x = 0: 1$          $x = 0: -\frac{1}{8}$

   $x = 2: 5$          $x = 2: -\frac{3}{2}$

---

**1.** $f'(-1) = -\frac{8}{25}$     **3.** $f'(-3) = -\frac{2}{3}$

   $f'(0) = 0$          $f'(-2)$ is undefined.

   $f'(1) = \frac{8}{25}$          $f'(-1) = \frac{2}{3}$

**5.** Increasing on $(-\infty, -1)$

   Decreasing on $(-1, \infty)$

**7.** Increasing on $(-1, 0)$ and $(1, \infty)$

   Decreasing on $(-\infty, -1)$ and $(0, 1)$
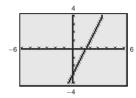
**9.** No critical numbers     **11.** Critical number: $x = 1$
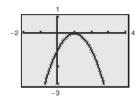
   Increasing on $(-\infty, \infty)$          Increasing on $(-\infty, 1)$

                              Decreasing on $(1, \infty)$
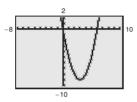
**13.** Critical number: $x = 3$

Decreasing on $(-\infty, 3)$

Increasing on $(3, \infty)$
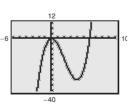
**15.** Critical numbers: $x = 0, x = 4$

Increasing on $(-\infty, 0)$
and $(4, \infty)$

Decreasing on $(0, 4)$
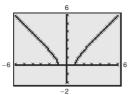
**17.** Critical numbers:
$x = -1, x = 1$
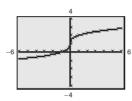
Decreasing on $(-\infty, -1)$

Increasing on $(1, \infty)$
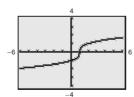
**19.** No critical numbers

Increasing on $(-\infty, \infty)$

**21.** No critical numbers
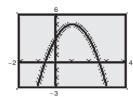
Increasing on $(-\infty, \infty)$

**23.** Critical number: $x = 1$

Increasing on $(-\infty, 1)$

Decreasing on $(1, \infty)$
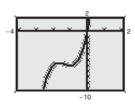
**25.** Critical numbers:
$x = -1, x = -\frac{5}{3}$

Increasing on $\left(-\infty, -\frac{5}{3}\right)$
and $(-1, \infty)$

Decreasing on $\left(-\frac{5}{3}, -1\right)$
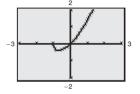
**27.** Critical numbers:
$x = -1, x = -\frac{2}{3}$

Decreasing on $\left(-1, -\frac{2}{3}\right)$

Increasing on $\left(-\frac{2}{3}, \infty\right)$
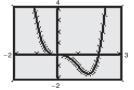
**29.** Critical numbers:
$x = 0, x = \frac{3}{2}$

Decreasing on $\left(-\infty, \frac{3}{2}\right)$

Increasing on $\left(\frac{3}{2}, \infty\right)$
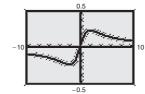
**31.** Critical numbers:
$x = 2, x = -2$

Decreasing on $(-\infty, -2)$
and $(2, \infty)$

Increasing on $(-2, 2)$
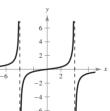
**33.** No critical numbers

Discontinuities: $x = \pm 4$

Increasing on $(-\infty, -4)$,
$(-4, 4)$, and $(4, \infty)$
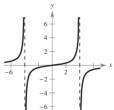
**35.** Critical number: $x = 0$

Discontinuity: $x = 0$

Increasing on $(-\infty, 0)$

Decreasing on $(0, \infty)$
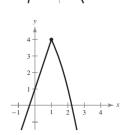
**37.** Critical number: $x = 1$

No discontinuity, but the
function is not differentiable
at $x = 1$.
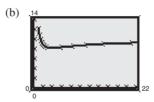
Increasing on $(-\infty, 1)$

Decreasing on $(1, \infty)$
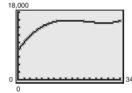
**39.** (a) Decreasing on $[1, 4.10)$

Increasing on $(4.10, \infty)$

(b)

(c) $C = 9$ (or \$900) when $x = 2$ and $x = 15$. Use an order
size of $x = 4$, which will minimize the cost $C$.

**41.** (a)



Increasing from 1970 to late 1986 and from late 1998 to 2004

Decreasing from late 1986 to late 1998

(b) $y' = 2.439t^2 - 111.4t + 1185.2$

Critical numbers: $t = 16.9, t = 28.8$

Therefore, the model is increasing from 1970 to late 1986 and from late 1998 to 2004 and decreasing from late 1986 to late 1998.

**43.** (a) $P = -\dfrac{1}{20,000}x^2 + 2.65x - 7500$

(b) Increasing on $[0, 26{,}500)$

Decreasing on $(26{,}500, 50{,}000]$

(c) The maximum profit occurs when the restaurant sells 26,500 hamburgers, the $x$-coordinate of the point at which the graph changes from increasing to decreasing.

## SECTION 3.2 *(page 223)*

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### Skills Review *(page 223)*

**1.** $0, \pm\frac{1}{2}$     **2.** $-2, 5$     **3.** 1     **4.** $0, 125$

**5.** $-4 \pm \sqrt{17}$     **6.** $1 \pm \sqrt{5}$

**7.** Negative     **8.** Positive     **9.** Positive

**10.** Negative     **11.** Increasing     **12.** Decreasing

---

**1.** Relative maximum: $(1, 5)$

**3.** Relative minimum: $(3, -9)$

**5.** Relative maximum: $\left(\frac{2}{3}, \frac{28}{9}\right)$

Relative minimum: $(1, 3)$
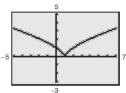
**7.** No relative extrema

**9.** Relative maximum: $(0, 15)$

Relative minimum: $(4, -17)$

**11.** Relative minima: $(-0.366, 0.75), (1.37, 0.75)$
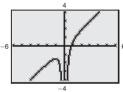
Relative maximum: $\left(\frac{1}{2}, \frac{21}{16}\right)$

**13.**



Relative minimum: $(1, 0)$
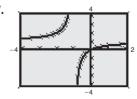
**15.**



Relative maximum: $\left(-1, -\frac{3}{2}\right)$

**17.**



No relative extrema

**19.** Minimum: $(2, 2)$

Maximum: $(-1, 8)$

**21.** Maximum: $(0, 5)$

Minimum: $(3, -13)$

**23.** Minima: $(-1, -4), (2, -4)$

Maxima: $(0, 0), (3, 0)$

**25.** Maximum: $(2, 1)$

Minimum: $\left(0, \frac{1}{3}\right)$

**27.** Maximum: $(-1, 5)$

Minimum: $(0, 0)$

**29.** Maximum: $(-7, 4)$

Minimum: $(1, 0)$

**31.** 2, absolute maximum

**33.** Maximum: $(5, 7)$

Minimum: $(2.69, -5.55)$
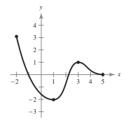
**35.** Maximum: $(2, 2.\overline{6})$

Minima: $(0, 0), (3, 0)$

**37.** Minimum: $(0, 0)$

Maximum: $(1, 2)$

**39.** Maximum: $\left(2, \frac{1}{2}\right)$

Minimum: $(0, 0)$

**41.** Maximum $\left| f''\left(\sqrt[3]{-10 + \sqrt{108}}\right) \right| \approx 1.47$

**43.** Maximum: $\left| f^{(4)}(0) \right| = \dfrac{56}{81}$

**45.** Answers will vary. Example:



**47.** 82 units     **49.** $2.15

**51.** (a) Population tends to increase each year, so the minimum population occurred in 1790 and the maximum population occurred in 2000.

(b) Maximum population: 278.968 million

Minimum population: 3.775 million

(c) The minimum population was about 3.775 million in 1790 and the maximum population was about 278.968 million in 2000.
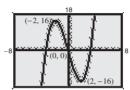
## SECTION 3.3  *(page 232)*

### Skills Review  *(PAGE 232)*

1. $f''(x) = 48x^2 - 54x$

2. $g''(s) = 12s^2 - 18s + 2$

3. $g''(x) = 56x^6 + 120x^4 + 72x^2 + 8$

4. $f''(x) = \dfrac{4}{9(x - 3)^{2/3}}$    5. $h''(x) = \dfrac{190}{(5x - 1)^3}$

6. $f''(x) = -\dfrac{42}{(3x + 2)^3}$    7. $x = \pm\dfrac{\sqrt{3}}{3}$
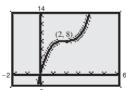
8. $x = 0, 3$    9. $t = \pm 4$    10. $x = 0, \pm 5$

1. Concave upward on $(-\infty, \infty)$

3. Concave upward on $\left(-\infty, -\frac{1}{2}\right)$
   Concave downward on $\left(-\frac{1}{2}, \infty\right)$

5. Concave upward on $(-\infty, -2)$ and $(2, \infty)$
   Concave downward on $(-2, 2)$

7. Concave upward on $(-\infty, 2)$
   Concave downward on $(2, \infty)$

9. Relative maximum: $(3, 9)$

11. Relative maximum: $(1, 3)$
    Relative minimum: $\left(\frac{7}{3}, \frac{49}{27}\right)$

13. Relative minimum: $(0, -3)$

15. Relative minimum: $(0, 1)$

17. Relative minima: $(-3, 0), (3, 0)$
    Relative maximum: $(0, 3)$

19. Relative maximum: $(0, 4)$

21. No relative extrema

23. Relative maximum: $(0, 0)$
    Relative minima: $(-0.5, -0.052), (1, -0.\overline{3})$

25. Relative maximum: $(2, 9)$
    Relative minimum: $(0, 5)$

27. Sign of $f'(x)$ on $(0, 2)$ is positive.
    Sign of $f''(x)$ on $(0, 2)$ is positive.

29. Sign of $f'(x)$ on $(0, 2)$ is negative.
    Sign of $f''(x)$ on $(0, 2)$ is negative.

31. $(3, 0)$    33. $(1, 0), (3, -16)$

35. No inflection points    37. $\left(\frac{3}{2}, -\frac{1}{16}\right), (2, 0)$

39.
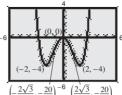
Relative maximum: $(-2, 16)$
Relative minimum: $(2, -16)$
Point of inflection: $(0, 0)$

41.

No relative extrema
Point of inflection: $(2, 8)$

43.
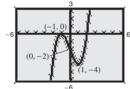
Relative maximum: $(0, 0)$
Relative minima: $(\pm 2, -4)$
Points of inflection:
$$\left(\pm\frac{2\sqrt{3}}{3}, -\frac{20}{9}\right)$$
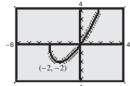
45.

Relative maximum: $(-1, 0)$
Relative minimum: $(1, -4)$
Point of inflection: $(0, -2)$
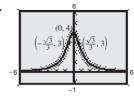
47.

Relative minimum: $(-2, -2)$
No inflection points

49.
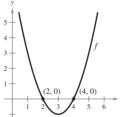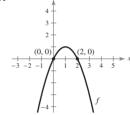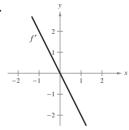
Relative maximum: $(0, 4)$
Points of inflection:
$$\left(\pm\frac{\sqrt{3}}{3}, 3\right)$$

51.


53.

**55.**
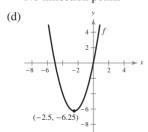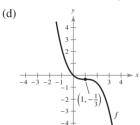


(a) $f'$: Positive on $(-\infty, 0)$

$f$: Increasing on $(-\infty, 0)$

(b) $f'$: Negative on $(0, \infty)$

$f$: Decreasing on $(0, \infty)$

(c) $f'$: Not increasing

$f$: Not concave upward

(d) $f'$: Decreasing on $(-\infty, \infty)$

$f$: Concave downward on $(-\infty, \infty)$

**57.** (a) $f'$: Increasing on $(-\infty, \infty)$

(b) $f$: Concave upward on $(-\infty, \infty)$

(c) Relative minimum: $(-2.5, -6.25)$

No inflection points

(d)
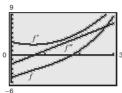


**59.** (a) $f'$: Increasing on $(-\infty, 1)$

Decreasing on $(1, \infty)$

(b) $f$: Concave upward on $(-\infty, 1)$

Concave downward on $(1, \infty)$

(c) No relative extrema

Point of inflection: $\left(1, -\frac{1}{3}\right)$

(d)



**61.** $(200, 320)$     **63.** $100$ units     **65.** 8:30 P.M.

**67.** $\sqrt{3} \approx 1.732/\text{yr}$

**69.**



Relative minimum: $(0, -5)$
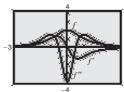
Relative maximum: $(3, 8.5)$

Point of inflection:

$\left(\frac{2}{3}, -3.2963\right)$

When $f'$ is positive, $f$ is increasing. When $f'$ is negative, $f$ is decreasing. When $f''$ is positive, $f$ is concave upward. When $f''$ is negative, $f$ is concave downward.

**71.**



Relative maximum: $(0, 2)$

Points of inflection:

$(0.58, 1.5), (-0.58, 1.5)$

When $f'$ is positive, $f$ is increasing. When $f'$ is negative, $f$ is decreasing. When $f''$ is positive, $f$ is concave upward. When $f''$ is negative, $f$ is concave downward.

**73.** 120 units

**75.** (a)



(b) November   (c) October   (d) October; April

**77.** (a) $S'$ is increasing and $S'' > 0$.

(b) $S'$ is increasing and positive and $S'' > 0$.

(c) $S'$ is constant and $S'' = 0$.

(d) $S' = 0$ and $S'' = 0$.

(e) $S' < 0$ and $S'' > 0$.

(f) $S' > 0$ and there are no restrictions on $S''$.

**79.** Answers will vary.

## SECTION 3.4   *(page 241)*

### Skills Review    *(page 241)*

**1.** $x + \frac{1}{2}y = 12$     **2.** $2xy = 24$     **3.** $xy = 24$

**4.** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 10$

**5.** $x = -3$     **6.** $x = -\frac{2}{3}, 1$     **7.** $x = \pm 5$

**8.** $x = 4$     **9.** $x = \pm 1$     **10.** $x = \pm 3$

**1.** 60, 60    **3.** 18, 9    **5.** $\sqrt{192}, \sqrt{192}$

**7.** $l = w = 25$ m    **9.** $l = w = 8$ ft

**11.** $x = 25$ ft, $y = \frac{100}{3}$ ft

**13.** (a) Proof

(b) $V_1 = 99$ in.$^3$

$V_2 = 125$ in.$^3$

$V_3 = 117$ in.$^3$

(c) 5 in. $\times$ 5 in. $\times$ 5 in.

**15.** $l = w = 2\sqrt[3]{5} \approx 3.42$

$h = 4\sqrt[3]{5} \approx 6.84$

**17.** $x = 5$ m, $y = 3\frac{1}{3}$ m    **19.** 1.056 ft$^3$

**21.** 9 in. by 9 in.

**23.** Length: 3 units

Width: 1.5 units

**25.** Length: $5\sqrt{2}$ units

Width: $\dfrac{5\sqrt{2}}{2}$ units

**27.** Radius: about 1.51 in.

Height: about 3.02 in.

**29.** $(1, 1)$    **31.** $\left(3.5, \dfrac{\sqrt{14}}{2}\right)$

**33.** 18 in. $\times$ 18 in. $\times$ 36 in.

**35.** Radius: $\sqrt[3]{\dfrac{562.5}{\pi}} \approx 5.636$ ft

Height: about 22.545 ft

**37.** Side of square: $\dfrac{10\sqrt{3}}{9 + 4\sqrt{3}}$

Side of triangle: $\dfrac{30}{9 + 4\sqrt{3}}$

**39.** Width of rectangle: $\dfrac{100}{\pi} \approx 31.8$ m
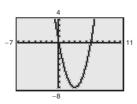
Length of rectangle: 50 m

**41.** $w = 8\sqrt{3}$ in., $h = 8\sqrt{6}$ in.

**43.** (a) $40,000 ($s = 40$)

(b) $20,000 ($s = 20$)

## MID-CHAPTER QUIZ *(page 244)*

**1.** Critical number: $x = 3$

Increasing on $(3, \infty)$

Decreasing on $(-\infty, 3)$

**2.** Critical numbers:

$x = -4, x = 0$

Increasing on
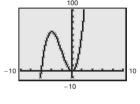
$(-\infty, 4)$ and $(0, \infty)$

Decreasing on $(-4, 0)$

**3.** Critical number: $x = 0$
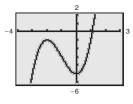
Increasing on $(-\infty, 0)$

Decreasing on $(0, \infty)$

**4.** Relative minimum: $(0, -5)$

Relative maximum:
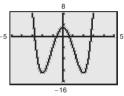
$(-2, -1)$
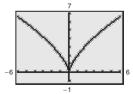
**5.** Relative minima:

$(2, -13), (-2, -13)$

Relative maximum: $(0, 3)$

**6.** Relative minimum: $(0, 0)$

**7.** Minimum: $(-1, -9)$    **8.** Minimum: $(3, -54)$

Maximum: $(1, -5)$        Maximum: $(-3, 54)$

**9.** Minimum: $(0, 0)$

Maximum: $(1, 0.5)$

**10.** Point of inflection: $(2, -2)$

Concave downward on $(-\infty, 2)$

Concave upward on $(2, \infty)$

**11.** Points of inflection: $(-2, -80)$ and $(2, -80)$

Concave downward on $(-2, 2)$

Concave upward on $(-\infty, -2)$ and $(2, \infty)$

**12.** Relative minimum: $(1, 9)$

Relative maximum: $(-2, 36)$

**13.** Relative minimum: $(1, 2)$

Relative maximum: $(-1, -2)$

**14.** $120,000 ($x = 120$)

**15.** 50 ft by 100 ft

**16.** (a) Late 1999; 2005

(b) Increasing from 1999 to late 1999.
Decreasing from late 1999 to 2005.

## SECTION 3.5  *(page 252)*

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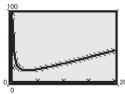
**Skills Review**    *(page 252)*

**1.** 1    **2.** $\frac{6}{5}$    **3.** 2    **4.** $\frac{1}{2}$
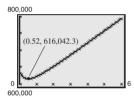
**5.** $\dfrac{dC}{dx} = 1.2 + 0.006x$    **6.** $\dfrac{dP}{dx} = 0.02x + 11$

**7.** $\dfrac{dR}{dx} = 14 - \dfrac{x}{1000}$    **8.** $\dfrac{dR}{dx} = 3.4 - \dfrac{x}{750}$

**9.** $\dfrac{dP}{dx} = -1.4x + 7$    **10.** $\dfrac{dC}{dx} = 4.2 + 0.003x^2$

---

**1.** 2000 units    **3.** 200 units    **5.** 200 units

**7.** 50 units    **9.** $60    **11.** $67.50

**13.** 3 units

$\overline{C}(3) = 17;\ \dfrac{dC}{dx} = 4x + 5;$ when $x = 3,\ \dfrac{dC}{dx} = 17$
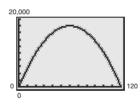


**15.** (a) $70    (b) About $40.63

**17.** The maximum profit occurs when $s = 10$ (or $10,000).

The point of diminishing returns occurs at $s = \frac{35}{6}$
(or $5833.33).

**19.** 200 players    **21.** $50

**23.** $C = $ cost under water $+$ cost on land

$= 25(5280)\sqrt{x^2 + 0.25} + 18(5280)(6 - x)$

$= 132,000\sqrt{x^2 + 0.25} + 570,240 - 95,040x$



The line should run from the power station to a point across
the river approximately 0.52 mile downstream.

$\left(\text{Exact: } \dfrac{9\sqrt{301}}{301}\ \text{mi}\right)$
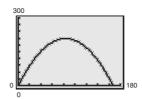
**25.** 60 mi/h

**27.** $-3$, elastic



Elastic: $(0, 60)$

Inelastic: $(60, 120)$
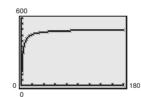
**29.** $-\frac{2}{3}$, inelastic



Elastic: $\left(0, 83\frac{1}{3}\right)$
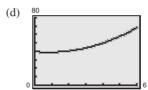
Inelastic: $\left(83\frac{1}{3}, 166\frac{2}{3}\right)$

**31.** $-\frac{25}{23}$, elastic



Elastic: $(0, \infty)$

**33.** (a) $-\frac{11}{14}$    (b) $x = 500$ units, $p = $10

(c) Answers will vary.

**35.** 500 units ($x = 5$)

**37.** No; when $p = 5,\ x = 350$ and $\eta = -\frac{5}{7}$.

Because $|\eta| = \frac{5}{7} < 1$, demand is inelastic.

**39.** (a) 2006

(b) 2001

(c) 2006: $11.25 billion/yr
2001: $0.32 billion/yr

(d)

**41.** Demand function: *a*

Revenue function: *c*

Cost function: *b*

Profit function: *d*

**43.** Answers will vary.    **45.** Answers will vary.

## SECTION 3.6    *(page 263)*

### Skills Review    *(page 263)*

**1.** 3    **2.** 1    **3.** $-11$    **4.** 4    **5.** $-\frac{1}{4}$

**6.** $-2$    **7.** 0    **8.** 1

**9.** $\overline{C} = \dfrac{150}{x} + 3$    **10.** $\overline{C} = \dfrac{1900}{x} + 1.7 + 0.002x$

$\dfrac{dC}{dx} = 3$    $\dfrac{dC}{dx} = 1.7 + 0.004x$

**11.** $\overline{C} = 0.005x + 0.5 + \dfrac{1375}{x}$    **12.** $\overline{C} = \dfrac{760}{x} + 0.05$

$\dfrac{dC}{dx} = 0.01x + 0.5$    $\dfrac{dC}{dx} = 0.05$

**1.** Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 1$

**3.** Vertical asymptotes: $x = -1, x = 2$

Horizontal asymptote: $y = 1$

**5.** Vertical asymptote: none

Horizontal asymptote: $y = \frac{3}{2}$

**7.** Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = \frac{1}{2}$

**9.** d    **10.** b    **11.** a    **12.** c

**13.** $\infty$    **15.** $-\infty$    **17.** $-\infty$    **19.** $-\infty$

**21.**

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ |
|---|---|---|---|---|
| $f(x)$ | 2.000 | 0.348 | 0.101 | 0.032 |

| $x$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|
| $f(x)$ | 0.010 | 0.003 | 0.001 |

$\lim\limits_{x \to \infty} \dfrac{x+1}{x\sqrt{x}} = 0$

**23.**

| $x$ | $10^0$ | $10^1$ | $10^2$ | $10^3$ |
|---|---|---|---|---|
| $f(x)$ | 0 | 49.5 | 49.995 | 49.99995 |

| $x$ | $10^4$ | $10^5$ | $10^6$ |
|---|---|---|---|
| $f(x)$ | 50.0 | 50.0 | 50.0 |

$\lim\limits_{x \to \infty} \dfrac{x^2 - 1}{0.02x^2} = 50$

**25.**
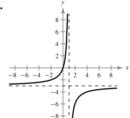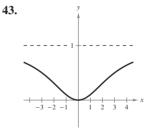
| $x$ | $-10^6$ | $-10^4$ | $-10^2$ | $10^0$ |
|---|---|---|---|---|
| $f(x)$ | $-2$ | $-2$ | $-1.9996$ | 0.8944 |

| $x$ | $10^2$ | $10^4$ | $10^6$ |
|---|---|---|---|
| $f(x)$ | 1.9996 | 2 | 2 |

$\lim\limits_{x \to -\infty} \dfrac{2x}{\sqrt{x^2 + 4}} = -2,\; \lim\limits_{x \to \infty} \dfrac{2x}{\sqrt{x^2 + 4}} = 2$

**27.** (a) $\infty$    (b) 5    (c) 0    **29.** (a) 0    (b) 1    (c) $\infty$

**31.** 2    **33.** 0    **35.** $-\infty$    **37.** $\infty$    **39.** 5

**41.**


**43.**


**45.**


**47.**


**49.**


**51.**

**53.**
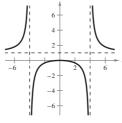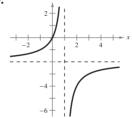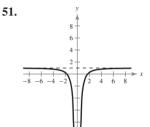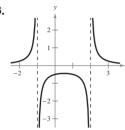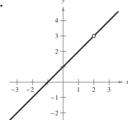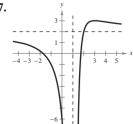


**55.**



**57.**



**59.** (a) $\overline{C} = 1.35 + \dfrac{4570}{x}$   (b) $47.05, $5.92   (c) $1.35

**61.** (a) $\overline{C} = 13.5 + \dfrac{45{,}750}{x}$
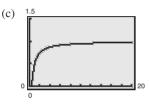
(b) $\overline{C}(100) = 471$;  $\overline{C}(1000) = 59.25$

(c) $13.50; The cost approaches $13.50 as the number of PDAs produced increases.

**63.** (a) 25%: $176 million; 50%: $528 million;

75%: $1584 million

(b) $\infty$; The limit does not exist, which means the cost increases without bound as the government approaches 100% seizure of illegal drugs entering the country.

**65.** (a)

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $P$ | 0.5 | 0.74 | 0.82 | 0.86 | 0.89 |

| $n$ | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| $P$ | 0.91 | 0.92 | 0.93 | 0.94 | 0.95 |

(b) 1

(c)



The percent of correct responses approaches 100% as the number of times the task is performed increases.

**67.** (a) $\overline{P} = 35.4 - \dfrac{15{,}000}{x}$
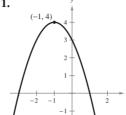
(b) $\overline{P}(1000) = $20.40$; $\overline{P}(10{,}000) = $33.90$;

$\overline{P}(100{,}000) = $35.25$

(c) $35.40; Explanations will vary.

## SECTION 3.7   *(page 273)*
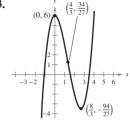
### Skills Review   *(page 273)*
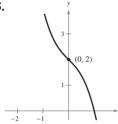
**1.** Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$

**2.** Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 0$

**3.** Vertical asymptote: $x = -3$
Horizontal asymptote: $y = 40$

**4.** Vertical asymptotes: $x = 1, x = 3$
Horizontal asymptote: $y = 1$

**5.** Decreasing on $(-\infty, -2)$
Increasing on $(-2, \infty)$

**6.** Increasing on $(-\infty, -4)$
Decreasing on $(-4, \infty)$

**7.** Increasing on $(-\infty, -1)$ and $(1, \infty)$
Decreasing on $(-1, 1)$

**8.** Decreasing on $(-\infty, 0)$ and $\left(\sqrt[3]{2}, \infty\right)$
Increasing on $\left(0, \sqrt[3]{2}\right)$

**9.** Increasing on $(-\infty, 1)$ and $(1, \infty)$

**10.** Decreasing on $(-\infty, -3)$ and $\left(\frac{1}{3}, \infty\right)$
Increasing on $\left(-3, \frac{1}{3}\right)$

**1.**



**3.**

**5.**



**7.**



**9.**



**11.**



**13.**



**15.**



**17.**
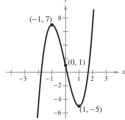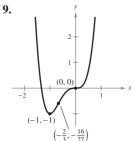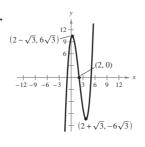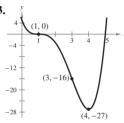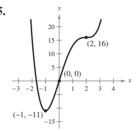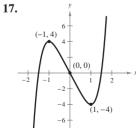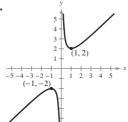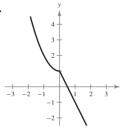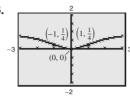


**19.**
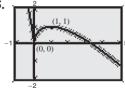


**21.**
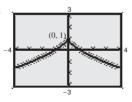


**23.**
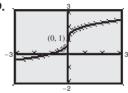


**25.**
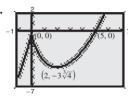


**27.**
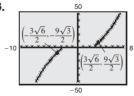


**29.**
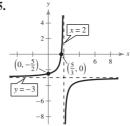


**31.**



**33.**
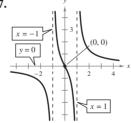


**35.**
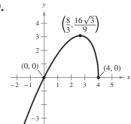


Domain: $(-\infty, 2) \cup (2, \infty)$

**37.**



Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

**39.**



Domain: $(-\infty, 4]$

**41.**
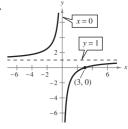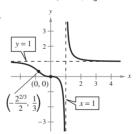


Domain: $(-\infty, 0) \cup (0, \infty)$

**43.**



Domain: $(-\infty, 1) \cup (1, \infty)$

**45.** Answers will vary.
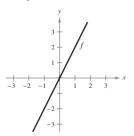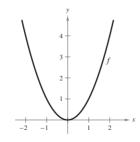
Sample answer: $f(x) = -x^3 + x^2 + x + 1$

**47.** Answers will vary.          **49.** Answers will vary.
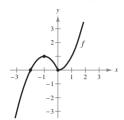
Sample answer:              Sample answer:
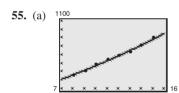


**51.** Answers will vary. Sample answer:
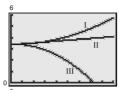


**53.** Answers will vary. Sample answer: $y = \dfrac{1}{x - 5}$

**55.** (a)



The model fits the data well.
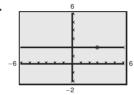
(b) \$1099.31

(c) No, because the benefits increase without bound as time approaches the year 2040 ($x = 50$), and the benefits are negative for the years past 2040.

**57.** (a)



(b) Models I and II

(c) Model I; Model III; Model I; Explanations will vary.

**59.**



The rational function has the common factor $3 - x$ in the numerator and denominator. At $x = 3$, there is a hole in the graph, not a vertical asymptote.

## SECTION 3.8   *(page 281)*

> **Skills Review**   *(page 281)*
>
> **1.** $\dfrac{dC}{dx} = 0.18x$     **2.** $\dfrac{dC}{dx} = 0.15$
>
> **3.** $\dfrac{dR}{dx} = 1.25 + 0.03\sqrt{x}$     **4.** $\dfrac{dR}{dx} = 15.5 - 3.1x$
>
> **5.** $\dfrac{dP}{dx} = -\dfrac{0.01}{\sqrt[3]{x^2}} + 1.4$     **6.** $\dfrac{dP}{dx} = -0.04x + 25$
>
> **7.** $\dfrac{dA}{dx} = \dfrac{\sqrt{3}}{2}x$     **8.** $\dfrac{dA}{dx} = 12x$     **9.** $\dfrac{dC}{dr} = 2\pi$
>
> **10.** $\dfrac{dP}{dw} = 4$     **11.** $\dfrac{dS}{dr} = 8\pi r$     **12.** $\dfrac{dP}{dx} = 2 + \sqrt{2}$
>
> **13.** $A = \pi r^2$     **14.** $A = x^2$
>
> **15.** $V = x^3$     **16.** $V = \frac{4}{3}\pi r^3$

**1.** $dy = 6x\,dx$     **3.** $dy = 12(4x - 1)^2\,dx$

**5.** $dy = \dfrac{-x}{\sqrt{9 - x^2}}\,dx$     **7.** $0.1005$     **9.** $-0.013245$

**11.** $dy = 0.6$          **13.** $dy = -0.04$

$\Delta y = 0.6305$          $\Delta y \approx -0.0394$

**15.**

| $dx = \triangle x$ | $dy$ | $\triangle y$ | $\triangle y - dy$ | $\dfrac{dy}{\triangle y}$ |
|---|---|---|---|---|
| 1.000 | 4.000 | 5.000 | 1.0000 | 0.8000 |
| 0.500 | 2.000 | 2.2500 | 0.2500 | 0.8889 |
| 0.100 | 0.400 | 0.4100 | 0.0100 | 0.9756 |
| 0.010 | 0.040 | 0.0401 | 0.0001 | 0.9975 |
| 0.001 | 0.004 | 0.0040 | 0.0000 | 1.0000 |

**17.**

| $dx = \triangle x$ | $dy$ | $\triangle y$ | $\triangle y - dy$ | $\dfrac{dy}{\triangle y}$ |
|---|---|---|---|---|
| 1.000 | $-0.25000$ | $-0.13889$ | 0.11111 | 1.79999 |
| 0.500 | $-0.12500$ | $-0.09000$ | 0.03500 | 1.38889 |
| 0.100 | $-0.02500$ | $-0.02324$ | 0.00176 | 1.07573 |
| 0.010 | $-0.00250$ | $-0.00248$ | 0.00002 | 1.00806 |
| 0.001 | $-0.00025$ | $-0.00025$ | 0.00000 | 1.00000 |

**19.**

| $dx = \triangle x$ | $dy$ | $\triangle y$ | $\triangle y - dy$ | $\dfrac{dy}{\triangle y}$ |
|---|---|---|---|---|
| 1.000 | 0.14865 | 0.12687 | $-0.02178$ | 1.17167 |
| 0.500 | 0.07433 | 0.06823 | $-0.00610$ | 1.08940 |
| 0.100 | 0.01487 | 0.01459 | $-0.00028$ | 1.01919 |
| 0.010 | 0.00149 | 0.00148 | $-0.00001$ | 1.00676 |
| 0.001 | 0.00015 | 0.00015 | 0.00000 | 1.00000 |

**21.** $y = 28x + 37$

For $\Delta x = -0.01, f(x + \Delta x) = -19.281302$ and

$y(x + \Delta x) = -19.28$

For $\Delta x = 0.01, f(x + \Delta x) = -18.721298$ and

$y(x + \Delta x) = -18.72$

**23.** $y = x$

For $\Delta x = -0.01, f(x + \Delta x) = -0.009999$ and

$y(x + \Delta x) = -0.01$
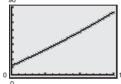
For $\Delta x = 0.01, f(x + \Delta x) = 0.009999$ and

$y(x + \Delta x) = 0.01$
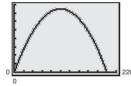
**25.** $dP = 1160$

Percent change: about 2.7%
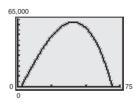
**27.** (a) $\Delta p = -0.25 = dp$     (b) $\Delta p = -0.25 = dp$

**29.** $5.20                    **31.** $7.50



**33.** $-$1250
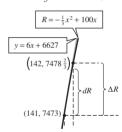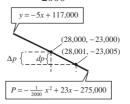


**35.** $R = -\frac{1}{3}x^2 + 100x$; $6
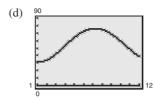


**37.** $P = -\dfrac{1}{2000}x^2 + 23x - 275,000$; $-$5



**39.** $\pm\dfrac{3}{8}$ in.$^2$, $\pm 0.0026$     **41.** $734.8 billion

**43.** $\dfrac{\sqrt{2}}{24} \approx 0.059$ m$^2$     **45.** True

## REVIEW EXERCISES FOR CHAPTER 3
*(page 287)*

**1.** $x = 1$     **3.** $x = 0, x = 1$

**5.** Increasing on $\left(-\frac{1}{2}, \infty\right)$

Decreasing on $\left(-\infty, -\frac{1}{2}\right)$

**7.** Increasing on $(-\infty, 3)$ and $(3, \infty)$

**9.** (a) $(1.38, 7.24)$     (b) $(1, 1.38), (7.24, 12)$

(c) Normal monthly temperature is rising from early January to early July and decreasing from early July to early January.

(d)



**11.** Relative maximum: $(0, -2)$

Relative minimum: $(1, -4)$

**13.** Relative minimum: $(8, -52)$

**15.** Relative maxima: $(-1, 1), (1, 1)$

Relative minimum: $(0, 0)$

**17.** Relative maximum: $(0, 6)$

**19.** Relative maximum: $(0, 0)$

Relative minimum: $(4, 8)$

**21.** Maximum: $(0, 6)$     **23.** Maxima: $(-2, 17), (4, 17)$

Minimum: $\left(-\frac{5}{2}, -\frac{1}{4}\right)$     Minima: $(-4, -15), (2, -15)$

**25.** Maximum: $(1, 3)$
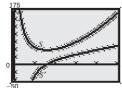
Minimum: $\left(3, 4\sqrt{3} - 9\right)$

**27.** Maximum: $\left(2, \dfrac{2\sqrt{5}}{5}\right)$     **29.** Maximum: $(1, 1)$

Minimum: $(0, 0)$     Minimum: $(-1, -1)$

**31.**



$r \approx 1.58$ in.

**33.** Concave upward on $(2, \infty)$

Concave downward on $(-\infty, 2)$

**35.** Concave upward on $\left(-\dfrac{2\sqrt{3}}{3}, \dfrac{2\sqrt{3}}{3}\right)$

Concave downward on $\left(-\infty, -\dfrac{2\sqrt{3}}{3}\right)$ and $\left(\dfrac{2\sqrt{3}}{3}, \infty\right)$
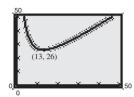
**37.** $(0, 0), (4, -128)$

**39.** $(0, 0), (1.0652, 4.5244), (2.5348, 3.5246)$

**41.** Relative maximum: $\left(-\sqrt{3}, 6\sqrt{3}\right)$

Relative minimum: $\left(\sqrt{3}, -6\sqrt{3}\right)$

**43.** Relative maxima: $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{1}{2}\right), \left(\dfrac{\sqrt{2}}{2}, \dfrac{1}{2}\right)$

Relative minimum: $(0, 0)$

**45.** $\left(50, 166\frac{2}{3}\right)$

**47.** 13, 13
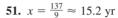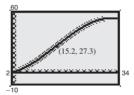


**49.** (a) Absolute maximum: $(4.30, 1765.98)$

Absolute minimum: $(34.19, 1472.33)$

(b) 1989

(c) The maximum number of daily newspapers in circulation was 1765.98 million in 1974 and the minimum number was 1472.33 million in 2004.

Circulation was changing at the greatest rate in 1989.

**51.** $x = \frac{137}{9} \approx 15.2$ yr



**53.** $s'(r) = -2cr$

$-2cr = 0 \implies r = 0$

$s''(r) = -2c < 0$ for all $r$

Therefore, $r = 0$ yields a maximum value of $s$.

**55.** $N = 85$ (maximizes revenue)     **57.** 125 units

**59.** Elastic: $(0, 75)$

Inelastic: $(75, 150)$

Demand is of unit elasticity when $x = 75$.
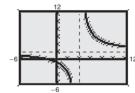
**61.** Elastic: $(0, 150)$

Inelastic: $(150, 300)$

Demand is of unit elasticity when $x = 150$.
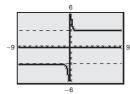
**63.** Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$
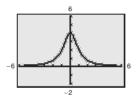


**65.** Vertical asymptote: $x = 0$

Horizontal asymptotes: $y = \pm 3$
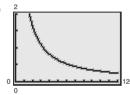


**67.** Horizontal asymptote: $y = 0$



**69.** $-\infty$     **71.** $\infty$     **73.** $\frac{2}{3}$     **75.** 0

**77.** (a)



(b) $\lim\limits_{s\to\infty} T = -0.03$

(c)



(d) Increasing: 1994 to 2005

(e) Increasing: 1994 to 1995 and 1997 to 2005

    Decreasing: 1995 to 1997

(f) Answers will vary.

**79.**



Intercepts: $(0, 0), (4, 0)$
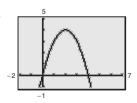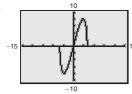
Relative maximum: $(2, 4)$

Domain: $(-\infty, \infty)$

**97.** $dS = \pm 1.8\pi$ in.$^2$

$dV = \pm 8.1\pi$ in.$^3$

**81.**



Intercepts: $(0, 0), (4, 0),$
$(-4, 0)$

Relative maximum: $(2\sqrt{2}, 8)$
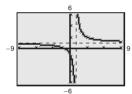
Relative minimum:
$(-2\sqrt{2}, -8)$

Point of inflection: $(0, 0)$

Domain: $[-4, 4]$

**99.**

| Quantity of output | Price | Total revenue | Marginal revenue |
|---|---|---|---|
| 1 | 14.00 | 14.00 | 10.00 |
| 2 | 12.00 | 24.00 | 6.00 |
| 3 | 10.00 | 30.00 | 4.00 |
| 4 | 8.50 | 34.00 | 1.00 |
| 5 | 7.00 | 35.00 | -2.00 |
| 6 | 5.50 | 33.00 | |

**83.**
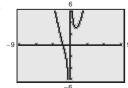


Intercepts: $(-1, 0), (0, -1)$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Domain: $(-\infty, 1) \cup (1, \infty)$

(a) $R = -1.43x^2 + 13.8x + 1.8$

(b) $\dfrac{dR}{dx} = -2.86x + 13.8;$

   $10.94, 8.08, 5.22, 2.36, -0.50, -3.36;$

   The model is a fairly good estimate.

(c) About 5 units of output: $(4.83, 35.09)$

**85.**



Intercept: $\left(-\sqrt[3]{2}, 0\right)$

Relative minimum: $(1, 3)$

Point of inflection: $\left(-\sqrt[3]{2}, 0\right)$

Vertical asymptote: $x = 0$

Domain: $(-\infty, 0) \cup (0, \infty)$

## CHAPTER TEST *(page 291)*

**1.** Critical number: $x = 0$

   Increasing on $(0, \infty)$

   Decreasing on $(-\infty, 0)$

**2.** Critical numbers: $x = -2, x = 2$

   Increasing on $(-\infty, -2)$ and $(2, \infty)$

   Decreasing on $(-2, 2)$

**3.** Critical number: $x = 5$
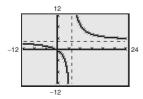
   Increasing on $(5, \infty)$

   Decreasing on $(-\infty, 5)$
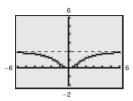
**4.** Relative minimum: $(3, -14)$

   Relative maximum: $(-3, 22)$

**87.** $dy = (1 - 2x)dx$    **89.** $dy = -\dfrac{x}{\sqrt{36 - x^2}}\,dx$

**91.** \$800    **93.** \$15.25

**95.** (a)



(b) Appears to be a positive correlation
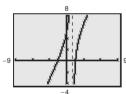
**5.** Relative minima: $(-1, -7)$ and $(1, -7)$

Relative maximum: $(0, -5)$

**6.** Relative maximum: $(0, 2.5)$

**7.** Minimum: $(-3, -1)$     **8.** Minimum: $(0, 0)$

Maximum: $(0, 8)$       Maximum: $(2.25, 9)$

**9.** Minimum: $\left(2\sqrt{3}, 2\sqrt{3}\right)$

Maximum: $(1, 6.5)$

**10.** Concave upward: $\left(\dfrac{\sqrt[3]{50}}{5}, \infty\right)$

Concave downward: $\left(-\infty, \dfrac{\sqrt[3]{50}}{5}\right)$

**11.** Concave upward: $\left(-\infty, -\dfrac{2\sqrt{2}}{3}\right)$ and $\left(\dfrac{2\sqrt{2}}{3}, \infty\right)$

Concave downward: $\left(-\dfrac{2\sqrt{2}}{3}, \dfrac{2\sqrt{2}}{3}\right)$

**12.** No point of inflection

**13.** $\left(\sqrt[3]{2}, -\dfrac{18\sqrt[3]{4}}{5}\right)$

**14.** Relative minimum: $(5.46, -135.14)$

Relative maximum: $(-1.46, 31.14)$

**15.** Relative minimum: $(3, -97.2)$

Relative maximum: $(-3, 97.2)$

**16.** Vertical asymptote: $x = 5$

Horizontal asymptote: $y = 3$



**17.** Horizontal asymptote: $y = 2$



**18.** Vertical asymptote: $x = 1$



**19.** 1    **20.** $\infty$    **21.** 3    **22.** $dy = 10x\, dx$
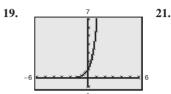
**23.** $dy = \dfrac{-4}{(x + 3)^2}\, dx$    **24.** $dy = 3(x + 4)^2\, dx$

**25.** $(312.5, 625)$
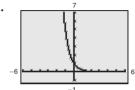
# CHAPTER 4
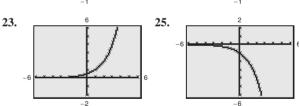
## SECTION 4.1   *(page 297)*

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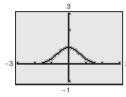
### Skills Review   *(page 297)*

**1.** Horizontal shift to the left two units

**2.** Reflection about the $x$-axis

**3.** Vertical shift down one unit

**4.** Reflection about the $y$-axis

**5.** Horizontal shift to the right one unit

**6.** Vertical shift up two units

**7.** Nonremovable discontinuity at $x = -4$

**8.** Continuous on $(-\infty, \infty)$

**9.** Discontinuous at $x = \pm 1$

**10.** Continuous on $(-\infty, \infty)$

**11.** 5    **12.** $\frac{4}{3}$    **13.** $-9, 1$    **14.** $2 \pm 2\sqrt{2}$

**15.** $1, -5$    **16.** $\frac{1}{2}, 1$

---

**1.** (a) 625    (b) 9    (c) $16\sqrt{2}$

(d) 9    (e) 125    (f) 4

**3.** (a) 3125    (b) $\frac{1}{5}$    (c) 625    (d) $\frac{1}{125}$

**5.** (a) $\frac{1}{5}$    (b) 27    (c) 5    (d) 4096

**7.** (a) 4    (b) $\dfrac{\sqrt{2}}{2} \approx 0.707$    (c) $\dfrac{1}{8}$    (d) $\dfrac{\sqrt{2}}{8} \approx 0.177$

**9.** (a) 0.907    (b) 348.912    (c) 1.796    (d) 1.308

**11.** 2 g    **13.** e    **14.** c    **15.** a

**16.** f    **17.** d    **18.** b

**19.**



**21.**



**23.**



**25.**

**27.**
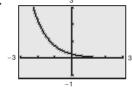

**29.**
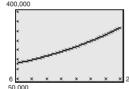

**31.** (a) $P(18) \approx 306.99$ million

(b) $P(22) \approx 320.72$ million

**33.** (a) $V(5) \approx \$80,634.95$    (b) $V(20) \approx \$161,269.89$

**35.** $36.93

**37.** (a)

| Year | 1998 | 1999 | 2000 | 2001 |
|---|---|---|---|---|
| Actual | 152,500 | 161,000 | 169,000 | 175,200 |
| Model | 149,036 | 158,709 | 169,009 | 179,978 |

| Year | 2002 | 2003 | 2004 | 2005 |
|---|---|---|---|---|
| Actual | 187,600 | 195,000 | 221,000 | 240,900 |
| Model | 191,658 | 204,097 | 217,343 | 231,448 |

The model fits the data well. Explanations will vary.

(b)


(c) 2009

## SECTION 4.2    *(page 305)*

**Skills Review**    *(page 305)*
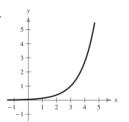
**1.** Continuous on $(-\infty, \infty)$

**2.** Discontinuous for $x = \pm 2$

**3.** Discontinuous for $x = \pm \sqrt{3}$

**4.** Removable discontinuity at $x = 4$

**5.** 0    **6.** 0    **7.** 4    **8.** $\frac{1}{2}$    **9.** $\frac{3}{2}$

**10.** 6    **11.** 0    **12.** 0

**1.** (a) $e^7$    (b) $e^{12}$    (c) $\dfrac{1}{e^6}$    (d) 1

**3.** (a) $e^5$    (b) $e^{5/2}$    (c) $e^6$    (d) $e^7$

**5.** f    **6.** e    **7.** d    **8.** b    **9.** c    **10.** a

**11.**


**13.**


**15.**


**17.**


**19.**
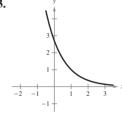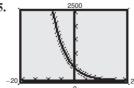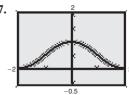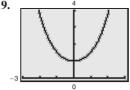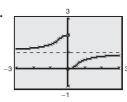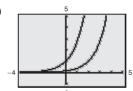

No horizontal asymptotes

Continuous on the entire
real number line

**21.**


Horizontal asymptote: $y = 1$
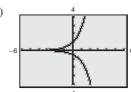
Discontinuous at $x = 0$

**23.** (a)
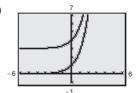

The graph of $g(x) = e^{x-2}$ is shifted horizontally two
units to the right.

(b)


The graph of $h(x) = -\frac{1}{2}e^x$ decreases at a slower rate
than $e^x$ increases.

(c)


The graph of $q(x) = e^x + 3$ is shifted vertically three
units upward.

**25.**

| $n$ | 1 | 2 | 4 | 12 |
|---|---|---|---|---|
| $A$ | 1343.92 | 1346.86 | 1348.35 | 1349.35 |

| $n$ | 365 | Continuous compounding |
|---|---|---|
| $A$ | 1349.84 | 1349.86 |

**27.**

| $n$ | 1 | 2 | 4 | 12 |
|---|---|---|---|---|
| $A$ | 2191.12 | 2208.04 | 2216.72 | 2222.58 |

| $n$ | 365 | Continuous compounding |
|---|---|---|
| $A$ | 2225.44 | 2225.54 |
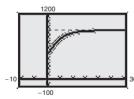
**29.**

| $t$ | 1 | 10 | 20 |
|---|---|---|---|
| $P$ | 96,078.94 | 67,032.00 | 44,932.90 |

| $t$ | 30 | 40 | 50 |
|---|---|---|---|
| $P$ | 30,119.42 | 20,189.65 | 13,533.53 |

**31.**

| $t$ | 1 | 10 | 20 |
|---|---|---|---|
| $P$ | 95,132.82 | 60,716.10 | 36,864.45 |

| $t$ | 30 | 40 | 50 |
|---|---|---|---|
| $P$ | 22,382.66 | 13,589.88 | 8251.24 |

**33.** $107,311.12

**35.** (a) 9%     (b) 9.20%     (c) 9.31%     (d) 9.38%

**37.** $12,500     **39.** $8751.92

**41.** (a) $849.53     (b) $421.12

$$\lim_{x\to\infty} p = 0$$

**43.** (a) 0.1535   (b) 0.4866   (c) 0.8111

**45.** (a) The model fits the data well.

(b) $y = 421.60x + 1504.6$; The linear model fits the data well, but the exponential model fits the data better.

(c) Exponential model: 2008

Linear model: 2010

**47.** (a)



(b) Yes, $\lim_{t\to\infty} \dfrac{925}{1 + e^{-0.3t}} = 925$

(c) $\lim_{t\to\infty} \dfrac{1000}{1 + e^{-0.3t}} = 1000$

Models similar to this logistic growth model where $y = \dfrac{a}{1 + be^{-ct}}$ have a limit of $a$ as $t\to\infty$.

**49.** (a) 0.731   (b) 11   (c) Yes, $\lim_{n\to\infty} \dfrac{0.83}{1 + e^{-0.2n}} = 0.83$

**51.** Amount earned:

(a) $5267.71

(b) $5255.81

(c) $5243.23

You should choose the certificate of deposit in part (a) because it earns more money than the others.

## SECTION 4.3   *(page 314)*

> **Skills Review**   *(page 314)*
>
> **1.** $\dfrac{1}{2}e^x(2x^2 - 1)$   **2.** $\dfrac{e^x(x + 1)}{x}$   **3.** $e^x(x - e^x)$
>
> **4.** $e^{-x}(e^{2x} - x)$   **5.** $-\dfrac{6}{7x^3}$   **6.** $6x - \dfrac{1}{6}$
>
> **7.** $6(2x^2 - x + 6)$   **8.** $\dfrac{t + 2}{2t^{3/2}}$
>
> **9.** Relative maximum: $\left(-\dfrac{4\sqrt{3}}{3}, \dfrac{16\sqrt{3}}{9}\right)$
>
> Relative minimum: $\left(\dfrac{4\sqrt{3}}{3}, -\dfrac{16\sqrt{3}}{9}\right)$
>
> **10.** Relative maximum: $(0, 5)$
>
> Relative minima: $(-1, 4), (1, 4)$

**1.** 3     **3.** $-1$     **5.** $5e^{5x}$     **7.** $-2xe^{-x^2}$
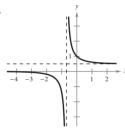
**9.** $\dfrac{2}{x^3}e^{-1/x^2}$     **11.** $e^{4x}(4x^2 + 2x + 4)$

**13.** $-\dfrac{6(e^x - e^{-x})}{(e^x + e^{-x})^4}$     **15.** $xe^x + e^x + 4e^{-x}$

**17.** $y = 2x - 3$     **19.** $y = \dfrac{4}{e^2}$     **21.** $y = 24x + 8$

**23.** $\dfrac{dy}{dx} = \dfrac{10 - e^y}{xe^y + 3}$     **25.** $\dfrac{dy}{dx} = \dfrac{e^{-x}(x^2 - 2x) + y}{4y - x}$

**27.** $6(3e^{3x} + 2e^{-2x})$     **29.** $5(e^{-x} - 10e^{-5x})$

**31.**



No relative extrema

No points of inflection
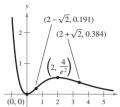
Horizontal asymptote to the right: $y = \frac{1}{2}$

Horizontal asymptote to the left: $y = 0$

Vertical asymptote: $x \approx -0.693$

**33.**
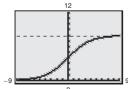


Relative minimum: $(0, 0)$

Relative maximum: $\left(2, \dfrac{4}{e^2}\right)$

Points of inflection: $\left(2 - \sqrt{2}, 0.191\right), \left(2 + \sqrt{2}, 0.384\right)$

Horizontal asymptote to the right: $y = 0$

**35.**
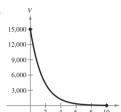


Horizontal asymptotes: $y = 0, y = 8$

**37.** $x = -\frac{1}{3}$　　**39.** $x = 9$

**41.** (a)



(b) $-\$5028.84/\text{yr}$　(c) $-\$406.89/\text{yr}$

(d) $v = -1497.2t + 15,000$

(e) In the exponential function, the initial rate of depreciation is greater than in the linear model. The linear model has a constant rate of depreciation.
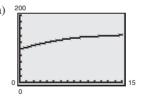
**43.** (a) 1.66 words/min/week　(b) 2.30 words/min/week

(c) 1.74 words/min/week

**45.** $t = 1$: $-24.3\%/\text{week}$

$t = 3$: $-8.9\%/\text{week}$

**47.** (a)
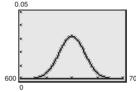


(b and c) 1996: 3.25 million people/yr

2000: 1.30 million people/yr

2005: 5.30 million people/yr

**49.** (a) $f(x) = \dfrac{1}{12.5\sqrt{2\pi}}e^{-(x-650)^2/312.5}$
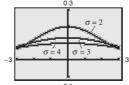
(b)



(c) $f'(x) = \dfrac{-4\sqrt{2}(x - 650)e^{-2(x-650)^2/625}}{15{,}625\sqrt{\pi}}$

(d) Answers will vary.
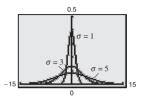
**51.**



As $\sigma$ increases, the graph becomes flatter.

**53.** Proof; maximum: $\left(0, \dfrac{1}{\sigma\sqrt{2\pi}}\right)$; answers will vary.

Sample answer:



## MID-CHAPTER QUIZ　*(page 316)*

**1.** 64　　**2.** $\frac{8}{27}$　　**3.** $3\sqrt[3]{3}$　　**4.** $\frac{16}{81}$　　**5.** 1024

**6.** 216　　**7.** 27　　**8.** $\sqrt{15}$　　**9.** $e^7$　　**10.** $e^{11/3}$

**11.** $e^6$　　**12.** $e^3$

**13.**



**14.**



**13.**



**15.**



**15.**



**16.**



**17.**



**17.**



**18.**



**19.** Answers will vary.     **21.** Answers will vary.
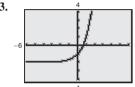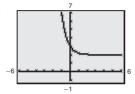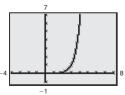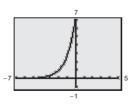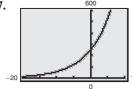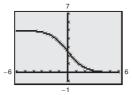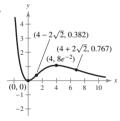


**19.** \$23.22     **20.** (a) \$3572.83     (b) \$3573.74

**21.** $5e^{5x}$     **22.** $e^{x-4}$     **23.** $5e^{x+2}$

**24.** $e^x(2 - x)$     **25.** $y = -2x + 1$

**26.**



Relative maximum: $(4, 8e^{-2})$

Relative minimum: $(0, 0)$

Points of inflection: $(4 - 2\sqrt{2}, 0.382), (4 + 2\sqrt{2}, 0.767)$

Horizontal asymptote to the right: $y = 0$

## SECTION 4.4   *(page 323)*

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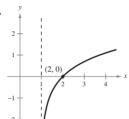
### Skills Review   *(page 323)*

**1.** $\frac{1}{4}$     **2.** 64     **3.** 729     **4.** $\frac{8}{27}$     **5.** 1
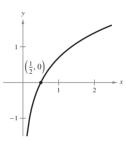
**6.** $81e^4$     **7.** $\frac{e^3}{2}$     **8.** $\frac{125}{8e^3}$     **9.** $x > -4$

**10.** Any real number $x$     **11.** $x < -1$ or $x > 1$

**12.** $x > 5$     **13.** \$3462.03     **14.** \$3374.65

---

**1.** $e^{0.6931. \cdot \cdot} = 2$     **3.** $e^{-1.6094 \cdot \cdot \cdot} = 0.2$     **5.** $\ln 1 = 0$

**7.** $\ln(0.0498. \, . \, .) = -3$     **9.** c     **10.** d

**11.** b     **12.** a
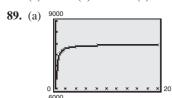
**23.** $x^2$     **25.** $5x + 2$     **27.** $2x - 1$

**29.** (a) 1.7917     (b) 0.4055     (c) 4.3944     (d) 0.5493

**31.** $\ln 2 - \ln 3$     **33.** $\ln x + \ln y + \ln z$

**35.** $\frac{1}{2}\ln(x^2 + 1)$     **37.** $\ln z + 2 \ln(z - 1)$

**39.** $\ln 3 + \ln x + \ln(x + 1) - 2 \ln(2x + 1)$

**41.** $\ln \dfrac{x - 2}{x + 2}$     **43.** $\ln \dfrac{x^3 y^2}{z^4}$     **45.** $\ln\left[\dfrac{x(x + 3)}{x + 4}\right]^3$

**47.** $\ln\left[\dfrac{x(x^2 + 1)}{x + 1}\right]^{3/2}$     **49.** $\ln\dfrac{(x + 1)^{1/3}}{(x - 1)^{2/3}}$

**51.** $x = 4$     **53.** $x = 1$

**55.** $x = \dfrac{e^{2.4}}{2} \approx 5.5116$     **57.** $x = \dfrac{e^{10/3}}{5} \approx 5.6063$

**59.** $x = \ln 4 - 1 \approx 0.3863$

**61.** $t = \dfrac{\ln 7 - \ln 3}{-0.2} \approx -4.2365$

**63.** $x = \frac{1}{2}\left(1 + \ln\frac{3}{2}\right) \approx 0.7027$

**65.** $x = -100 \ln\frac{3}{4} \approx 28.7682$

**67.** $x = \dfrac{\ln 15}{2 \ln 5} \approx 0.8413$     **69.** $t = \dfrac{\ln 2}{\ln 1.07} \approx 10.2448$

**71.** $t = \dfrac{\ln 3}{12 \ln[1 + (0.07/12)]} \approx 15.7402$

**73.** $t = \dfrac{\ln 30}{3 \ln[16 - (0.878/26)]} \approx 0.4092$

**75.** (a) 8.15 yr     (b) 12.92 yr

**77.** (a) 14.21 yr     (b) 13.89 yr

    (c) 13.86 yr     (d) 13.86 yr

**79.** (a) About 896 units     (b) About 136 units

**81.** (a) $P(25) \approx 210{,}650$     (b) 2023

**83.** 9395 yr     **85.** 12,484 yr

**87.** (a) 80     (b) 57.5     (c) 10 mo

**89.** (a)
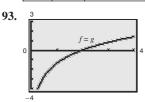


    (b) $y = 7955.6$; This means that the orchard's yield approaches but does not reach 7955.6 pounds per acre as it increases in age.

    (c) About 6.53 yr

**91.**
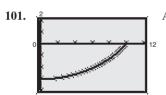
| $x$ | $y$ | $\dfrac{\ln x}{\ln y}$ | $\ln \dfrac{x}{y}$ | $\ln x - \ln y$ |
|---|---|---|---|---|
| 1 | 2 | 0 | $-0.6931$ | $-0.6931$ |
| 3 | 4 | 0.7925 | $-0.2877$ | $-0.2877$ |
| 10 | 5 | 1.4307 | 0.6931 | 0.6931 |
| 4 | 0.5 | $-2$ | 2.0794 | 2.0794 |

**93.**



**95.** False. $f(x) = \ln x$ is undefined for $x \leq 0$.

**97.** False. $f\left(\dfrac{x}{2}\right) = f(x) - f(2)$     **99.** False. $u = v^2$

**101.**

     Answers will vary.

## SECTION 4.5    *(page 332)*

**Skills Review**    *(page 332)*

**1.** $2 \ln(x + 1)$     **2.** $\ln x + \ln(x + 1)$

**3.** $\ln x - \ln(x + 1)$     **4.** $3[\ln x - \ln(x - 3)]$

**5.** $\ln 4 + \ln x + \ln(x - 7) - 2 \ln x$

**6.** $3 \ln x + \ln(x + 1)$

**7.** $-\dfrac{y}{x + 2y}$     **8.** $\dfrac{3 - 2xy + y^2}{x(x - 2y)}$

**9.** $-12x + 2$     **10.** $-\dfrac{6}{x^4}$

**1.** 3     **3.** 2     **5.** $\dfrac{2}{x}$     **7.** $\dfrac{2x}{x^2 + 3}$     **9.** $\dfrac{1}{2(x - 4)}$

**11.** $\dfrac{4}{x}(\ln x)^3$     **13.** $2 \ln x + 2$     **15.** $\dfrac{2x^2 - 1}{x(x^2 - 1)}$

**17.** $\dfrac{1}{x(x + 1)}$     **19.** $\dfrac{2}{3(x^2 - 1)}$     **21.** $-\dfrac{4}{x(4 + x^2)}$

**23.** $e^{-x}\left(\dfrac{1}{x} - \ln x\right)$     **25.** $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$     **27.** $e^{x(\ln 2)}$

**29.** $\dfrac{1}{\ln 4} \ln x$     **31.** 1.404     **33.** 5.585     **35.** $-0.631$

**37.** $-2.134$     **39.** $(\ln 3)3^x$     **41.** $\dfrac{1}{x \ln 2}$

**43.** $(2 \ln 4)4^{2x-3}$     **45.** $\dfrac{2x + 6}{(x^2 + 6x) \ln 10}$

**47.** $2^x(1 + x \ln 2)$     **49.** $y = x - 1$

**51.** $y = \dfrac{1}{27 \ln 3}x - \dfrac{1}{\ln 3} + 3$     **53.** $\dfrac{2xy}{3 - 2y^2}$
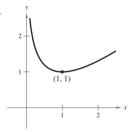
**55.** $\dfrac{y(1 - 6x^2)}{1 + y}$     **57.** $y = x - 1$

**59.** $\dfrac{1}{2x}$     **61.** $\dfrac{1}{x}$     **63.** $(\ln 5)^2 5^x$
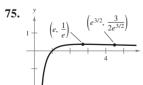
**65.** $\dfrac{d\beta}{dI} = \dfrac{10}{(\ln 10)I}$, so for $I = 10^{-4}$, the rate of change is about 43,429.4 db/w/cm$^2$.

**67.** $2, y = 2x - 1$     **69.** $-\frac{8}{5}, y = -\frac{8}{5}x - 4$

**71.** $\dfrac{1}{\ln 2}, y = \dfrac{1}{\ln 2}x - \dfrac{1}{\ln 2}$

**73.**



Relative minimum: $(1, 1)$

**75.**
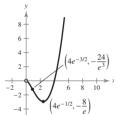


Relative maximum: $\left(e, \dfrac{1}{e}\right)$

Point of inflection: $\left(e^{3/2}, \dfrac{3}{2e^{3/2}}\right)$

**77.**



Relative minimum: $\left(4e^{-1/2}, \dfrac{-8}{e}\right)$

Point of inflection: $\left(4e^{-3/2}, \dfrac{-24}{e^3}\right)$

**79.** $-\dfrac{1}{p}, -\dfrac{1}{10}$

**81.** $p = 1000e^{-x}$

$\dfrac{dp}{dx} = -1000e^{-x}$
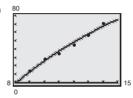
At $p = 10$, rate of change $= -10$.

$\dfrac{dp}{dx}$ and $\dfrac{dx}{dp}$ are reciprocals of each other.

**83.** (a) $\overline{C} = \dfrac{500 + 300x - 300\ln x}{x}$

(b) Minimum of 279.15 at $e^{8/3}$

**85.** (a)



(b) \$10.1625 billion/yr

**87.** (a) $I = 10^{8.3} \approx 199{,}526{,}231.5$

(b) $I = 10^{6.3} \approx 1{,}995{,}262.315$

(c) $10^R$

(d) $\dfrac{dR}{dI} = \dfrac{1}{I\ln(10)}$

**89.** Answers will vary.

## SECTION 4.6   *(page 341)*

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**Skills Review**   *(page 341)*

**1.** $-\dfrac{1}{4}\ln 2$    **2.** $\dfrac{1}{5}\ln\dfrac{10}{3}$    **3.** $-\dfrac{\ln(25/16)}{0.01}$

**4.** $-\dfrac{\ln(11/16)}{0.02}$    **5.** $7.36e^{0.23t}$    **6.** $1.296e^{0.072t}$

**7.** $-33.6e^{-1.4t}$    **8.** $-0.025e^{-0.001t}$    **9.** $4$

**10.** $12$    **11.** $2x + 1$    **12.** $x^2 + 1$

---

**1.** $y = 2e^{0.1014t}$    **3.** $y = 4e^{-0.4159t}$

**5.** $y = 0.6687e^{0.4024t}$    **7.** $y = 10e^{2t}$, exponential growth

**9.** $y = 30e^{-4t}$, exponential decay

**11.** *Amount after 1000 years: 6.48 g*

*Amount after 10,000 years: 0.13 g*

**13.** *Initial quantity: 6.73 g*

*Amount after 1000 years: 5.96 g*

**15.** *Initial quantity: 2.16 g*

*Amount after 10,000 years: 1.62 g*

**17.** 68%    **19.** 15,642 yr

**21.** $k_1 = \dfrac{\ln 4}{12} \approx 0.1155$, so $y_1 = 5e^{0.1155t}$.

$k_2 = \dfrac{1}{6}$, so $y_2 = 5(2)^{t/6}$

Explanations will vary.

**23.** (a) 1350    (b) $\dfrac{5\ln 2}{\ln 3} \approx 3.15$ hr

(c) No. Answers will vary.

**25.** *Time to double:* 5.78 yr

Amount after 10 years: $3320.12

Amount after 25 years: $20,085.54

**27.** *Annual rate:* 8.66%

Amount after 10 years: $1783.04

Amount after 25 years: $6535.95

**29.** *Annual rate:* 9.50%

*Time to double:* 7.30 yr

Amount after 25 years: $5375.51

**31.** *Initial investment:* $6376.28

*Time to double:* 15.40 yr
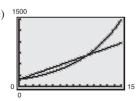
Amount after 25 years: $19,640.33

**33.** $49,787.07    **35.** (a) Answers will vary.    (b) 6.17%

**37.**

| Number of compoundings/yr | 4 | 12 |
|---|---|---|
| Effective yield | 5.095% | 5.116% |

| Number of compoundings/yr | 365 | Continuous |
|---|---|---|
| Effective yield | 5.127% | 5.127% |

**39.** Answers will vary.

**41.** (a) $1486.1 million    (b) $964.4 million

(c)
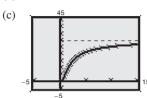


$t = 0$ corresponds to 1996.

Answers will vary.

**43.** (a) $C = 30$

$k = \ln\left(\frac{1}{6}\right) \approx -1.7918$

(b) $30e^{-0.35836} = 20.9646$ or 20,965 units

(c)



**45.** About 36 days    **47.** $496,806

**49.** (a) $C = \frac{625}{64}$

$k = \frac{1}{100}\ln\frac{4}{5}$

(b) $x = 448$ units; $p = $3.59$

**51.** 2046    **53.** Answers will vary.

## REVIEW EXERCISES FOR CHAPTER 4
*(page 348)*

**1.** 8    **3.** 64    **5.** 1    **7.** $\frac{1}{6}$    **9.** $e^{10}$

**11.** $e^3$    **13.** $f(4) = 128$    **15.** $f(10) \approx 1.219$
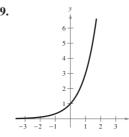
**17.** (a) 1999: $P(9) \approx $179.8 million

2003: $P(13) \approx $352.1 million
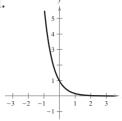
2005: $P(15) \approx $492.8 million

(b) Answers will vary.
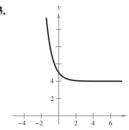
**19.**
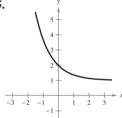


**21.**
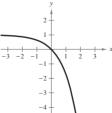


**23.**



**25.**
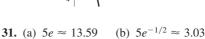


**27.**
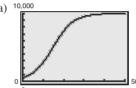


**29.** $7500

Explanations will vary.

**31.** (a) $5e \approx 13.59$    (b) $5e^{-1/2} \approx 3.03$

(c) $5e^9 \approx 40,515.42$

**33.** (a) $6e^{-3.4} \approx 0.2002$    (b) $6e^{-10} \approx 0.0003$

(c) $6e^{-20} \approx 1.2367 \times 10^{-8}$

**35.** (a)



(b) $P \approx 1049$ fish

(c) Yes, $P$ approaches 10,000 fish as $t$ approaches $\infty$.

(d) The population is increasing most rapidly at the inflection point, which occurs around $t = 15$ months.

**37.**

| $n$ | 1 | 2 | 4 | 12 |
|---|---|---|---|---|
| $A$ | $1216.65 | $1218.99 | $1220.19 | $1221.00 |

| $n$ | 365 | Continuous compounding |
|---|---|---|
| $A$ | $1221.39 | $1221.40 |

**39.** b    **41.** (a) 6.14%    (b) 6.17%    **43.** $10,338.10

**45.** 1990: $P(0) = 29.7$ million

2000: $P(10) \approx 32.8$ million

2005: $P(15) \approx 34.5$ million

**47.** $8xe^{x^2}$    **49.** $\dfrac{1 - 2x}{e^{2x}}$    **51.** $4e^{2x}$    **53.** $-\dfrac{10e^{2x}}{(1 + e^{2x})^2}$

**55.**



No relative extrema

No points of inflection

Horizontal asymptote: $y = 0$

**57.**
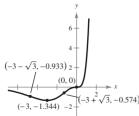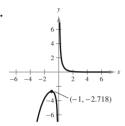


Relative minimum: $(-3, -1.344)$

Inflection points: $(0, 0), (-3 + \sqrt{3}, -0.574),$
and $(-3 - \sqrt{3}, -0.933)$

Horizontal asymptote: $y = 0$

**59.**
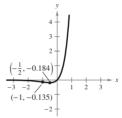


Relative maximum: $(-1, -2.718)$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 0$

**61.**



Relative minimum: $\left(-\dfrac{1}{2}, -0.184\right)$
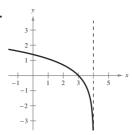
Inflection point: $(-1, -0.135)$
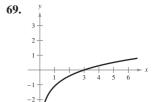
Horizontal asymptote: $y = 0$

**63.** $e^{2.4849} \approx 12$    **65.** $\ln 4.4816 \approx 1.5$

**67.**



**69.**
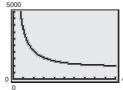


**71.** $\ln x + \frac{1}{2} \ln(x - 1)$    **73.** $2 \ln x - 3 \ln(x + 1)$

**75.** $3[\ln(1 - x) - \ln 3 - \ln x]$    **77.** 3

**79.** $e^{3e^{-1}} \approx 3.0151$    **81.** 1    **83.** $\frac{1}{2}(\ln 6 + 1) \approx 1.3959$

**85.** $\dfrac{3 + \sqrt{13}}{2} \approx 3.3028$    **87.** $-\dfrac{\ln(0.25)}{1.386} \approx 1.0002$

**89.** $\dfrac{\ln 1.1}{\ln 1.21} = 0.5$    **91.** $100 \ln\left(\dfrac{25}{4}\right) \approx 183.2581$

**93.** (a)
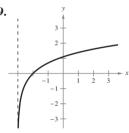


(b) A 30-year term has a smaller monthly payment, but the total amount paid is higher due to more interest.
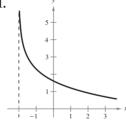
**95.** $\dfrac{2}{x}$    **97.** $\dfrac{1}{x} + \dfrac{1}{x - 1} - \dfrac{1}{x - 2} = \dfrac{x^2 - 4x + 2}{x(x - 2)(x - 1)}$

**99.** 2    **101.** $\dfrac{1 - 3 \ln x}{x^4}$    **103.** $\dfrac{4x}{3(x^2 - 2)}$

**105.** $\dfrac{2}{x} + \dfrac{1}{2(x + 1)}$    **107.** $\dfrac{1}{1 + e^x}$

**109.**
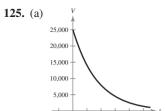


No relative extrema
No points of inflection

**111.**



No relative extrema
No points of inflection

**113.** 2 **115.** 0 **117.** 1.594 **119.** 1.500

**121.** $\dfrac{2}{(2x-1)\ln 3}$ **123.** $-\dfrac{2}{x\ln 2}$

**125.** (a)



$t = 2$: $14,062.50

(b) $t = 1$: $-$5394.04$/yr (c) $t \approx 5.6$ yr

$t = 4$: $-$2275.61$/yr

**127.** $A = 500e^{-0.01277t}$ **129.** 27.9 yr
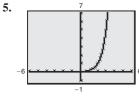
**131.** $1048.2 million
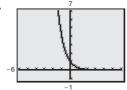
## CHAPTER TEST (page 352)

**1.** 1 **2.** $\frac{1}{256}$ **3.** $e^{9/2}$ **4.** $e^2$

**5.**



**6.**



**7.**



**8.**



**9.**



**10.**
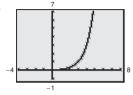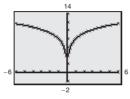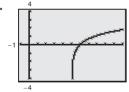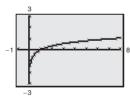


**11.** $\ln 3 - \ln 2$ **12.** $\frac{1}{2}\ln(x + y)$ **13.** $\ln(x + 1) - \ln y$

**14.** $\ln[y(x + 1)]$ **15.** $\ln\dfrac{8}{(x-1)^2}$ **16.** $\ln\dfrac{x^2 y}{z + 4}$

**17.** $x \approx 3.197$ **18.** $x \approx 1.750$ **19.** $x \approx 58.371$

**20.** (a) 17.67 yr (b) 17.36 yr

(c) 17.33 yr (d) 17.33 yr

**21.** $-3e^{-3x}$ **22.** $7e^{x+2} + 2$

**23.** $\dfrac{2x}{3 + x^2}$ **24.** $\dfrac{2}{x(x + 2)}$

**25.** (a) $828.58 million (b) $24.95 million/yr

**26.** 59.4% **27.** 39.61 yr

## CHAPTER 5

### SECTION 5.1 (page 362)

**Skills Review** (page 362)

**1.** $x^{-1/2}$ **2.** $(2x)^{4/3}$ **3.** $5^{1/2}x^{3/2} + x^{5/2}$

**4.** $x^{-1/2} + x^{-2/3}$ **5.** $(x + 1)^{5/2}$ **6.** $x^{1/6}$

**7.** $-12$ **8.** $-10$ **9.** 14 **10.** 14

**1–7.** Answers will vary. **9.** $6x + C$

**11.** $\dfrac{5}{3}t^3 + C$ **13.** $-\dfrac{5}{2x^2} + C$ **15.** $u + C$

**17.** $et + C$ **19.** $\dfrac{2}{5}y^{5/2} + C$

| Rewrite | Integrate | Simplify |
|---|---|---|
| **21.** $\displaystyle\int x^{1/3}\,dx$ | $\dfrac{x^{4/3}}{4/3} + C$ | $\dfrac{3}{4}x^{4/3} + C$ |
| **23.** $\displaystyle\int x^{-3/2}\,dx$ | $\dfrac{x^{-1/2}}{-1/2} + C$ | $-\dfrac{2}{\sqrt{x}} + C$ |
| **25.** $\dfrac{1}{2}\displaystyle\int x^{-3}\,dx$ | $\dfrac{1}{2}\left(\dfrac{x^{-2}}{-2}\right) + C$ | $-\dfrac{1}{4x^2} + C$ |

**27.** $\dfrac{x^2}{2} + 3x + C$ **29.** $\dfrac{1}{4}x^4 + 2x + C$

**31.** $\frac{3}{4}x^{4/3} - \frac{3}{4}x^{2/3} + C$ **33.** $\frac{3}{5}x^{5/3} + C$

**35.** $-\dfrac{1}{3x^3} + C$ **37.** $2x - \dfrac{1}{2x^2} + C$
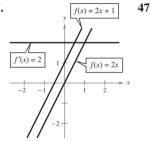
**39.** $\dfrac{3}{4}u^4 + \dfrac{1}{2}u^2 + C$ **41.** $x^3 + \dfrac{x^2}{2} - 2x + C$
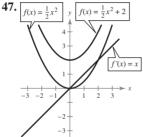
**43.** $\dfrac{2}{7}y^{7/2} + C$

**45.**



$f(x) = 2x + 1$
$f'(x) = 2$
$f(x) = 2x$

**47.**



$f(x) = \frac{1}{2}x^2$     $f(x) = \frac{1}{2}x^2 + 2$
$f'(x) = x$

**49.** $f(x) = 2x^2 + 6$     **51.** $f(x) = x^2 - 2x - 1$

**53.** $f(x) = -\dfrac{1}{x^2} + \dfrac{1}{x} + \dfrac{1}{2}$     **55.** $y = -\dfrac{5}{2}x^2 - 2x + 2$

**57.** $f(x) = x^2 - 6$     **59.** $f(x) = x^2 + x + 4$

**61.** $f(x) = \frac{9}{4}x^{4/3}$     **63.** $C = 85x + 5500$

**65.** $C = \frac{1}{10}\sqrt{x} + 4x + 750$

**67.** $R = 225x - \frac{3}{2}x^2,\ p = 225 - \frac{3}{2}x$

**69.** $P = -9x^2 + 1650x$     **71.** $P = -12x^2 + 805x + 68$

**73.** $s(t) = -16t^2 + 6000$; about 19.36 sec

**75.** (a) $C = x^2 - 12x + 125$     (b) $2025

$\overline{C} = x - 12 + \dfrac{125}{x}$

(c) $125 is fixed.

$1900 is variable.

Examples will vary.

**77.** (a) $P(t) = 52.73t^2 + 2642.7t + 69{,}903.25$

(b) 273,912; Yes, this seems reasonable. Explanations will vary.

**79.** (a) $I(t) = -0.0625t^4 + 1.773t^3 - 9.67t^2$

$+ 21.03t - 0.212$ (in millions)

(b) 20.072 million; No, this does not seem reasonable. Explanations will vary. Sample answer: A sharp decline from 863 million users to about 20 million users from the year 2004 to the year 2012 does not seem to follow the trend over the past few years, which is always increasing.

## SECTION 5.2  *(page 372)*

**Skills Review**  *(page 372)*

**1.** $\frac{1}{2}x^4 + x + C$     **2.** $\frac{3}{2}x^2 + \frac{2}{3}x^{3/2} - 4x + C$

**3.** $-\dfrac{1}{x} + C$     **4.** $-\dfrac{1}{6t^2} + C$

**5.** $\frac{4}{7}t^{7/2} + \frac{2}{5}t^{5/2} + C$     **6.** $\frac{4}{5}x^{5/2} - \frac{2}{3}x^{3/2} + C$

**7.** $\dfrac{5x^3 - 4}{2x} + C$     **8.** $\dfrac{-6x^2 + 5}{3x^3} + C$

**9.** $\frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C$

**10.** $\frac{1}{7}x^7 - \frac{4}{5}x^5 + \frac{1}{2}x^4 + \frac{4}{3}x^3 - 2x^2 + x + C$

**11.** $-\dfrac{5(x - 2)^4}{16}$     **12.** $-\dfrac{1}{12(x - 1)^2}$

**13.** $9(x^2 + 3)^{2/3}$     **14.** $-\dfrac{5}{(1 - x^3)^{1/2}}$

| $\displaystyle\int u^n \dfrac{du}{dx}\,dx$ | $u$ | $\dfrac{du}{dx}$ |
|---|---|---|
| **1.** $\displaystyle\int (5x^2 + 1)^2(10x)\,dx$ | $5x^2 + 1$ | $10x$ |
| **3.** $\displaystyle\int \sqrt{1 - x^2}\,(-2x)\,dx$ | $1 - x^2$ | $-2x$ |
| **5.** $\displaystyle\int \left(4 + \dfrac{1}{x^2}\right)^5\!\left(\dfrac{-2}{x^3}\right) dx$ | $4 + \dfrac{1}{x^2}$ | $-\dfrac{2}{x^3}$ |
| **7.** $\displaystyle\int (1 + \sqrt{x})^3\!\left(\dfrac{1}{2\sqrt{x}}\right) dx$ | $1 + \sqrt{x}$ | $\dfrac{1}{2\sqrt{x}}$ |

**9.** $\frac{1}{5}(1 + 2x)^5 + C$     **11.** $\frac{2}{3}(4x^2 - 5)^{3/2} + C$

**13.** $\frac{1}{5}(x - 1)^5 + C$     **15.** $\dfrac{(x^2 - 1)^8}{8} + C$

**17.** $-\dfrac{1}{3(1 + x^3)} + C$     **19.** $-\dfrac{1}{2(x^2 + 2x - 3)} + C$

**21.** $\sqrt{x^2 - 4x + 3} + C$     **23.** $-\frac{15}{8}(1 - u^2)^{4/3} + C$

**25.** $4\sqrt{1 + y^2} + C$     **27.** $-3\sqrt{2t + 3} + C$

**29.** $-\dfrac{1}{2}\sqrt{1 - x^4} + C$     **31.** $-\dfrac{1}{24}\left(1 + \dfrac{4}{t^2}\right)^3 + C$

**33.** $\dfrac{(x^3 + 3x + 9)^2}{6} + C$     **35.** $\frac{1}{4}(6x^2 - 1)^4 + C$

**37.** $-\frac{2}{45}(2 - 3x^3)^{5/2} + C$     **39.** $\sqrt{x^2 + 25} + C$

**41.** $\frac{2}{3}\sqrt{x^3 + 3x + 4} + C$

**43.** (a) $\frac{1}{3}x^3 - x^2 + x + C_1 = \frac{1}{3}(x - 1)^3 + C_2$

(b) Answers differ by a constant: $C_1 = C_2 - \frac{1}{3}$
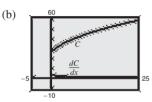
(c) Answers will vary.

**45.** (a) $\dfrac{1}{6}x^6 - \dfrac{1}{2}x^4 + \dfrac{1}{2}x^2 + C_1 = \dfrac{(x^2 - 1)^3}{6} + C_2$

(b) Answers differ by a constant: $C_1 = C_2 - \dfrac{1}{6}$

(c) Answers will vary.

**47.** $f(x) = \frac{1}{3}[5 - (1 - x^2)^{3/2}]$

**49.** (a) $C = 8\sqrt{x + 1} + 18$

(b)
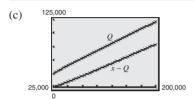


**51.** $x = \frac{1}{3}(p^2 - 25)^{3/2} + 24$     **53.** $x = \dfrac{6000}{\sqrt{p^2 - 16}} + 3000$

**55.** (a) $h = \sqrt{17.6t^2 + 1} + 5$     (b) 26 in.

**57.** (a) $Q = (x - 24,999)^{0.95} + 24,999$

(b)

| $x$ | 25,000 | 50,000 | 100,000 | 150,000 |
|---|---|---|---|---|
| $Q$ | 25,000 | 40,067.14 | 67,786.18 | 94,512.29 |
| $x - Q$ | 0 | 9932.86 | 32,213.82 | 55,487.71 |

(c)



**59.** $-\frac{2}{3}x^{3/2} + \frac{2}{3}(x + 1)^{3/2} + C$

## SECTION 5.3   *(page 379)*
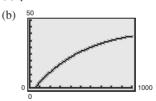
### Skills Review   *(page 379)*

**1.** $\left(\frac{5}{2}, \infty\right)$     **2.** $(-\infty, 2) \cup (3, \infty)$

**3.** $x + 2 - \dfrac{2}{x + 2}$     **4.** $x - 2 + \dfrac{1}{x - 4}$

**5.** $x + 8 + \dfrac{2x - 4}{x^2 - 4x}$     **6.** $x^2 - x - 4 + \dfrac{20x + 22}{x^2 + 5}$

**7.** $\frac{1}{4}x^4 - \frac{1}{x} + C$     **8.** $\frac{1}{2}x^2 + 2x + C$

**9.** $\frac{1}{2}x^2 - \frac{4}{x} + C$     **10.** $-\frac{1}{x} - \frac{3}{2x^2} + C$

**1.** $e^{2x} + C$     **3.** $\frac{1}{4}e^{4x} + C$     **5.** $-\frac{9}{2}e^{-x^2} + C$

**7.** $\frac{5}{3}e^{x^3} + C$     **9.** $\frac{1}{3}e^{x^3 + 3x^2 - 1} + C$     **11.** $-5e^{2-x} + C$

**13.** $\ln|x + 1| + C$     **15.** $-\frac{1}{2}\ln|3 - 2x| + C$

**17.** $\frac{2}{3}\ln|3x + 5| + C$

**19.** $\ln\sqrt{x^2 + 1} + C$     **21.** $\frac{1}{3}\ln|x^3 + 1| + C$

**23.** $\frac{1}{2}\ln|x^2 + 6x + 7| + C$     **25.** $\ln|\ln x| + C$

**27.** $\ln|1 - e^{-x}| + C$     **29.** $-\frac{1}{2}e^{2/x} + C$     **31.** $2e^{\sqrt{x}} + C$

**33.** $\frac{1}{2}e^{2x} - 4e^x + 4x + C$     **35.** $-\ln(1 + e^{-x}) + C$

**37.** $-2\ln|5 - e^{2x}| + C$

**39.** $e^x + 2x - e^{-x} + C$; Exponential Rule and General Power Rule

**41.** $-\frac{2}{3}(1 - e^x)^{3/2} + C$; Exponential Rule

**43.** $-\dfrac{1}{x - 1} + C$; General Power Rule

**45.** $2e^{2x-1} + C$; Exponential Rule

**47.** $\frac{1}{4}x^2 - 4\ln|x| + C$; General Power Rule and Logarithmic Rule

**49.** $2\ln(e^x + 1) + C$; Logarithmic Rule

**51.** $\frac{1}{2}x^2 + 3x + 8\ln|x - 1| + C$; General Power Rule and Logarithmic Rule

**53.** $\ln|e^x + x| + C$; Logarithmic Rule

**55.** $f(x) = \frac{1}{2}x^2 + 5x + 8\ln|x - 1| - 8$

**57.** (a) $P(t) = 1000[1 + \ln(1 + 0.25t)^{12}]$

(b) $P(3) \approx 7715$ bacteria     (c) $t \approx 6$ days

**59.** (a) $p = -50e^{-x/500} + 45.06$

(b)



The price increases as the demand increases.

(c) 387

**61.** (a) $S = -7241.22e^{-t/4.2} + 42,721.88$ (in dollars)

(b) $38,224.03

**63.** False. $\ln x^{1/2} = \frac{1}{2}\ln x$

## MID-CHAPTER QUIZ   *(page 381)*

**1.** $3x + C$     **2.** $5x^2 + C$     **3.** $-\dfrac{1}{4x^4} + C$

**4.** $\dfrac{x^3}{3} - x^2 + 15x + C$     **5.** $\dfrac{x^3}{3} + 2x^2 + C$

**6.** $\dfrac{(6x + 1)^4}{4} + C$     **7.** $\dfrac{(x^2 - 5x)^2}{2} + C$

**8.** $-\dfrac{1}{2(x^3 + 3)^2} + C$     **9.** $\frac{2}{15}(5x + 2)^{3/2} + C$

**10.** $f(x) = 8x^2 + 1$     **11.** $f(x) = 3x^3 + 4x - 2$

**12.** (a) $9.03     (b) $509.03

**13.** $f(x) = \frac{2}{3}x^3 + x + 1$     **14.** $e^{5x + 4} + C$

**15.** $\dfrac{x^2}{2} + e^{2x} + C$     **16.** $e^{x^3} + C$     **17.** $\ln|2x - 1| + C$

**18.** $-\ln|x^2 + 3| + C$     **19.** $3\ln|x^3 + 2x^2| + C$

**20.** (a) 1000 bolts     (b) About 8612 bolts

## SECTION 5.4   *(page 391)*

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### Skills Review   *(page 391)*

**1.** $\frac{3}{2}x^2 + 7x + C$     **2.** $\frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + C$

**3.** $\dfrac{1}{5}\ln|x| + C$     **4.** $-\dfrac{1}{6e^{6x}} + C$     **5.** $-\dfrac{8}{5}$
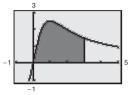
**6.** $-\frac{62}{3}$     **7.** $C = 0.008x^{5/2} + 29{,}500x + C$

**8.** $R = x^2 + 9000x + C$

**9.** $P = 25{,}000x - 0.005x^2 + C$
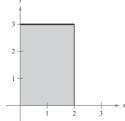
**10.** $C = 0.01x^3 + 4600x + C$

---

**1.**



Positive

**3.**



Area = 6

**5.**



Area = 8

**7.**



Area = $\frac{35}{2}$

**9.**
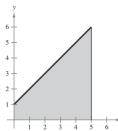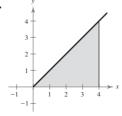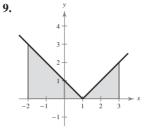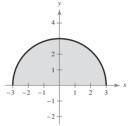


Area = $\frac{13}{2}$

**11.**



Area = $\dfrac{9\pi}{2}$

**13.** (a) 8     (b) 4     (c) $-24$     (d) 0

**15.** $\dfrac{1}{6}$     **17.** $\dfrac{1}{2}$     **19.** $6\left(1 - \dfrac{1}{e^2}\right)$     **21.** $8\ln 2 + \dfrac{15}{2}$

**23.** 1     **25.** $-\frac{5}{2}$     **27.** $\frac{14}{3}$     **29.** $-\frac{15}{4}$     **31.** $-4$

**33.** $\frac{2}{3}$     **35.** $-\frac{27}{20}$     **37.** 2     **39.** $\frac{1}{2}(1 - e^{-2}) \approx 0.432$

**41.** $\dfrac{e^3 - e}{3} \approx 5.789$     **43.** $\dfrac{1}{3}\left[(e^2 + 1)^{3/2} - 2\sqrt{2}\right] \approx 7.157$
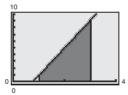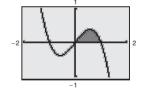
**45.** $\frac{1}{8}\ln 17 \approx 0.354$     **47.** 4     **49.** 4

**51.** $\frac{1}{2}\ln 5 - \frac{1}{2}\ln 8 \approx -0.235$

**53.** $2\ln(2 + e^3) - 2\ln 3 \approx 3.993$

**55.** Area = 10     **57.** Area = $\frac{1}{4}$
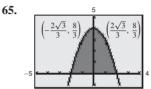




**59.** Area = $\ln 9$



**61.** 10     **63.** $4\ln 3 \approx 4.394$

**65.**
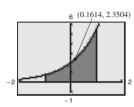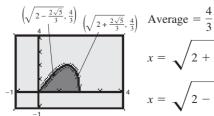


Average = $\dfrac{8}{3}$

$x = \pm\dfrac{2\sqrt{3}}{3} \approx \pm 1.155$

**67.**



Average $= e - e^{-1} \approx 2.3504$

$x = \ln\left(\dfrac{e - e^{-1}}{2}\right) \approx 0.1614$

**69.**



Average $= \dfrac{4}{3}$
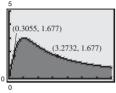
$x = \sqrt{2 + \dfrac{2\sqrt{5}}{3}} \approx 1.868$

$x = \sqrt{2 - \dfrac{2\sqrt{5}}{3}} \approx 0.714$

**71.**



Average $= \dfrac{3}{7}\ln 50 \approx 1.677$

$x \approx 0.3055$

$x \approx 3.2732$

**73.** Even     **75.** Neither odd nor even

**77.** (a) $\dfrac{1}{3}$   (b) $\dfrac{2}{3}$   (c) $-\dfrac{1}{3}$

Explanations will vary.

**79.** \$6.75     **81.** \$22.50     **83.** \$3.97     **85.** \$1925.23

**87.** \$16,605.21     **89.** \$2500     **91.** \$4565.65

**93.** (a) \$137,000     (b) \$214,720.93     (c) \$338,393.53

**95.** \$2623.94     **97.** About 144.36 thousand kg

**99.** $\dfrac{2}{3}\sqrt{7} - \dfrac{1}{3}$     **101.** $\dfrac{39}{200}$
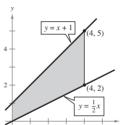
## SECTION 5.5   *(page 400)*

**Skills Review**   *(page 400)*
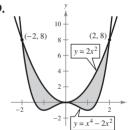
**1.** $-x^2 + 3x + 2$     **2.** $-2x^2 + 4x + 4$

**3.** $-x^3 + 2x^2 + 4x - 5$     **4.** $x^3 - 6x - 1$

**5.** $(0, 4), (4, 4)$     **6.** $(1, -3), (2, -12)$

**7.** $(-3, 9), (2, 4)$     **8.** $(-2, -4), (0, 0), (2, 4)$

**9.** $(1, -2), (5, 10)$     **10.** $(1, e)$

**1.** 36     **3.** 9     **5.** $\dfrac{3}{2}$
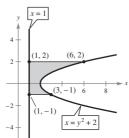
**7.**
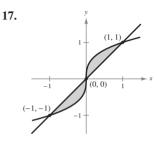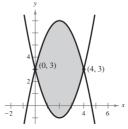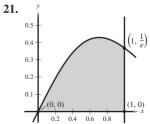


**9.**



**11.**



**13.** d

**15.**



Area $= \dfrac{4}{5}$

**17.**



Area $= \dfrac{1}{2}$

**19.**



Area $= \dfrac{64}{3}$

**21.**



Area $= -\dfrac{1}{2}e^{-1} + \dfrac{1}{2}$

**23.**



Area $= \dfrac{7}{3} + 8\ln 2$

**25.**



Area $= (2e + \ln 2) - 2e^{1/2}$

**27.**



Area $= \frac{9}{2}$

**29.**



Area $= 18$

**31.**
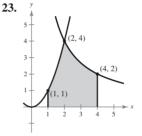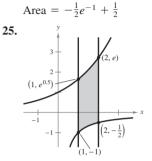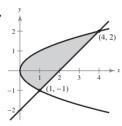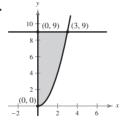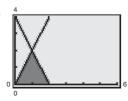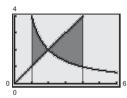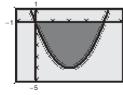


Area $= \int_0^1 2x\,dx + \int_1^2 (4 - 2x)\,dx$

**33.**
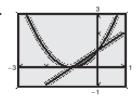


Area $= \int_1^2 \left( \frac{4}{x} - x \right) dx + \int_2^4 \left( x - \frac{4}{x} \right) dx$

**35.**



Area $= \frac{32}{3}$

**37.**
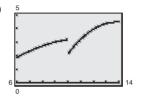


Area $= \frac{1}{6}$

**39.** 8

**41.** Consumer surplus $= 1600$

Producer surplus $= 400$

**43.** Consumer surplus $= 500$

Producer surplus $= 2000$

**45.** Offer 2 is better because the cumulative salary (area under the curve) is greater.

**47.** $R_1$, \$4.68 billion

**49.** \$300.6 million; Explanations will vary.

**51.** (a)



(b) 2.472 fewer pounds

**53.** Consumer surplus $=$ \$625,000

Producer surplus $=$ \$1,375,000

**55.** \$337.33 million

**57.**

| Quintile | Lowest | 2nd | 3rd | 4th | Highest |
|----------|--------|-----|-----|-----|---------|
| Percent | 2.81 | 6.98 | 14.57 | 27.01 | 45.73 |

**59.** Answers will vary.

## SECTION 5.6   *(page 407)*

**Skills Review**   *(page 407)*

**1.** $\frac{1}{6}$   **2.** $\frac{3}{20}$   **3.** $\frac{7}{40}$   **4.** $\frac{13}{12}$   **5.** $\frac{61}{30}$   **6.** $\frac{53}{18}$

**7.** $\frac{2}{3}$   **8.** $\frac{4}{7}$   **9.** 0   **10.** 5
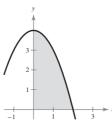
**1.** Midpoint Rule: 2

Exact area: 2

**3.** Midpoint Rule: 0.6730

Exact area: $\frac{2}{3} \approx 0.6667$

**5.** Midpoint Rule: 5.375

Exact area: $\frac{16}{3} \approx 5.333$
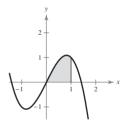
**7.** Midpoint Rule: 6.625

Exact area: $\frac{20}{3} \approx 6.667$





**9.** Midpoint Rule: 17.25

Exact area: $\frac{52}{3} \approx 17.33$
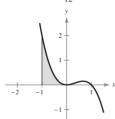
**11.** Midpoint Rule: 0.7578

Exact area: 0.75
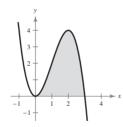
**13.** Midpoint Rule: 0.5703

Exact area: $\frac{7}{12} \approx 0.5833$



**15.** Midpoint Rule: 6.9609

Exact area: 6.75



**17.** Area $\approx 54.6667$,

$n = 31$

**19.** Area $\approx 4.16$,

$n = 5$

**21.** Area $\approx 0.9163$,

$n = 5$

**23.** Midpoint Rule: 1.5

Exact area: 1.5

**25.** Midpoint Rule: 25

Exact area: $\frac{76}{3} \approx 25.33$

**27.** Exact: 4

Trapezoidal Rule: 4.0625

Midpoint Rule: 3.9688

The Midpoint Rule is better in this example.

**29.** 1.1167     **31.** 1.55

**33.**

| $n$ | Midpoint Rule | Trapezoidal Rule |
|---|---|---|
| 4 | 15.3965 | 15.6055 |
| 8 | 15.4480 | 15.5010 |
| 12 | 15.4578 | 15.4814 |
| 16 | 15.4613 | 15.4745 |
| 20 | 15.4628 | 15.4713 |

**35.** 4.8103

**37.** Answers will vary. Sample answers:

(a) 966 ft$^2$     (b) 966 ft$^2$
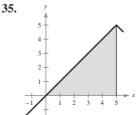
**39.** Midpoint Rule: 3.1468

Trapezoidal Rule: 3.1312

Graphing utility: 3.141593

# REVIEW EXERCISES FOR CHAPTER 5

**1.** $16x + C$     **3.** $\frac{2}{3}x^3 + \frac{5}{2}x^2 + C$     **5.** $x^{2/3} + C$

**7.** $\frac{3}{7}x^{7/3} + \frac{3}{2}x^2 + C$     **9.** $\frac{4}{9}x^{9/2} - 2\sqrt{x} + C$

**11.** $f(x) = \frac{3}{2}x^2 + x - 2$     **13.** $f(x) = \frac{1}{6}x^4 - 8x + \frac{33}{2}$

**15.** (a) 2.5 sec     (b) 100 ft

(c) 1.25 sec     (d) 75 ft

**17.** $x + 5x^2 + \frac{25}{3}x^3 + C$ or $\frac{1}{15}(1 + 5x)^3 + C_1$

**19.** $\frac{2}{5}\sqrt{5x - 1} + C$     **21.** $\frac{1}{2}x^2 - x^4 + C$

**23.** $\frac{1}{4}(x^4 - 2x)^2 + C$

**25.** (a) 30.5 board-feet     (b) 125.2 board-feet

**27.** $-e^{-3x} + C$     **29.** $\frac{1}{2}e^{x^2 - 2x} + C$

**31.** $-\frac{1}{3}\ln|1 - x^3| + C$     **33.** $\frac{2}{3}x^{3/2} + 2x + 2x^{1/2} + C$

**35.**
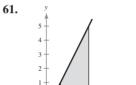


Area $= \frac{25}{2}$

**37.** $A = 4$     **39.** $A = \frac{32}{3}$     **41.** $A = \frac{8}{3}$     **43.** $A = 2 \ln 2$

**45.** (a) 13     (b) 7     (c) 11     (d) 50

**47.** 16     **49.** $\frac{422}{5}$     **51.** 0     **53.** 2

**55.** $\frac{1}{8}$     **57.** 3.899     **59.** 0

**61.**



Area $= 6$

**63.**
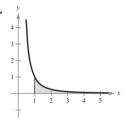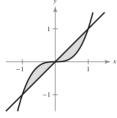


Area $= \frac{10}{3}$

**65.** Increases by $700.25

**67.** Average value: $\frac{2}{5}$; $x = \frac{25}{4}$

**69.** Average value: $\frac{1}{3}(-1 + e^3) \approx 6.362$; $x \approx 3.150$

**71.** $520.54; Explanations will vary.

**73.** (a) $B = -0.01955t^2 + 0.6108t - 1.818$

(b) According to the model, the price of beef per pound will never surpass $3.25. The highest price is approximately $2.95 per pound in 2005, and after that the prices decrease.

**75.** $17,492.94

**77.** $\int_{-2}^{2} 6x^5 \, dx = 0$

(Odd function)

**79.** $\int_{-2}^{-1} \frac{4}{x^2} \, dx = \int_{1}^{2} \frac{4}{x^2} \, dx = 2$

(Symmetric about $y$-axis)

**81.**



Area $= \frac{4}{5}$

**83.**



Area $= \frac{1}{2}$

**85.**



Area $= 16$

**87.**



Area $= \frac{64}{3}$

**89.**



Area $= \frac{9}{2}$
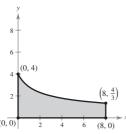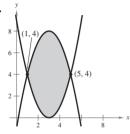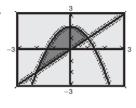
**91.** Consumer surplus: 11,250

Producer surplus: 14,062.5

**93.** About \$1236.39 million less

**95.** About \$11,237.24 million more

**97.** $n = 4$: 13.3203

$n = 20$: 13.7167

**99.** $n = 4$: 0.7867

$n = 20$: 0.7855

**101.** Answers will vary. Sample answer: 381.6 mi$^2$

## CHAPTER TEST    *(page 417)*

**1.** $3x^3 - 2x^2 + 13x + C$

**2.** $\dfrac{(x + 1)^3}{3} + C$

**3.** $\dfrac{2(x^4 - 7)^{3/2}}{3} + C$

**4.** $\dfrac{10x^{3/2}}{3} - 12x^{1/2} + C$

**5.** $5e^{3x} + C$

**6.** $\ln|x^3 - 11x| + C$

**7.** $f(x) = e^x + x$

**8.** $f(x) = \ln|x| + 2$

**9.** 8    **10.** 18    **11.** $\frac{2}{3}$
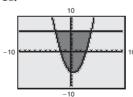
**12.** $2\sqrt{5} - 2\sqrt{2} \approx 1.644$    **13.** $\frac{1}{4}(e^{12} - 1) \approx 40{,}688.4$
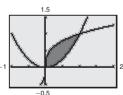
**14.** $\ln 6 \approx 1.792$

**15.** (a) $S = \dfrac{15.7}{0.23}e^{0.23t} + 1679.49$    (b) \$2748.08 million

**16.**



Area $= \frac{343}{6} \approx 57.167$

**17.**
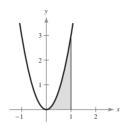


Area $= \frac{5}{12}$

**18.** Consumer surplus $= 20$ million

Producer surplus $= 8$ million
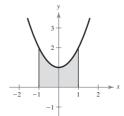
**19.** Midpoint Rule: $\frac{63}{64} \approx 0.9844$

Exact area: 1



**20.** Midpoint Rule: $\frac{21}{8} = 2.625$

Exact area: $\frac{8}{3} = 2.\overline{6}$



# CHAPTER 6

## SECTION 6.1    *(page 426)*

**Skills Review**    *(page 426)*

**1.** $\dfrac{1}{x + 1}$    **2.** $\dfrac{2x}{x^2 - 1}$    **3.** $3x^2 e^{x^3}$

**4.** $-2xe^{-x^2}$    **5.** $e^x(x^2 + 2x)$    **6.** $e^{-2x}(1 - 2x)$

**7.** $\frac{64}{3}$    **8.** $\frac{4}{3}$    **9.** 36    **10.** 8

**1.** $u = x; dv = e^{3x}\,dx$    **3.** $u = \ln 2x; dv = x\,dx$

**5.** $\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$    **7.** $-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

**9.** $x\ln 2x - x + C$    **11.** $\frac{1}{4}e^{4x} + C$

**13.** $\frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C$    **15.** $\frac{1}{2}e^{x^2} + C$

**17.** $-xe^{-x} - e^{-x} + C$    **19.** $2x^2e^x - 4e^x x + 4e^x + C$

**21.** $\frac{1}{2}t^2\ln(t + 1) - \frac{1}{2}\ln|t + 1| - \frac{1}{4}t^2 + \frac{1}{2}t + C$

**23.** $xe^x - 2e^x + C$    **25.** $-e^{1/t} + C$

**27.** $\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C$

**29.** $\frac{1}{3}(\ln x)^3 + C$    **31.** $-\frac{1}{x}(\ln x + 1) + C$

**33.** $\frac{2}{3}x(x - 1)^{3/2} - \frac{4}{15}(x - 1)^{5/2} + C$

**35.** $\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + C$    **37.** $\dfrac{e^{2x}}{4(2x + 1)} + C$

**39.** $e(2e - 1) \approx 12.060$    **41.** $-12e^{-2} + 4 \approx 2.376$
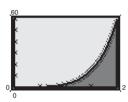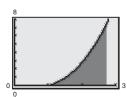
**43.** $\frac{5}{36}e^6 + \frac{1}{36} \approx 56.060$    **45.** $2\ln 2 - 1 \approx 0.386$

**47.** Area $= 2e^2 + 6 \approx 20.778$



**49.** Area $= \frac{1}{9}(2e^3 + 1) \approx 4.575$



**51.** Proof    **53.** $\dfrac{e^{5x}}{125}(25x^2 - 10x + 2) + C$
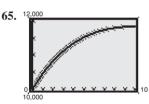
**55.** $-\dfrac{1}{x}(1 + \ln x) + C$    **57.** $1 - 5e^{-4} \approx 0.908$

**59.** $\frac{1}{4}(e^2 + 1) \approx 2.097$    **61.** $\frac{3}{128} - \frac{379}{128}e^{-8} \approx 0.022$

**63.** $\dfrac{1,171,875}{256}\pi \approx 14,381.070$

**65.**



(a) Increase   (b) 113,212 units   (c) 11,321 units/yr

**67.** (a) $3.2\ln 2 - 0.2 \approx 2.018$

(b) $12.8\ln 4 - 7.2\ln 3 - 1.8 \approx 8.035$

**69.** $\$18,482.03$    **71.** $\$931,265.10$    **73.** $\$4103.07$

**75.** (a) $\$1,200,000$    (b) $\$1,094,142.27$    **77.** $\$45,957.78$

**79.** (a) $\$17,378.62$    (b) $\$3681.26$    **81.** $4.254$

## SECTION 6.2   (page 436)

**Skills Review**   (page 436)

**1.** $(x - 4)(x + 4)$    **2.** $(x - 5)(x + 5)$

**3.** $(x - 4)(x + 3)$    **4.** $(x - 2)(x + 3)$

**5.** $x(x - 2)(x + 1)$    **6.** $x(x - 2)^2$

**7.** $(x - 2)(x - 1)^2$    **8.** $(x - 3)(x - 1)^2$

**9.** $x + \dfrac{1}{x - 2}$    **10.** $2x - 2 - \dfrac{1}{1 - x}$

**11.** $x^2 - x - 2 - \dfrac{2}{x - 2}$

**12.** $x^2 - x + 3 - \dfrac{4}{x + 1}$

**13.** $x + 4 + \dfrac{6}{x - 1},\quad x \neq -1$

**14.** $x + 3 + \dfrac{1}{x + 1},\quad x \neq 1$

**1.** $\dfrac{5}{x - 5} - \dfrac{3}{x + 5}$    **3.** $\dfrac{9}{x - 3} - \dfrac{1}{x}$    **5.** $\dfrac{1}{x - 5} + \dfrac{3}{x + 2}$

**7.** $\dfrac{3}{x} - \dfrac{5}{x^2}$    **9.** $\dfrac{1}{3(x - 2)} + \dfrac{1}{(x - 2)^2}$

**11.** $\dfrac{8}{x + 1} - \dfrac{1}{(x + 1)^2} + \dfrac{2}{(x + 1)^3}$    **13.** $\frac{1}{2}\ln\left|\dfrac{x - 1}{x + 1}\right| + C$

**15.** $\frac{1}{4}\ln\left|\dfrac{x + 4}{x - 4}\right| + C$    **17.** $\ln\left|\dfrac{2x - 1}{x}\right| + C$

**19.** $\ln\left|\dfrac{x - 10}{x}\right| + C$    **21.** $\ln\left|\dfrac{x - 1}{x + 2}\right| + C$

**23.** $\frac{3}{2}\ln|2x - 1| - 2\ln|x + 1| + C$

**25.** $\ln\left|\dfrac{x(x + 2)}{x - 2}\right| + C$    **27.** $\frac{1}{2}(3\ln|x - 4| - \ln|x|) + C$

**29.** $2\ln|x - 1| + \dfrac{1}{x - 1} + C$

**31.** $\ln|x| + 2\ln|x + 1| + \dfrac{1}{x + 1} + C$

**33.** $\frac{1}{6}\ln\frac{4}{7} \approx -0.093$    **35.** $-\frac{4}{5} + 2\ln\frac{5}{3} \approx 0.222$
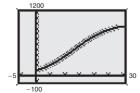
**37.** $\frac{1}{2} - \ln 2 \approx -0.193$    **39.** $4\ln 2 + \frac{1}{2} \approx 3.273$

**41.** $12 - \frac{7}{2}\ln 7 \approx 5.189$    **43.** $5\ln 2 - \ln 5 \approx 1.856$

**45.** $24 \ln 3 - 36 \ln 2 \approx 1.413$

**47.** $\dfrac{1}{2a}\left(\dfrac{1}{a + x} + \dfrac{1}{a - x}\right)$    **49.** $\dfrac{1}{a}\left(\dfrac{1}{x} + \dfrac{1}{a - x}\right)$

**51.** Divide $x^2$ by $(x - 5)$ because the degree of the numerator is greater than the degree of the denominator.

**53.** $y = \dfrac{1000}{1 + 9e^{-0.1656t}}$



**55.** $1.077$ thousand    **57.** $11,408$ million; $1426$ million

**59.** The rate of growth is increasing on $[0, 3]$ for *P. aurelia* and on $[0, 2]$ for *P. caudatum*; the rate of growth is decreasing on $[3, \infty)$ for *P. aurelia* and on $[2, \infty)$ for *P. caudatum*; *P. aurelia* has a higher limiting population.

**61.** Answers will vary.

## SECTION 6.3  *(page 447)*

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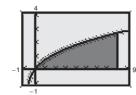
### Skills Review  *(page 447)*

**1.** $x^2 + 8x + 16$    **2.** $x^2 - 2x + 1$

**3.** $x^2 + x + \frac{1}{4}$    **4.** $x^2 - \frac{2}{3}x + \frac{1}{9}$

**5.** $\dfrac{2}{x} - \dfrac{2}{x + 2}$    **6.** $-\dfrac{3}{4x} + \dfrac{3}{4(x - 4)}$

**7.** $\dfrac{3}{2(x - 2)} - \dfrac{2}{x^2} - \dfrac{3}{2x}$    **8.** $\dfrac{4}{x} - \dfrac{3}{x + 1} + \dfrac{2}{x - 2}$

**9.** $2e^x(x - 1) + C$    **10.** $x^3 \ln x - \dfrac{x^3}{3} + C$

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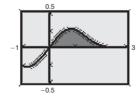
**1.** $\dfrac{1}{9}\left(\dfrac{2}{2 + 3x} + \ln|2 + 3x|\right) + C$

**3.** $\dfrac{2(3x - 4)}{27}\sqrt{2 + 3x} + C$    **5.** $\ln\left(x^2 + \sqrt{x^4 - 9}\right) + C$

**7.** $\dfrac{1}{2}(x^2 - 1)e^{x^2} + C$    **9.** $\ln\left|\dfrac{x}{1 + x}\right| + C$

**11.** $-\dfrac{1}{3}\ln\left|\dfrac{3 + \sqrt{x^2 + 9}}{x}\right| + C$

**13.** $-\dfrac{1}{2}\ln\left|\dfrac{2 + \sqrt{4 - x^2}}{x}\right| + C$

**15.** $\frac{1}{4}x^2(-1 + 2 \ln x) + C$    **17.** $3x^2 - \ln(1 + e^{3x^2}) + C$

**19.** $\frac{1}{4}\left(x^2\sqrt{x^4 - 4} - 4\ln|x^2 + \sqrt{x^4 - 4}|\right) + C$
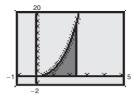
**21.** $\dfrac{1}{27}\left[\dfrac{4}{2 + 3t} - \dfrac{2}{(2 + 3t)^2} + \ln|2 + 3t|\right] + C$

**23.** $\dfrac{\sqrt{3}}{3}\ln\left|\dfrac{\sqrt{3 + s} - \sqrt{3}}{\sqrt{3 + s} + \sqrt{3}}\right| + C$

**25.** $-\frac{1}{2}x(2 - x) + \ln|x + 1| + C$

**27.** $\dfrac{1}{8}\left[\dfrac{-1}{2(3 + 2x)^2} + \dfrac{2}{(3 + 2x)^3} - \dfrac{9}{4(3 + 2x)^4}\right] + C$

**29.** $-\dfrac{\sqrt{1 - x^2}}{x} + C$    **31.** $\dfrac{1}{9}x^3(-1 + 3\ln x) + C$

**33.** $\dfrac{1}{27}\left(3x - \dfrac{25}{3x - 5} + 10\ln|3x - 5|\right) + C$

**35.** $\frac{1}{9}(3\ln x - 4\ln|4 + 3\ln x|) + C$

**37.** Area $= \frac{40}{3}$
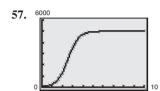


Area $= 13.\overline{3}$

**39.** Area $= \dfrac{1}{2}\left[4 + \ln\left(\dfrac{2}{1 + e^4}\right)\right]$



Area $\approx 0.3375$

**41.** Area $= \frac{1}{4}\left[21\sqrt{5} - 8\ln\left(\sqrt{5} + 3\right) + 8\ln 2\right]$



Area $\approx 9.8145$

**43.** $\dfrac{-2\sqrt{2} + 4}{3} \approx 0.3905$    **45.** $-\dfrac{5}{9} + \ln\dfrac{9}{4} \approx 0.2554$

**47.** $12\left(2 + \ln\left|\dfrac{2}{1 + e^2}\right|\right) \approx 6.7946$

**49.** $-\frac{15}{4} + 8\ln 4 \approx 7.3404$    **51.** $(x^2 - 2x + 2)e^x + C$

**53.** $-\left(\dfrac{1}{x} + \ln\left|\dfrac{x}{x + 1}\right|\right) + C$    **55.** (a) $0.483$    (b) $0.283$

**57.**



Average value: 401.40

**59.** $1138.43     **61.** $0.50 billion/yr

## MID-CHAPTER QUIZ   (page 449)

**1.** $\frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C$     **2.** $3x \ln x - 3x + C$

**3.** $\frac{1}{2}x^2 \ln x + x \ln x - \frac{1}{4}x^2 - x + C$

**4.** $\frac{2}{3}x(x + 3)^{3/2} - \frac{4}{15}(x + 3)^{5/2} + C$

**5.** $\frac{x^2}{4} \ln x - \frac{x^2}{8} + C$     **6.** $-\frac{1}{2}e^{-2x}\left(x^2 + x + \frac{1}{2}\right) + C$

**7.** Yes, $673,108.31 > $650,000.

**8.** $\ln\left|\frac{x - 5}{x + 5}\right| + C$     **9.** $3 \ln|x + 4| - 2 \ln|x - 2| + C$

**10.** $5 \ln|x + 1| + \frac{6}{x + 1} + C$     **11.** $y = \frac{100,000}{1 + 3e^{-0.01186t}}$

**12.** $\frac{1}{4}(2x - \ln|1 + 2x|) + C$     **13.** $10 \ln\left|\frac{x}{0.1 + 0.2x}\right| + C$

**14.** $\ln\left|x + \sqrt{x^2 - 16}\right| - \frac{\sqrt{x^2 - 16}}{x} + C$

**15.** $\frac{1}{2} \ln\left|\frac{\sqrt{4 + 9x} - 2}{\sqrt{4 + 9x} + 2}\right| + C$

**16.** $\frac{1}{4}[4x^2 - \ln(1 + e^{4x^2})] + C$     **17.** $x^2e^{x^2+1} + C$

**18.** About 515 stores     **19.** $\frac{8}{e} - 4 \approx -1.0570$

**20.** $e - 2 \approx 0.7183$     **21.** $\ln 4 + 2 \ln 5 - 2 \ln 2 \approx 3.2189$

**22.** $15(\ln 9 - \ln 5) \approx 8.8168$

**23.** $\frac{\sqrt{5}}{18} \approx 0.1242$     **24.** $\frac{1}{4}\left(\ln\frac{17}{19} - \ln\frac{7}{9}\right) \approx 0.0350$

## SECTION 6.4   (page 456)

### Skills Review   (page 456)

**1.** $\frac{2}{x^3}$     **2.** $-\frac{96}{(2x + 1)^4}$     **3.** $-\frac{12}{x^4}$     **4.** $6x - 4$

**5.** $16e^{2x}$     **6.** $e^{x^2}(4x^2 + 2)$     **7.** $(3, 18)$

**8.** $(1, 8)$     **9.** $n < -5\sqrt{10},\ n > 5\sqrt{10}$

**10.** $n < -5, n > 5$

| | Exact Value | Trapezoidal Rule | Simpson's Rule |
|---|---|---|---|
| **1.** | 2.6667 | 2.7500 | 2.6667 |
| **3.** | 8.4000 | 9.0625 | 8.4167 |
| **5.** | 4.0000 | 4.0625 | 4.0000 |
| **7.** | 0.6931 | 0.6941 | 0.6932 |
| **9.** | 5.3333 | 5.2650 | 5.3046 |
| **11.** | 12.6667 | 12.6640 | 12.6667 |
| **13.** | 0.6931 | 0.6970 | 0.6933 |

**15.** (a) 0.783     (b) 0.785     **17.** (a) 3.283     (b) 3.240

**19.** (a) 0.749     (b) 0.771     **21.** (a) 0.877     (b) 0.830

**23.** (a) 1.879     (b) 1.888

**25.** $21,831.20; $21,836.98     **27.** $678.36

**29.** $0.3413 = 34.13\%$     **31.** $0.4999 = 49.99\%$

**33.** 89,500 ft²

**35.** (a) $|E| \leq 0.5$     (b) $|E| = 0$

**37.** (a) $|E| \leq \frac{5e}{64} \approx 0.212$     (b) $|E| \leq \frac{13e}{1024} \approx 0.035$

**39.** (a) $n = 71$     (b) $n = 1$

**41.** (a) $n = 3280$     (b) $n = 60$     **43.** 19.5215

**45.** 3.6558     **47.** 23.375     **49.** 416.1 ft

**51.** (a) 17.171 billion board-feet/yr

     (b) 17.082 billion board-feet/yr

     (c) The results are approximately equal.

**53.** 58.912 mg     **55.** 1878 subscribers

## SECTION 6.5   (page 468)

### Skills Review   (page 468)

**1.** 9     **2.** 3     **3.** $-\frac{1}{8}$     **4.** Limit does not exist.

**5.** Limit does not exist.     **6.** $-4$

**7.** (a) $\frac{32}{3}b^3 - 16b^2 + 8b - \frac{4}{3}$     (b) $-\frac{4}{3}$

**8.** (a) $\frac{b^2 - b - 11}{(b - 2)^2(b - 5)}$     (b) $\frac{11}{20}$

**9.** (a) $\ln\left(\frac{5 - 3b^2}{b + 1}\right)$     (b) $\ln 5 \approx 1.609$

**10.** (a) $e^{-3b^2}(e^{6b^2} + 1)$     (b) 2

**1.** Improper; The integrand has an infinite discontinuity when $x = \frac{2}{3}$ and $0 \leq \frac{2}{3} \leq 1$.

**3.** Not improper; continuous on $[0, 1]$

**5.** Improper because the integrand has an infinite discontinuity when $x = 0$ and $0 \leq 0 \leq 4$; converges; 4

**7.** Improper because the integrand has an infinite discontinuity when $x = 1$ and $0 \le 1 \le 2$; converges; 6

**9.** Improper because the upper limit of integration is infinite; converges; 1

**11.** Converges; 1     **13.** Diverges     **15.** Diverges

**17.** Diverges     **19.** Diverges     **21.** Converges; 0

**23.** Diverges     **25.** Converges; 6     **27.** Diverges

**29.** Converges; 0     **31.** Converges; $\ln\left(\dfrac{4 + \sqrt{7}}{3}\right) \approx 0.7954$

**33.** 1

**35.**

| $x$ | 1 | 10 | 25 | 50 |
|-----|---|----|----|----|
| $xe^{-x}$ | 0.3679 | 0.0005 | 0.0000 | 0.0000 |

**37.**

| $x$ | 1 | 10 | 25 | 50 |
|-----|---|----|----|----|
| $x^2 e^{-(1/2)x}$ | 0.6065 | 0.6738 | 0.0023 | 0.0000 |

**39.** 2     **41.** $\frac{1}{4}$

**43.** (a) 0.9495     (b) 0.0974     (c) 0.0027

**45.** $66,666.67     **47.** Yes, $360,000 < $400,000.

**49.** (a) $4,637,228     (b) $5,555,556

**51.** (a) $748,367.34     (b) $808,030.14     (c) $900,000.00

## REVIEW EXERCISES FOR CHAPTER 6
(page 474)

**1.** $2\sqrt{x} \ln x - 4\sqrt{x} + C$     **3.** $xe^x + C$

**5.** $x^2 e^{2x} - xe^{2x} + \frac{1}{2}e^{2x} + C$     **7.** $90,634.62

**9.** $865,958.50

**11.** (a) $8847.97, $7869.39, $7035.11     (b) $1,995,258.71

**13.** $90,237.67     **15.** $\dfrac{1}{5}\ln\left|\dfrac{x}{x+5}\right| + C$

**17.** $6\ln|x + 2| - 5\ln|x - 3| + C$

**19.** $x - \frac{25}{8}\ln|x + 5| + \frac{9}{8}\ln|x - 3| + C$

**21.** (a) $y = \dfrac{10,000}{1 + 7e^{-0.106873t}}$

(b)

| Time, $t$ | 0 | 3 | 6 | 12 | 24 |
|-----------|---|---|---|----|----|
| Sales, $y$ | 1250 | 1645 | 2134 | 3400 | 6500 |

(c) $t \approx 28$ weeks

**23.** $\dfrac{1}{9}\left(\dfrac{2}{2 + 3x} + \ln|2 + 3x|\right) + C$

**25.** $\sqrt{x^2 + 25} - 5\ln\left|\dfrac{5 + \sqrt{x^2 + 25}}{x}\right| + C$

**27.** $\dfrac{1}{4}\ln\left|\dfrac{x - 2}{x + 2}\right| + C$     **29.** $\dfrac{8}{3}$

**31.** $2\sqrt{1 + x} + \ln\left|\dfrac{\sqrt{1 + x} - 1}{\sqrt{1 + x} + 1}\right| + C$

**33.** $(x - 5)^3 e^{x-5} - 3(x - 5)^2 e^{x-5} + 6(x - 6)e^{x-5} + C$

**35.** (a) 0.675     (b) 0.290     **37.** 0.705     **39.** 0.741

**41.** 0.376     **43.** 0.289     **45.** 9.0997     **47.** 0.017

**49.** Converges; 1     **51.** Diverges

**53.** Converges; 2     **55.** Converges; 2

**57.** (a) $989,050.57     (b) $1,666,666.67

**59.** (a) 0.441     (b) 0.119     (c) 0.015

## CHAPTER TEST     (page 477)

**1.** $xe^{x+1} - e^{x+1} + C$     **2.** $3x^3 \ln x - x^3 + C$

**3.** $-3x^2 e^{-x/3} - 18xe^{-x/3} - 54e^{-x/3} + C$

**4.** $1.95 per share     **5.** $\ln\left|\dfrac{x - 9}{x + 9}\right| + C$

**6.** $\dfrac{1}{3}\ln|3x + 1| + \dfrac{1}{3(3x + 1)} + C$

**7.** $2\ln|x| - \ln|x + 2| + C$

**8.** $\dfrac{1}{4}\left(\dfrac{7}{7 + 2x} + \ln|7 + 2x|\right) + C$

**9.** $x^3 - \ln|1 + e^{x^3}| + C$

**10.** $-\dfrac{2}{75}(2 - 5x^2)\sqrt{1 + 5x^2} + C$

**11.** $-1 + \frac{3}{2}\ln 3 \approx 0.6479$     **12.** $4\ln\left(\frac{48}{13}\right) \approx 5.2250$

**13.** $4\ln\left[3\left(\sqrt{17} - 4\right)\right] + \sqrt{17} - 5 \approx -4.8613$

**14.** Trapezoid Rule: 0.2100; Exact: 0.2055

**15.** Simpson Rule: 41.3606; Exact: 41.1711

**16.** Converges; $\frac{1}{3}$     **17.** Converges; 12     **18.** Diverges

**19.** (a) $498.75     (b) Plan B, because $149 < $498.75.
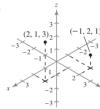
# CHAPTER 7

## SECTION 7.1     (page 485)
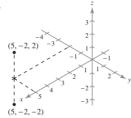
> **Skills Review**     (page 485)
>
> **1.** $2\sqrt{5}$     **2.** 5     **3.** 8     **4.** 8     **5.** $(4, 7)$
>
> **6.** $(1, 0)$     **7.** $(0, 3)$     **8.** $(-1, 1)$
>
> **9.** $(x - 2)^2 + (y - 3)^2 = 4$
>
> **10.** $(x - 1)^2 + (y - 4)^2 = 25$

**1.**

**3.**

**5.** $A(2, 3, 4)$, $B(-1, -2, 2)$    **7.** $(-3, 4, 5)$

**9.** $(10, 0, 0)$    **11.** $0$    **13.** $3\sqrt{2}$    **15.** $\sqrt{206}$

**17.** $(2, -5, 3)$    **19.** $\left(\frac{1}{2}, \frac{1}{2}, -1\right)$    **21.** $(6, -3, 5)$

**23.** $(1, 2, 1)$    **25.** $3, 3\sqrt{5}, 6$; right triangle

**27.** $2, 2\sqrt{5}, 2\sqrt{2}$; neither right nor isosceles

**29.** $(0, 0, 5)$, $(2, 2, 6)$, $(2, -4, 9)$

**31.** $x^2 + (y - 2)^2 + (z - 2)^2 = 4$

**33.** $\left(x - \frac{3}{2}\right)^2 + (y - 2)^2 + (z - 1)^2 = \frac{21}{4}$

**35.** $(x - 1)^2 + (y - 1)^2 + (z - 5)^2 = 9$

**37.** $(x - 1)^2 + (y - 3)^2 + z^2 = 10$

**39.** $(x + 2)^2 + (y - 1)^2 + (z - 1)^2 = 1$

**41.** Center: $\left(\frac{5}{2}, 0, 0\right)$    **43.** Center: $(1, -3, -4)$
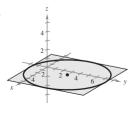
Radius: $\frac{5}{2}$    Radius: $5$
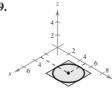
**45.** Center: $(1, 3, 2)$

Radius: $\dfrac{5\sqrt{2}}{2}$

**47.**

**49.**

**51.**

**53.**

**55.** (a)    (b)

**57.** (a)    (b)
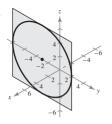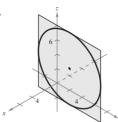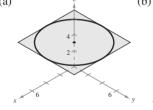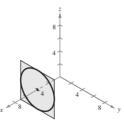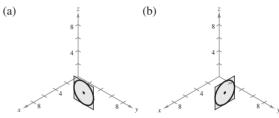
**59.** $(3, 3, 3)$    **61.** $x^2 + y^2 + z^2 = 6806.25$

## SECTION 7.2    *(page 494)*

**Skills Review**    *(page 494)*

**1.** $(4, 0)$, $(0, 3)$    **2.** $\left(-\frac{4}{3}, 0\right)$, $(0, -8)$

**3.** $(1, 0)$, $(0, -2)$    **4.** $(-5, 0)$, $(0, -5)$

**5.** $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 + 1 = 0$

**6.** $(x - 4)^2 + (y + 2)^2 - (z + 3)^2 = 0$
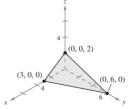
**7.** $(x + 1)^2 + (y - 1)^2 - z = 0$

**8.** $(x - 3)^2 + (y + 5)^2 + (z + 13)^2 = 1$
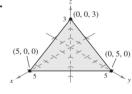
**9.** $x^2 + y^2 + z^2 = \frac{1}{4}$    **10.** $x^2 + y^2 + z^2 = 4$
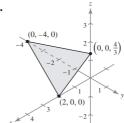
**1.**
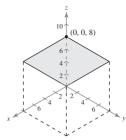
**3.**
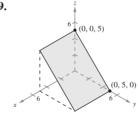
**5.**
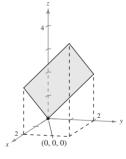
**7.**

**9.**

**11.**

**13.** $\dfrac{6\sqrt{14}}{7}$    **15.** $\dfrac{8\sqrt{14}}{7}$    **17.** $\dfrac{13\sqrt{29}}{29}$    **19.** $\dfrac{28\sqrt{29}}{29}$

**21.** Perpendicular    **23.** Parallel    **25.** Parallel

**27.** Neither parallel nor perpendicular    **29.** Perpendicular

**31.** c    **32.** e    **33.** f    **34.** b    **35.** d    **36.** a

**37.** Trace in $xy$-plane $(z = 0)$: $y = x^2$ (parabola)

Trace in plane $y = 1$: $x^2 - z^2 = 1$ (hyperbola)

Trace in $yz$-plane $(x = 0)$: $y = -z^2$ (parabola)

**39.** Trace in $xy$-plane $(z = 0)$: $\dfrac{x^2}{4} + y^2 = 1$ (ellipse)

Trace in $xz$-plane $(y = 0)$: $\dfrac{x^2}{4} + z^2 = 1$ (ellipse)

Trace in $yz$-plane $(x = 0)$: $y^2 + z^2 = 1$ (circle)

**41.** Ellipsoid    **43.** Hyperboloid of one sheet

**45.** Elliptic paraboloid    **47.** Hyperbolic paraboloid

**49.** Hyperboloid of two sheets    **51.** Elliptic cone

**53.** Hyperbolic paraboloid

**55.** $(20, 0, 0)$    **57.** $(0, 0, 20)$

**59.** (a)

| Year | 1999 | 2000 | 2001 |
|---|---|---|---|
| $x$ | 6.2 | 6.1 | 5.9 |
| $y$ | 7.3 | 7.1 | 7.0 |
| $z$ (actual) | 7.8 | 7.7 | 7.4 |
| $z$ (approximated) | 7.8 | 7.7 | 7.5 |

| Year | 2002 | 2003 | 2004 |
|---|---|---|---|
| $x$ | 5.8 | 5.6 | 5.5 |
| $y$ | 7.0 | 6.9 | 6.9 |
| $z$ (actual) | 7.3 | 7.2 | 6.9 |
| $z$ (approximated) | 7.3 | 7.1 | 7.0 |

The approximated values of $z$ are very close to the actual values.

(b) According to the model, increases in consumption of milk types $y$ and $z$ will correspond to an increase in consumption of milk type $x$.
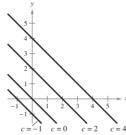
## SECTION 7.3    *(page 502)*

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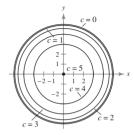
**Skills Review**    *(page 502)*

**1.** 11    **2.** $-16$    **3.** 7

**4.** 4    **5.** $(-\infty, \infty)$

**6.** $(-\infty, -3) \cup (-3, 0) \cup (0, \infty)$

**7.** $[5, \infty)$    **8.** $\left(-\infty, -\sqrt{5}\,\right] \cup \left[\sqrt{5}, \infty\right)$
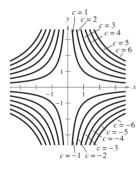
**9.** 55.0104    **10.** 6.9165

---

**1.** (a) $\dfrac{3}{2}$    (b) $-\dfrac{1}{4}$    (c) 6    (d) $\dfrac{5}{y}$    (e) $\dfrac{x}{2}$    (f) $\dfrac{5}{t}$

**3.** (a) 5    (b) $3e^2$    (c) $2e^{-1}$

(d) $5e^y$    (e) $xe^2$    (f) $te^t$

**5.** (a) $\frac{2}{3}$    (b) 0    **7.** (a) $90\pi$    (b) $50\pi$

**9.** (a) \$20,655.20    (b) \$1,397,672.67

**11.** (a) 0    (b) 6

**13.** (a) $x^2 + 2x\,\Delta x + (\Delta x)^2 - 2y$    (b) $-2$, $\Delta y \neq 0$

**15.** Domain: all points $(x, y)$ inside and on the circle

$x^2 + y^2 = 16$

Range: $[0, 4]$

**17.** Domain: all points $(x, y)$ such that $y \neq 0$

Range: $(0, \infty)$

**19.** All points inside and on the circle $x^2 + y^2 = 4$

**21.** All points $(x, y)$

**23.** All points $(x, y)$ such that $x \neq 0$ and $y \neq 0$

**25.** All points $(x, y)$ such that $y \geq 0$

**27.** The half-plane below the line $y = -x + 4$

**29.** b    **30.** d    **31.** a    **32.** c

**33.** The level curves are parallel lines.    **35.** The level curves are circles.
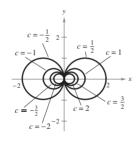
**37.** The level curves are hyperbolas.



**39.** The level curves are circles.



**41.** 135,540 units    **43.** (a) $15,250    (b) $18,425

**45.**

| $\dfrac{I}{R}$ | 0 | 0.03 | 0.05 |
|---|---|---|---|
| 0 | $2593.74 | $1929.99 | $1592.33 |
| 0.28 | $2004.23 | $1491.34 | $1230.42 |
| 0.35 | $1877.14 | $1396.77 | $1152.40 |

**47.** (a) C    (b) A    (c) B

**49.** (a) $.663 earnings per share

(b) $x$; Explanations will vary. Sample answer: The $x$-variable has a greater influence on the earnings per share because the absolute value of its coefficient is larger than the absolute value of the coefficient of the $y$-term.

**51.** Answers will vary.

## SECTION 7.4   (page 513)

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### Skills Review    (page 513)

**1.** $\dfrac{x}{\sqrt{x^2 + 3}}$    **2.** $-6x(3 - x^2)^2$    **3.** $e^{2t+1}(2t + 1)$

**4.** $\dfrac{e^{2x}(2 - 3e^{2x})}{\sqrt{1 - e^{2x}}}$    **5.** $-\dfrac{2}{3 - 2x}$    **6.** $\dfrac{3(t^2 - 2)}{2t(t^2 - 6)}$

**7.** $-\dfrac{10x}{(4x - 1)^3}$    **8.** $-\dfrac{(x + 2)^2(x^2 + 8x + 27)}{(x^2 - 9)^3}$

**9.** $f'(2) = 8$    **10.** $g'(2) = \frac{7}{2}$

---

**1.** $\dfrac{\partial z}{\partial x} = 3$; $\dfrac{\partial z}{\partial y} = 5$

**3.** $f_x(x, y) = 3$; $f_y(x, y) = -12y$

**5.** $f_x(x, y) = \dfrac{1}{y}$; $f_y(x, y) = -\dfrac{x}{y^2}$

**7.** $f_x(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}$; $f_y(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}$

**9.** $\dfrac{\partial z}{\partial x} = 2xe^{2y}$; $\dfrac{\partial z}{\partial y} = 2x^2e^{2y}$

**11.** $h_x(x, y) = -2xe^{-(x^2+y^2)}$; $h_y(x, y) = -2ye^{-(x^2+y^2)}$

**13.** $\dfrac{\partial z}{\partial x} = -\dfrac{2y}{x^2 - y^2}$; $\dfrac{\partial z}{\partial y} = \dfrac{2x}{x^2 - y^2}$

**15.** $f_x(x, y) = 3xye^{x-y}(2 + x)$

**17.** $g_x(x, y) = 3y^2e^{y-x}(1 - x)$     **19.** 9

**21.** $f_x(x, y) = 6x + y$, 13; $f_y(x, y) = x - 2y$, 0

**23.** $f_x(x, y) = 3ye^{3xy}$, 12; $f_y(x, y) = 3xe^{3xy}$, 0

**25.** $f_x(x, y) = -\dfrac{y^2}{(x - y)^2}, -\dfrac{1}{4}$; $f_y(x, y) = \dfrac{x^2}{(x - y)^2}, \dfrac{1}{4}$

**27.** $f_x(x, y) = \dfrac{2x}{x^2 + y^2}$, 2; $f_y(x, y) = \dfrac{2y}{x^2 + y^2}$, 0

**29.** $w_x = yz$

$w_y = xz$

$w_z = xy$

**31.** $w_x = -\dfrac{2z}{(x + y)^2}$

$w_y = -\dfrac{2z}{(x + y)^2}$

$w_z = \dfrac{2}{x + y}$

**33.** $w_x = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}, \dfrac{2}{3}$

$w_y = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}, -\dfrac{1}{3}$

$w_z = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}, \dfrac{2}{3}$

**35.** $w_x = \dfrac{x}{x^2 + y^2 + z^2}, \dfrac{3}{25}$

$w_y = \dfrac{y}{x^2 + y^2 + z^2}, 0$

$w_z = \dfrac{z}{x^2 + y^2 + z^2}, \dfrac{4}{25}$

**37.** $w_x = 2z^2 + 3yz$, 2

$w_y = 3xz - 12yz$, 30

$w_z = 4xz + 3xy - 6y^2$, $-1$

**39.** $(-6, 4)$     **41.** $(1, 1)$

**43.** (a) 2    (b) 1     **45.** (a) $-2$    (b) $-2$

**47.** $\dfrac{\partial^2 z}{\partial x^2} = 2$

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = -2$

$\dfrac{\partial^2 z}{\partial y^2} = 6$

**49.** $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{e^{2xy}(2x^2y^2 - 2xy + 1)}{2x^3}$

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = ye^{2xy}$

$\dfrac{\partial^2 z}{\partial y^2} = xe^{2xy}$

**51.** $\dfrac{\partial^2 z}{\partial x^2} = 6x$

$\dfrac{\partial^2 z}{\partial y^2} = -8$

$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = 0$

**53.** $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{2}{(x-y)^3}$

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{-2}{(x-y)^3}$

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{2}{(x-y)^3}$

**55.** $f_{xx}(x, y) = 12x^2 - 6y^2,\ 12$

$f_{xy}(x, y) = -12xy,\ 0$

$f_{yy}(x, y) = -6x^2 + 2,\ -4$

$f_{yx}(x, y) = -12xy,\ 0$

**57.** $f_{xx}(x, y) = -\dfrac{1}{(x-y)^2},\ -1$

$f_{xy}(x, y) = \dfrac{1}{(x-y)^2},\ 1$

$f_{yy}(x, y) = -\dfrac{1}{(x-y)^2},\ -1$

$f_{yx}(x, y) = \dfrac{1}{(x-y)^2},\ 1$

**59.** (a) At $(120, 160)$, $\dfrac{\partial C}{\partial x} \approx 154.77$

At $(120, 160)$, $\dfrac{\partial C}{\partial y} \approx 193.33$

(b) Racing bikes; Explanations will vary. Sample answer: The $y$-variable has a greater influence on the cost because the absolute value of its coefficient is larger than the absolute value of the coefficient of the $x$-term.

**61.** (a) About 113.72    (b) About 97.47

**63.** Complementary

**65.** (a) $\dfrac{\partial z}{\partial x} = 1.25;\ \dfrac{\partial z}{\partial y} = -0.125$

(b) For every increase of 1.25 gallons of whole milk, there is an increase of one gallon of reduced-fat (1%) and skim milks. For every decrease of 0.125 gallon of whole milk, there is an increase of one gallon of reduced-fat (2%) milk.

**67.** $IQ_M(M, C) = \dfrac{100}{C},\ IQ_M(12, 10) = 10$; For a child that has a current mental age of 12 years and chronological age of 10 years, the IQ is increasing at a rate of 10 IQ points for every increase of 1 year in the child's mental age.

$IQ_C(M, C) = \dfrac{-100M}{C^2},\ IQ_C(12, 10) = -12$; For a child that has a current mental age of 12 years and chronological age of 10 years, the IQ is decreasing at a rate of 12 IQ points for every increase of 1 year in the child's chronological age.

**69.** An increase in either price will cause a decrease in the number of applicants.

**71.** Answers will vary.

## SECTION 7.5   (page 522)

### Skills Review   (page 522)

**1.** $(3, 2)$    **2.** $(11, 6)$    **3.** $(1, 4)$    **4.** $(4, 4)$

**5.** $(5, 2)$    **6.** $(3, -2)$    **7.** $(0, 0), (-1, 0)$

**8.** $(-2, 0), (2, -2)$

**9.** $\dfrac{\partial z}{\partial x} = 12x^2$    $\dfrac{\partial^2 z}{\partial y^2} = -6$

$\dfrac{\partial z}{\partial y} = -6y$    $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

$\dfrac{\partial^2 z}{\partial x^2} = 24x$    $\dfrac{\partial^2 z}{\partial y \partial x} = 0$

**10.** $\dfrac{\partial z}{\partial x} = 10x^4$    $\dfrac{\partial^2 z}{\partial y^2} = -6y$

$\dfrac{\partial z}{\partial y} = -3y^2$    $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

$\dfrac{\partial^2 z}{\partial x^2} = 40x^3$    $\dfrac{\partial^2 z}{\partial y \partial x} = 0$

**11.** $\dfrac{\partial z}{\partial x} = 4x^3 - \dfrac{\sqrt{xy}}{2x}$    $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\sqrt{xy}}{4y^2}$

$\dfrac{\partial z}{\partial y} = -\dfrac{\sqrt{xy}}{2y} + 2$    $\dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{\sqrt{xy}}{4xy}$

$\dfrac{\partial^2 z}{\partial x^2} = 12x^2 + \dfrac{\sqrt{xy}}{4x^2}$    $\dfrac{\partial^2 z}{\partial y \partial x} = -\dfrac{\sqrt{xy}}{4xy}$

**12.** $\dfrac{\partial z}{\partial x} = 4x - 3y$    $\dfrac{\partial^2 z}{\partial y^2} = 2$

$\dfrac{\partial z}{\partial y} = 2y - 3x$    $\dfrac{\partial^2 z}{\partial x \partial y} = -3$

$\dfrac{\partial^2 z}{\partial x^2} = 4$    $\dfrac{\partial^2 z}{\partial y \partial x} = -3$

**13.** $\dfrac{\partial z}{\partial x} = y^3 e^{xy^2}$    $\dfrac{\partial^2 z}{\partial y^2} = 4x^2 y^3 e^{xy^2} + 6xy e^{xy^2}$

$\dfrac{\partial z}{\partial y} = 2xy^2 e^{xy^2} + e^{xy^2}$    $\dfrac{\partial^2 z}{\partial x \partial y} = 2xy^4 e^{xy^2} + 3y^2 e^{xy^2}$

$\dfrac{\partial^2 z}{\partial x^2} = y^5 e^{xy^2}$    $\dfrac{\partial^2 z}{\partial y \partial x} = 2xy^4 e^{xy^2} + 3y^2 e^{xy^2}$

**14.** $\dfrac{\partial z}{\partial x} = e^{xy}(xy + 1)$    $\dfrac{\partial^2 z}{\partial y^2} = x^3 e^{xy}$

$\dfrac{\partial z}{\partial y} = x^2 e^{xy}$    $\dfrac{\partial^2 z}{\partial x \partial y} = xe^{xy}(xy + 2)$

$\dfrac{\partial^2 z}{\partial x^2} = ye^{xy}(xy + 2)$    $\dfrac{\partial^2 z}{\partial y \partial x} = xe^{xy}(xy + 2)$

**1.** Critical point: $(-2, -4)$

No relative extrema

$(-2, -4, 1)$ is a saddle point.

**3.** Critical point: $(0, 0)$

Relative minimum: $(0, 0, 1)$

**5.** Relative minimum: $(1, 3, 0)$

**7.** Relative minimum: $(-1, 1, -4)$

**9.** Relative maximum: $(8, 16, 74)$

**11.** Relative minimum: $(2, 1, -7)$

**13.** Saddle point: $(-2, -2, -8)$

**15.** Saddle point: $(0, 0, 0)$

**17.** Relative maximum: $\left(\dfrac{1}{2}, \dfrac{1}{2}, e^{1/2}\right)$

Relative minimum: $\left(-\dfrac{1}{2}, -\dfrac{1}{2}, -e^{1/2}\right)$

**19.** Saddle point: $(0, 0, 4)$

**21.** Insufficient information

**23.** $f(x_0, y_0)$ is a saddle point.

**25.** Relative minima: $(a, 0, 0)$, $(0, b, 0)$

Second-Partials Test fails at $(a, 0)$ and $(0, b)$.

**27.** Saddle point: $(0, 0, 0)$

Second-Partials Test fails at $(0, 0)$.

**29.** Relative minimum: $(0, 0, 0)$

Second-Partials Test fails at $(0, 0)$.

**31.** Relative minimum: $(1, -3, 0)$

**33.** 10, 10, 10     **35.** 10, 10, 10     **37.** $x_1 = 3$, $x_2 = 6$

**39.** $p_1 = 2500$, $p_2 = 3000$     **41.** $x_1 \approx 94$, $x_2 \approx 157$

**43.** 32 in. × 16 in. × 16 in.

**45.** Base dimensions: 2 ft × 2 ft;
Height: 1.5 ft; Minimum cost: $1.80
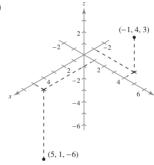
**47.** Proof     **49.** $x = 1.25$, $y = 2.5$; $4.625 million

**51.** True
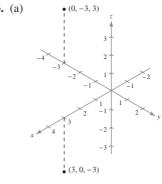
## MID-CHAPTER QUIZ   *(page 525)*

**1.** (a)



(b) 3     (c) $\left(0, \dfrac{5}{2}, 1\right)$
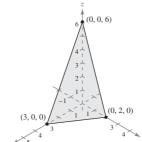
**2.** (a)



(b) $3\sqrt{14}$     (c) $\left(2, \dfrac{5}{2}, -\dfrac{3}{2}\right)$

**3.** (a)
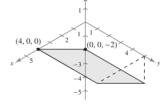


(b) $3\sqrt{6}$     (c) $\left(\dfrac{3}{2}, -\dfrac{3}{2}, 0\right)$

**4.** $(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = 16$

**5.** $(x - 1)^2 + (y - 4)^2 + (z + 2)^2 = 11$

**6.** Center: $(4, 1, 3)$; radius: 7

**7.**



**8.**
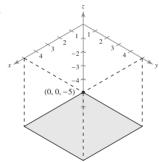
**9.**



**10.** Ellipsoid

**11.** Hyperboloid of two sheets    **12.** Elliptic paraboloid

**13.** $f(1, 0) = 1$
   $f(4, -1) = -5$

**14.** $f(1, 0) = 2$
   $f(4, -1) = 3\sqrt{7}$

**15.** $f(1, 0) = 0$
   $f(4, -1) = 0$

**16.** (a) Between 30° and 50°    (b) Between 40° and 80°
   (c) Between 70° and 90°

**17.** $f_x = 2x - 3$; $f_x(-2, 3) = -7$
   $f_y = 4y - 1$; $f_y(-2, 3) = 11$

**18.** $f_x = \dfrac{y(3 + y)}{(x + y)^2}$; $f_x(-2, 3) = 18$

   $f_y = \dfrac{-2xy - y^2 - 3x}{(x + y)^2}$; $f_y(-2, 3) = 9$

**19.** Critical point: $(1, 0)$
   Relative minimum: $(1, 0, -3)$

**20.** Critical points: $(0, 0), \left(\frac{4}{3}, \frac{4}{3}\right)$
   Relative maximum: $\left(\frac{4}{3}, \frac{4}{3}, \frac{59}{27}\right)$
   Saddle point: $(0, 0, 1)$

**21.** $x = 80$, $y = 20$; $20,000$

**22.** $x^2 + y^2 + z^2 = 3963^2$

Lines of longitude would be traces in planes passing through the $z$-axis. Each trace is a circle. Lines of latitude would be traces in planes parallel to the equator. They are circles.

## SECTION 7.6 *(page 532)*

**Skills Review**    *(page 532)*

**1.** $\left(\frac{7}{8}, \frac{1}{12}\right)$    **2.** $\left(-\frac{1}{24}, -\frac{7}{8}\right)$    **3.** $\left(\frac{55}{12}, -\frac{25}{12}\right)$

**4.** $\left(\frac{22}{23}, -\frac{3}{23}\right)$    **5.** $\left(\frac{5}{3}, \frac{1}{3}, 0\right)$    **6.** $\left(\frac{14}{19}, -\frac{10}{19}, -\frac{32}{57}\right)$

**7.** $f_x = 2xy + y^2$    **8.** $f_x = 50y^2(x + y)$
   $f_y = x^2 + 2xy$       $f_y = 50y(x + y)(x + 2y)$

**9.** $f_x = 3x^2 - 4xy + yz$    **10.** $f_x = yz + z^2$
   $f_y = -2x^2 + xz$          $f_y = xz + z^2$
   $f_z = xy$                  $f_z = xy + 2xz + 2yz$

**1.** $f(5, 5) = 25$    **3.** $f(2, 2) = 8$    **5.** $f(\sqrt{2}, 1) = 1$

**7.** $f(25, 50) = 2600$    **9.** $f(1, 1) = 2$

**11.** $f(2, 2) = e^4$    **13.** $f(9, 6, 9) = 432$

**15.** $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$    **17.** $f\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) = \sqrt{3}$

**19.** $f(8, 16, 8) = 1024$

**21.** $f\left(\sqrt{\frac{10}{3}}, \frac{1}{2}\sqrt{\frac{10}{3}}, \sqrt{\frac{5}{3}}\right) = \dfrac{5\sqrt{15}}{9}$

**23.** $x = 4$, $y = \frac{2}{3}$, $z = 2$    **25.** 40, 40, 40    **27.** $\frac{S}{3}, \frac{S}{3}, \frac{S}{3}$

**29.** $3\sqrt{2}$    **31.** $\sqrt{3}$    **33.** 36 in. × 18 in. × 18 in.

**35.** Length = width = $\sqrt[3]{360} \approx 7.1$ ft

   Height = $\dfrac{480}{360^{2/3}} \approx 9.5$ ft

**37.** $x_1 = 752.5$, $x_2 = 1247.5$
   To minimize cost, let $x_1 = 753$ units and $x_2 = 1247$ units.

**39.** (a) $x = 50\sqrt{2} \approx 71$    (b) Answers will vary.
      $y = 200\sqrt{2} \approx 283$

**41.** (a) $f\left(\frac{3125}{6}, \frac{6250}{3}\right) \approx 147{,}314$    (b) 1.473
   (c) 184,142 units

**43.** $x = \sqrt[3]{0.065} \approx 0.402$ L
   $y = \frac{1}{2}\sqrt[3]{0.065} \approx 0.201$ L
   $z = \frac{1}{3}\sqrt[3]{0.065} \approx 0.134$ L

**45.** (a) $x = 52$, $y = 48$    (b) 64 dogs

**47.** (a) 50 ft × 120 ft    (b) $2400

**49.** Stock G: $157,791.67
   Stock P: $8500.00
   Stock S: $133,708.33

**51.** (a) Cable television: $1200
      Newspaper: $600
      Radio: $900
   (b) About 3718 responses

## SECTION 7.7 *(page 542)*

**Skills Review**    *(page 542)*
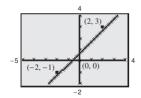
**1.** 5.0225    **2.** 0.0189

**3.** $S_a = 2a - 4 - 4b$    **4.** $S_a = 8a - 6 - 2b$
   $S_b = 12b - 8 - 4a$       $S_b = 18b - 4 - 2a$
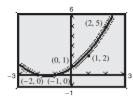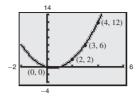
**5.** 15    **6.** 42    **7.** $\frac{25}{12}$

**8.** 14    **9.** 31    **10.** 95

**1.** (a) $y = \frac{3}{4}x + \frac{4}{3}$    (b) $\frac{1}{6}$

**3.** (a) $y = -2x + 4$    (b) 2

**5.** $y = x + \frac{2}{3}$                    **7.** $y = -2.3x - 0.9$



**9.** $y = 0.7x + 1.4$    **11.** $y = x + 4$

**13.** $y = -0.65x + 1.75$    **15.** $y = 0.8605x + 0.163$

**17.** $y = -1.1824x + 6.385$

**19.** $y = 0.4286x^2 + 1.2x + 0.74$    **21.** $y = x^2 - x$



**23.** Linear: $y = 1.4x + 6$

Quadratic: $y = 0.12x^2 + 1.7x + 6$

The quadratic model is a better fit.

**25.** Linear: $y = -68.9x + 754$

Quadratic: $y = 2.82x^2 - 83.0x + 763$

The quadratic model is a better fit.
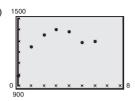
**27.** (a) $y = -240x + 685$    (b) 349    (c) $.77

**29.** (a) $y = 13.8x + 22.1$    (b) 44.18 bushels/acre

**31.** (a) $y = -0.238t + 11.93$;

In 2010, $y \approx 4.8$ deaths per 1000 live births

(b) $y = 0.0088t^2 - 0.458t + 12.66$;

In 2010, $y \approx 6.8$ deaths per 1000 live births

**33.** (a)
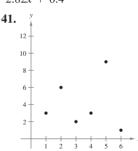


(b) $y = -28.415t^2 + 208.33t + 1025.1$

(c) Sample answer: The quadratic model has an "$r$-value" of about 0.95 ($r^2 \approx 0.91$) and the linear model has an "$r$-value" of about 0.58. Because $0.95 > 0.58$, the quadratic model is a better fit for the data.

**35.** Linear: $y = 3.757x + 9.03$

Quadratic: $y = 0.006x^2 + 3.63x + 9.4$
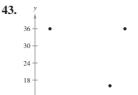
Either model is a good fit for the data.

**37.** Quadratic: $y = -0.087x^2 + 2.82x + 0.4$

**39.**



Positive correlation, $r \approx 0.9981$

**41.**



No correlation, $r = 0$

**43.**



No correlation, $r \approx 0.0750$

**45.** False; The data modeled by $y = 3.29x - 4.17$ have a positive correlation.

**47.** True    **49.** True    **51.** Answers will vary.
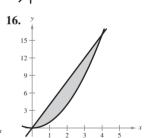
## SECTION 7.8    *(page 551)*

### Skills Review    *(page 551)*

**1.** 1    **2.** 6    **3.** 42    **4.** $\frac{1}{2}$    **5.** $\frac{19}{4}$

**6.** $\frac{16}{3}$    **7.** $\frac{1}{7}$    **8.** 4    **9.** $\ln 5$    **10.** $\ln(e - 1)$

**11.** $\frac{e}{2}(e^4 - 1)$    **12.** $\frac{1}{2}\left(1 - \frac{1}{e^2}\right)$

**13.**



**14.**



**15.**



**16.**

**1.** $\dfrac{3x^2}{2}$    **3.** $y \ln|2y|$    **5.** $x^2\left(2 - \tfrac{1}{2}x^2\right)$    **7.** $\dfrac{y^3}{2}$
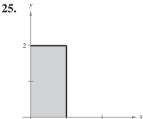
**9.** $e^{x^2} - \dfrac{e^{x^2}}{x^2} + \dfrac{1}{x^2}$    **11.** 3    **13.** 36    **15.** $\dfrac{1}{2}$

**17.** $\dfrac{148}{3}$    **19.** 5    **21.** 64    **23.** 4

**25.**



$$\int_0^1 \int_0^2 dy\, dx = \int_0^2 \int_0^1 dx\, dy = 2$$

**27.**
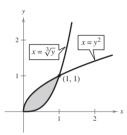


$$\int_0^1 \int_{2y}^2 dx\, dy = \int_0^2 \int_0^{x/2} dy\, dx = 1$$

**29.**
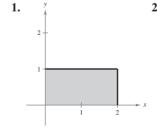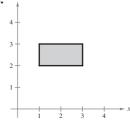


$$\int_0^2 \int_{x/2}^1 dy\, dx = \int_0^1 \int_0^{2y} dx\, dy = 1$$

**31.**
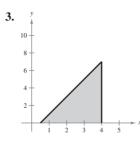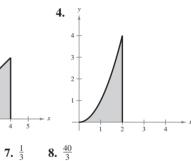


$$\int_0^1 \int_{y^2}^{\sqrt[3]{y}} dx\, dy = \int_0^1 \int_{x^3}^{\sqrt{x}} dy\, dx = \dfrac{5}{12}$$

**33.** $\tfrac{1}{2}(e^9 - 1) \approx 4051.042$    **35.** 24    **37.** $\dfrac{16}{3}$

**39.** $\dfrac{8}{3}$    **41.** 36    **43.** 5    **45.** 2    **47.** 0.6588

**49.** 8.1747    **51.** 0.4521    **53.** 1.1190    **55.** True

## SECTION 7.9    *(page 559)*

**Skills Review**    *(page 559)*

**1.**

**2.**

**3.**

**4.**

**5.** 1    **6.** 6    **7.** $\dfrac{1}{3}$    **8.** $\dfrac{40}{3}$

**9.** $\dfrac{28}{3}$    **10.** $\dfrac{7}{6}$

**1.**

10

**3.**

$\dfrac{1}{54}$

**5.**

$\dfrac{1}{3}$

**7.**
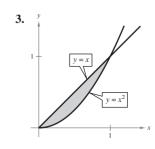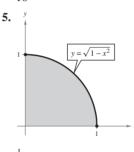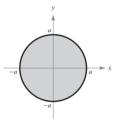
$\pi a^2$

**9.** $\displaystyle\int_0^3\int_0^5 xy\,dy\,dx = \int_0^5\int_0^3 xy\,dx\,dy = \frac{225}{4}$

**11.** $\displaystyle\int_0^2\int_x^{2x} \frac{y}{x^2+y^2}\,dy\,dx = \int_0^2\int_{y/2}^{y} \frac{y}{x^2+y^2}\,dx\,dy$

$\displaystyle + \int_2^4\int_{y/2}^{2} \frac{y}{x^2+y^2}\,dx\,dy = \ln\frac{5}{2}$

**13.** 4    **15.** 4    **17.** 12    **19.** $\frac{3}{8}$    **21.** $\frac{40}{3}$    **23.** 4

**25.** $\frac{32}{3}$    **27.** 10,000    **29.** 2    **31.** $\frac{8}{3}$    **33.** \$75,125

**35.** \$13,400    **37.** 25,645.24
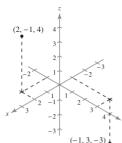
## REVIEW EXERCISES FOR CHAPTER 7
*(page 565)*

**1.**



(2, –1, 4)    (–1, 3, –3)
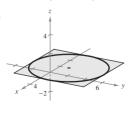
**3.** $\sqrt{110}$    **5.** $(-1, 4, 6)$

**7.** $x^2 + (y-1)^2 + z^2 = 25$
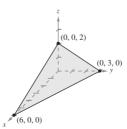
**9.** $(x-2)^2 + (y-3)^2 + (z-2)^2 = 17$

**11.** Center: $(-2, 1, 4)$; radius: 4

**13.**



**15.**



(0, 0, 2)    (0, 3, 0)    (6, 0, 0)

**17.**



(4, 0, 0)    (0, 0, –2)
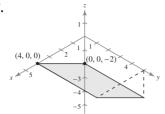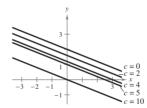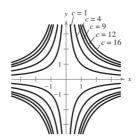
**19.** Sphere    **21.** Ellipsoid    **23.** Elliptic paraboloid

**25.** Top half of a circular cone

**27.** (a) 18    (b) 0    (c) $-245$    (d) $-32$

**29.** The domain is the set of all points inside or on the circle $x^2 + y^2 = 1$, and the range is $[0, 1]$.

**31.** The level curves are lines of slope $-\frac{2}{5}$.



$c = 0$
$c = 2$
$c = 4$
$c = 5$
$c = 10$

**33.** The level curves are hyperbolas.



$c = 1$
$c = 4$
$c = 9$
$c = 12$
$c = 16$

**35.** (a) As the color darkens from light green to dark green, the average yearly precipitation increases.

(b) The small eastern portion containing Davenport

(c) The northwestern portion containing Sioux City

**37.** Southwest to northeast    **39.** \$2.50

**41.** $f_x = 2xy + 3y + 2$

$f_y = x^2 + 3x - 5$

**43.** $z_x = \dfrac{2x}{y^2}$

$z_y = \dfrac{-2x^2}{y^3}$

**45.** $f_x = \dfrac{2}{2x + 3y}$

$f_y = \dfrac{3}{2x + 3y}$

**47.** $f_x = ye^x + e^y$

$f_y = xe^y + e^x$

**49.** $w_x = yz^2$

$w_y = xz^2$

$w_z = 2xyz$

**51.** (a) $z_x = 3$    (b) $z_y = -4$

**53.** (a) $z_x = -2x$    (b) $z_y = -2y$

At $(1, 2, 3)$, $z_x = -2$.    At $(1, 2, 3)$, $z_y = -4$.

**55.** $f_{xx} = 6$

$f_{yy} = 12y$

$f_{xy} = f_{yx} = -1$

**57.** $f_{xx} = f_{yy} = f_{xy} = f_{yx} = \dfrac{-1}{4(1 + x + y)^{3/2}}$

**59.** $C_x(500, 250) = 99.50$

$C_y(500, 250) = 140$

**61.** (a) $A_w = 43.095w^{-0.575}h^{0.725}$

$A_h = 73.515w^{0.425}h^{-0.275}$

(b) $\approx 47.35$;

The surface area of an average human body increases approximately 47.35 square centimeters per pound for a human who weighs 180 pounds and is 70 inches tall.

**63.** Critical point: $(0, 0)$

Relative minimum: $(0, 0, 0)$

**65.** Critical point: $(-2, 3)$

Saddle point: $(-2, 3, 1)$

**67.** Critical points: $(0, 0), \left(\frac{1}{6}, \frac{1}{12}\right)$

Relative minimum: $\left(\frac{1}{6}, \frac{1}{12}, -\frac{1}{432}\right)$

Saddle point: $(0, 0, 0)$

**69.** Critical points: $(1, 1), (-1, -1). (1, -1), (-1, 1)$

Relative minimum: $(1, 1, -2)$

Relative maximum: $(-1, -1, 6)$

Saddle points: $(1, -1, 2), (-1, 1, 2)$
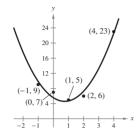
**71.** (a) $R = -x_1^2 - 0.5x_2^2 + 100x_1 + 200x_2$

(b) $x_1 = 50, x_2 = 200$    (c) \$22,500.00

**73.** At $\left(\frac{4}{3}, \frac{1}{3}\right)$, the relative maximum is $\frac{16}{27}$.

At $(0, 1)$, the relative minimum is 0.
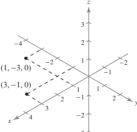
**75.** At $\left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)$, the relative maximum is $\frac{32}{27}$.

**77.** At $\left(\frac{4}{3}, \frac{10}{3}, \frac{14}{3}\right)$, the relative minimum is $\frac{104}{3}$.

**79.** At $\left(2\sqrt{2}, 2\sqrt{2}, \sqrt{2}\right)$, the relative maximum is 8.

**81.** $f(49.4, 253) \approx 13,202$

**83.** (a) $y = \frac{60}{59}x - \frac{15}{59}$    (b) 2.746

**85.** (a) $y = 14x + 19$    (b) 21.8 bushels/acre

**87.** $y = 1.71x^2 - 2.57x + 5.56$

**89.** 1    **91.** $\frac{7}{4}$
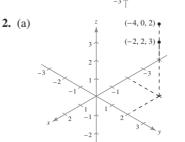
**93.** $\displaystyle\int_{-2}^{2}\int_{5}^{9-x^2} dy\, dx = \int_{5}^{9}\int_{-\sqrt{9-y}}^{\sqrt{9-y}} dx\, dy = \frac{32}{3}$

**95.** $\displaystyle\int_{-3}^{6}\int_{1/3(x+3)}^{\sqrt{x+3}} dy\, dx = \int_{0}^{3}\int_{3y-3}^{y^2-3} dx\, dy = \frac{9}{2}$

**97.** $\frac{4096}{9}$    **99.** 0.0833 mi

## CHAPTER TEST   *(page 569)*

**1.** (a)

(b) $2\sqrt{2}$

(c) $(2, -2, 0)$

**2.** (a)
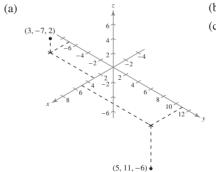
(b) 3

(c) $(-3, 1, 2.5)$

**3.** (a)

(b) $14\sqrt{2}$

(c) $(4, 2, -2)$

**4.** Center: $(10, -5, 5)$; radius: 5    **5.** Plane

**6.** Elliptic cone    **7.** Hyperbolic paraboloid

**8.** $f(3, 3) = 19$    **9.** $f(3, 3) = \frac{3}{2}$    **10.** $f(3, 3) = 0$

$f(1, 1) = 3$    $f(1, 1) = \frac{3}{2}$    $f(1, 1) = 0$

**11.** $f_x = 6x + 9y^2$; $f_x(10, -1) = 69$

$f_y = 18xy$; $f_y(10, -1) = -180$

**12.** $f_x = (x + y)^{1/2} + \dfrac{x}{2(x + y)^{1/2}}$; $f_x(10, -1) = \dfrac{14}{3}$

$f_y = \dfrac{x}{2(x + y)^{1/2}}$; $f_y(10, -1) = \dfrac{5}{3}$

**13.** Critical point: $(1, -2)$; Relative minimum: $(1, -2, -23)$

**14.** Critical points: $(0, 0), (1, 1), (-1, -1)$

Saddle point: $(0, 0, 0)$

Relative maxima: $(1, 1, 2), (-1, -1, 2)$

**15.** (a) $x = 4000$ units of labor, $y = 500$ units of capital

(b) About 128,613 units

**16.** $y = -1.839x^2 + 31.70x + 73.6$

**17.** $\frac{3}{2}$    **18.** 1    **19.** $\frac{4}{3}$ units$^2$    **20.** $\frac{11}{6}$

# CHAPTER 8

## SECTION 8.1   *(page 576)*

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**Skills Review**   *(page 576)*

   **1.** 35 cm$^2$    **2.** 12 in.$^2$    **3.** $c = 13$    **4.** $b = 4$

   **5.** $b = 15$    **6.** $a = 6$    **7.** Equilateral triangle

   **8.** Isosceles triangle    **9.** Right triangle

   **10.** Isosceles triangle and right triangle

---

**1.** (a) $405°, -315°$    (b) $319°, -401°$

**3.** (a) $660°, -60°$    (b) $20°, -340°$

**5.** (a) $\dfrac{19\pi}{9}, -\dfrac{17\pi}{9}$    (b) $\dfrac{8\pi}{3}, -\dfrac{4\pi}{3}$

**7.** (a) $\dfrac{7\pi}{4}, -\dfrac{\pi}{4}$    (b) $\dfrac{28\pi}{15}, -\dfrac{32\pi}{15}$

**9.** $\dfrac{\pi}{6}$    **11.** $\dfrac{3\pi}{2}$    **13.** $\dfrac{7\pi}{4}$    **15.** $-\dfrac{\pi}{9}$    **17.** $-\dfrac{3\pi}{2}$

**19.** $\dfrac{11\pi}{6}$    **21.** $450°$    **23.** $420°$    **25.** $-15°$

**27.** $405°$    **29.** $570°$    **31.** $-\dfrac{3\pi}{2}$    **33.** $\dfrac{4\pi}{5}$

**35.** $c = 10, \theta = 60°$    **37.** $a = 4\sqrt{3}, \theta = 30°$

**39.** $\theta = 40°$    **41.** $s = \sqrt{3}, \theta = 60°$

**43.** $4\sqrt{3}$ in.$^2$    **45.** $\dfrac{25\sqrt{3}}{4}$ ft$^2$    **47.** 18 ft

**49.**

| $r$ | 8 ft | 15 in. | 85 cm | 24 in. | $\dfrac{12{,}963}{\pi}$ mi |
|---|---|---|---|---|---|
| $s$ | 12 ft | 24 in. | 200.28 cm | 96 in. | 8642 mi |
| $\theta$ | 1.5 | 1.6 | $\dfrac{3\pi}{4}$ | 4 | $\dfrac{2\pi}{3}$ |

**51.** (a) $-\dfrac{4\pi}{9}$    (b) $\dfrac{10\pi}{9}$ ft    **53.** $\dfrac{4900\pi}{3}$ ft$^2$

**55.** False. An obtuse angle is between $90°$ and $180°$.

**57.** True

## SECTION 8.2   *(page 587)*

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**Skills Review**   *(page 587)*

   **1.** $\dfrac{3\pi}{4}$    **2.** $\dfrac{7\pi}{4}$    **3.** $-\dfrac{7\pi}{6}$    **4.** $-\dfrac{5\pi}{3}$

---

**5.** $-\dfrac{2\pi}{3}$    **6.** $-\dfrac{5\pi}{4}$    **7.** $3\pi$    **8.** $\dfrac{13\pi}{6}$

**9.** $x = 0, 1$    **10.** $x = 0, -\dfrac{1}{2}$

**11.** $x = -\dfrac{1}{2}, 1$    **12.** $x = -1, 3$    **13.** $x = 1$

**14.** $x = -1, \dfrac{1}{2}$    **15.** $x = 2, 3$    **16.** $x = -2, 1$
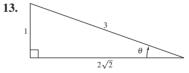
**17.** $t = 10$    **18.** $t = \dfrac{7}{2}$    **19.** $t = \dfrac{445}{8}$    **20.** $t = 7$

---

**1.** $\sin \theta = \dfrac{4}{5},$    $\csc \theta = \dfrac{5}{4}$

   $\cos \theta = \dfrac{3}{5},$    $\sec \theta = \dfrac{5}{3}$

   $\tan \theta = \dfrac{4}{3},$    $\cot \theta = \dfrac{3}{4}$

**3.** $\sin \theta = -\dfrac{5}{13},$    $\csc \theta = -\dfrac{13}{5}$

   $\cos \theta = -\dfrac{12}{13},$    $\sec \theta = -\dfrac{13}{12}$

   $\tan \theta = \dfrac{5}{12},$    $\cot \theta = \dfrac{12}{5}$

**5.** $\sin \theta = \dfrac{1}{2},$    $\csc \theta = 2$

   $\cos \theta = -\dfrac{\sqrt{3}}{2},$    $\sec \theta = -\dfrac{2\sqrt{3}}{3}$

   $\tan \theta = -\dfrac{\sqrt{3}}{3},$    $\cot \theta = -\sqrt{3}$

**7.** $\csc \theta = 2$    **9.** $\cot \theta = \dfrac{4}{3}$    **11.** $\sec \theta = \dfrac{17}{15}$

**13.**



$\csc \theta = 3$

$\cos \theta = \dfrac{2\sqrt{2}}{3},$    $\sec \theta = \dfrac{3\sqrt{2}}{4}$
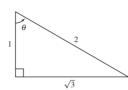
$\tan \theta = \dfrac{\sqrt{2}}{4},$    $\cot \theta = 2\sqrt{2}$

**15.**
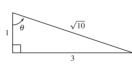


$\sin \theta = \dfrac{\sqrt{3}}{2},$    $\csc \theta = \dfrac{2\sqrt{3}}{3}$

$\cos \theta = \dfrac{1}{2}$

$\tan \theta = \sqrt{3},$    $\cot \theta = \dfrac{\sqrt{3}}{3}$

**17.**



$\sin \theta = \dfrac{3\sqrt{10}}{10},$    $\csc \theta = \dfrac{\sqrt{10}}{3}$

$\cos \theta = \dfrac{\sqrt{10}}{10},$    $\sec \theta = \sqrt{10}$

$\cot \theta = \dfrac{1}{3}$
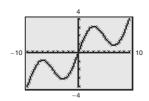
**19.** Quadrant IV     **21.** Quadrant I     **23.** Quadrant II

| Function | $\theta$ (deg) | $\theta$ (rad) | Function Value |
|---|---|---|---|
| **25.** sin | 30° | $\dfrac{\pi}{6}$ | $\dfrac{1}{2}$ |
| **27.** tan | 60° | $\dfrac{\pi}{3}$ | $\sqrt{3}$ |
| **29.** cot | 45° | $\dfrac{\pi}{4}$ | 1 |

**31.** (a) $\sin 60° = \dfrac{\sqrt{3}}{2}$     (b) $\sin\left(-\dfrac{2\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

$\cos 60° = \dfrac{1}{2}$     $\cos\left(-\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}$

$\tan 60° = \sqrt{3}$     $\tan\left(-\dfrac{2\pi}{3}\right) = \sqrt{3}$

**33.** (a) $\sin\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{2}$     (b) $\sin 150° = \dfrac{1}{2}$

$\cos\left(-\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$     $\cos 150° = -\dfrac{\sqrt{3}}{2}$

$\tan\left(-\dfrac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{3}$     $\tan 150° = -\dfrac{\sqrt{3}}{3}$

**35.** (a) $\sin 225° = -\dfrac{\sqrt{2}}{2}$     (b) $\sin(-225°) = \dfrac{\sqrt{2}}{2}$

$\cos 225° = -\dfrac{\sqrt{2}}{2}$     $\cos(-225°) = -\dfrac{\sqrt{2}}{2}$

$\tan 225° = 1$     $\tan(-225°) = -1$

**37.** (a) $\sin 750° = \dfrac{1}{2}$     (b) $\sin 510° = \dfrac{1}{2}$

$\cos 750° = \dfrac{\sqrt{3}}{2}$     $\cos 510° = -\dfrac{\sqrt{3}}{2}$

$\tan 750° = \dfrac{\sqrt{3}}{3}$     $\tan 510° = -\dfrac{\sqrt{3}}{3}$

**39.** (a) 0.1736     (b) 5.7588

**41.** (a) 0.3640     (b) 0.3640

**43.** (a) $-0.3420$     (b) $-0.3420$

**45.** (a) 2.0070     (b) 2.0000

**47.** (a) $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$     (b) $\dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

**49.** (a) $\dfrac{\pi}{3}, \dfrac{2\pi}{3}$     (b) $\dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

**51.** (a) $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$     (b) $\dfrac{5\pi}{6}, \dfrac{11\pi}{6}$     **53.** $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

**55.** $0, \dfrac{\pi}{4}, \pi, \dfrac{5\pi}{4}, 2\pi$     **57.** $\dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$     **59.** $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$

**61.** $0, \dfrac{\pi}{2}, \pi, 2\pi$     **63.** $\dfrac{100\sqrt{3}}{3}$     **65.** $\dfrac{25\sqrt{3}}{3}$

**67.** 15.5572     **69.** About 443.2 m; about 323.3 m

**71.** About 19.3 ft     **73.** About 1.3 mi

**75.** (a) 102.6°F     (b) 102.1°F     (c) 100.6°F

At 4 P.M. the following afternoon, the patient's temperature should return to normal. This is determined by setting the function equal to 98.6 and solving for $t$.

**77.**

| $x$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0 | 2.7021 | 2.7756 | 1.2244 | 1.2979 | 4 |
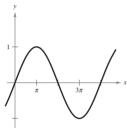


## SECTION 8.3     *(page 596)*
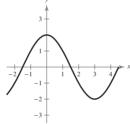
**Skills Review**     *(page 596)*

**1.** 14     **2.** 10     **3.** 0     **4.** 0     **5.** 1

**6.** $-\dfrac{\sqrt{3}}{3}$     **7.** $-\dfrac{1}{2}$     **8.** $-\dfrac{\sqrt{3}}{2}$     **9.** $\dfrac{1}{2}$

**10.** $-\dfrac{\sqrt{3}}{2}$     **11.** 0.9659     **12.** $-0.6428$

**13.** $-0.9962$     **14.** 0.6428     **15.** 0.9744

**16.** 0.3090     **17.** $-0.6494$     **18.** $-0.8391$

**1.** Period: $\pi$     **3.** Period: $4\pi$

Amplitude: 2     Amplitude: $\frac{3}{2}$

**5.** Period: 2     **7.** Period: $2\pi$

Amplitude: $\frac{1}{2}$     Amplitude: 2

**9.** Period: $\dfrac{\pi}{5}$     **11.** Period: $3\pi$

Amplitude: 2     Amplitude: $\dfrac{1}{2}$

**13.** Period: $\frac{1}{2}$

Amplitude: 3

**15.** $\pi$     **17.** $\dfrac{2\pi}{5}$     **19.** 6     **21.** c; $\pi$     **22.** e; $\pi$

**23.** f; 2     **24.** a; $2\pi$     **25.** b; $4\pi$     **26.** d; $2\pi$

**27.**



**29.**



**31.**



**33.**



**35.**



**37.**



**39.**



**41.**



**43.**



**45.**



**47.**
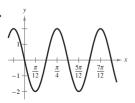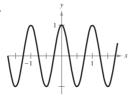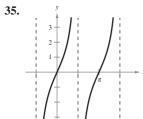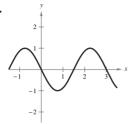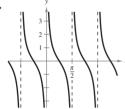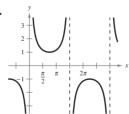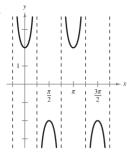
| $x$ | $-0.1$ | $-0.01$ | $-0.001$ |
|---|---|---|---|
| $f(x)$ | 1.9471 | 1.9995 | 2.0000 |

| $x$ | 0.001 | 0.01 | 0.1 |
|---|---|---|---|
| $f(x)$ | 2.0000 | 1.9995 | 1.9471 |

$$\lim_{x \to 0} \frac{\sin 4x}{2x} = 2$$

**49.**

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ |
|---|---|---|---|
| $f(x)$ | 0.1997 | 0.2000 | 0.2000 |

| $x$ | 0.001 | 0.01 | 0.1 |
|---|---|---|---|
| $f(x)$ | 0.2000 | 0.2000 | 0.1997 |

$$\lim_{x \to 0} \frac{\sin x}{5x} = \frac{1}{5}$$

**51.**

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ |
|---|---|---|---|
| $f(x)$ | $-0.1499$ | $-0.0150$ | $-0.0015$ |

| $x$ | 0.001 | 0.01 | 0.1 |
|---|---|---|---|
| $f(x)$ | 0.0015 | 0.0150 | 0.1499 |

$$\lim_{x \to 0} \frac{3(1 - \cos x)}{x} = 0$$

**53.**

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ |
|---|---|---|---|
| $f(x)$ | 2.0271 | 2.0003 | 2.0000 |

| $x$ | 0.001 | 0.01 | 0.1 |
|---|---|---|---|
| $f(x)$ | 2.0000 | 2.0003 | 2.0271 |

$$\lim_{x \to 0} \frac{\tan 2x}{x} = 2$$
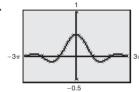
**55.**

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ |
|---|---|---|---|
| $f(x)$ | $-0.0997$ | $-0.0100$ | $-0.0010$ |

| $x$ | 0.001 | 0.01 | 0.1 |
|---|---|---|---|
| $f(x)$ | 0.0010 | 0.0100 | 0.0997 |
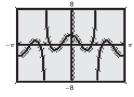
$$\lim_{x \to 0} \frac{\sin^2 x}{x} = 0$$

**57.**



$$\lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}$$

**59.**



$$\lim_{x \to 0} \frac{\sin 5x}{\sin 2x} = \frac{5}{2}$$

**61.** $a = 2, d = 1$     **63.** $a = -4, d = 4$

**65.** b     **66.** d     **67.** a     **68.** c

**69.** (a) 6 sec    (b) 10
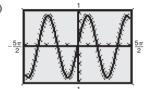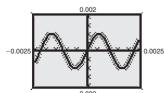
(c)
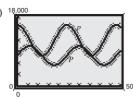


**71.** (a) $\frac{1}{440}$    (b) 440

(c)
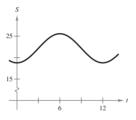


**73.** (a)



(b) As the population of the prey increases, the population of the predator increases as well. At some point, the predator eliminates the prey faster than the prey can reproduce, and the prey population decreases rapidly. As the prey becomes scarce, the predator population decreases, releasing the prey from predator pressure, and the cycle begins again.
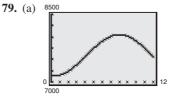
**75.**



**77.** $P(8930) \approx 0.9977$

$E(8930) \approx -0.4339$

$I(8930) \approx -0.6182$

**79.** (a)



(b) Yes; June, July, August, and September

**81.** (a) May; November    (b) About 8 days

**83.** (a) A    (b) B    (c) B

(d) The frequency is the inverse of the period.

**85.** False. The amplitude is $|-3| = 3$.     **87.** True

## MID-CHAPTER QUIZ  *(page 600)*

**1.** $\frac{\pi}{12}$    **2.** $\frac{7\pi}{12}$    **3.** $-\frac{4\pi}{9}$    **4.** $\frac{7\pi}{36}$    **5.** $120°$

**6.** $48°$    **7.** $-240°$    **8.** $165°$    **9.** $-\frac{\sqrt{2}}{2}$

**10.** $-\frac{\sqrt{3}}{2}$    **11.** $-\frac{\sqrt{3}}{3}$    **12.** $1$    **13.** $2$    **14.** $-1$

**15.** $\frac{\pi}{4}, \frac{5\pi}{4}$    **16.** $0, 2\pi$    **17.** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
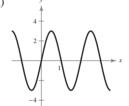
**18.** $\theta = 30°; a = 5\sqrt{3}$    **19.** $\theta = 40°; a \approx 10.285$
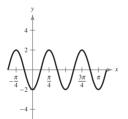
**20.** $\theta = 50°; a \approx 3.356$    **21.** $d \approx 286.8$ ft

**22.** (a)                        (b) 2

**23.** (a)



(b) $\dfrac{\pi}{2}$

**24.** (a)



(b) 3

**25.** (a)
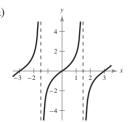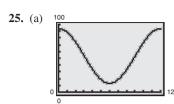


(b) Maximum: December

Minimum: June

## SECTION 8.4  *(page 607)*

**Skills Review**    *(page 607)*

**1.** $f'(x) = 9x^2 - 4x + 4$      **2.** $g'(x) = 12x^2(x^3 + 4)^3$

**3.** $f'(x) = 3x^2 + 2x + 1$      **4.** $g'(x) = \dfrac{2(5 - x^2)}{(x^2 + 5)^2}$

**5.** Relative minimum: $(-2, -3)$

**6.** Relative maximum: $\left(-2, \frac{22}{3}\right)$

  Relative minimum: $\left(2, -\frac{10}{3}\right)$

**7.** $x = \dfrac{\pi}{3}, x = \dfrac{2\pi}{3}$    **8.** $x = \dfrac{2\pi}{3}, x = \dfrac{4\pi}{3}$

**9.** $x = \pi$    **10.** No solution

**1.** $-3 \cos x$    **3.** $2x + \sin x$    **5.** $\dfrac{2}{\sqrt{x}} - 3 \sin x$

**7.** $-t^2 \sin t + 2t \cos t$    **9.** $-\dfrac{t \sin t + \cos t}{t^2}$

**11.** $\sec^2 x + 2x$    **13.** $e^{x^2} \sec x(\tan x + 2x)$

**15.** $-3 \sin 3x + 2 \sin x \cos x$    **17.** $\pi \tan \pi x \sec \pi x$

**19.** $\sin \dfrac{1}{x} - \dfrac{1}{x} \cos \dfrac{1}{x}$    **21.** $12 \sec^2 4x$

**23.** $16 \sec^2 4x \tan 4x$      **25.** $2e^{2x}(\cos 2x + \sin 2x)$

**27.** $-2 \cos x \sin x = -\sin 2x$

**29.** $-4 \cos x \sin x = -2 \sin 2x$

**31.** $2 \sin 2x + 2 \sin x \cos x = 3 \sin 2x$

**33.** $\sec^2 x - 1 = \tan^2 x$

**35.** $\sin^2 x \cos x - \sin^4 x \cos x = \sin^2 x \cos^3 x$

**37.** $\dfrac{2 \cos x}{\sin x} = 2 \cot x$      **39.** $y = 2x + \dfrac{\pi}{2} - 1$

**41.** $y = 4x - 4\pi$      **43.** $y = -2x + \frac{3}{2}\pi - 1$

**45.** $y = 0$      **47.** $\dfrac{\cos x}{2 \sin 2y}; 0$

**49.** $y'' + y = (-2 \sin x - 3 \cos x) + (2 \sin x + 3 \cos x) = 0$

**51.** $y'' + 4y = (-4 \cos 2x - 4 \sin 2x)$

    $+ 4(\cos 2x + \sin 2x) = 0$

**53.** $\frac{5}{4}$; one complete cycle      **55.** 2; two complete cycles

**57.** 1; one complete cycle

**59.** Relative maximum: $\left(\dfrac{\pi}{3}, \dfrac{3\sqrt{3}}{2}\right)$

  Relative minimum: $\left(\dfrac{5\pi}{3}, -\dfrac{3\sqrt{3}}{2}\right)$

**61.** Relative maximum: $\left(\dfrac{5\pi}{3}, \dfrac{5\pi}{3} + \sqrt{3}\right)$

  Relative minimum: $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3} - \sqrt{3}\right)$

**63.** Relative maximum: $\left(\dfrac{\pi}{4}, 1.5509\right)$

  Relative minimum: $\left(\dfrac{5\pi}{4}, -35.8885\right)$
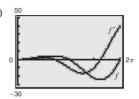
**65.** (a) $h'(t)$ is a maximum when $t = 0$, or at midnight.

  (b) $h'(t)$ is a minimum when $t = \frac{1}{2}$, or at noon.

**67.** August; about 8049 thousand, or 8,049,000, workers

**69.** July; 14 hr

**71.** (a)
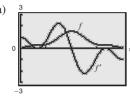


(b) 0, 2.2889, 5.0870

(c) $f' > 0$ on $(0, 2.2889)$, $(5.0870, 2\pi)$

  $f' < 0$ on $(2.2889, 5.0870)$

**73.** (a)
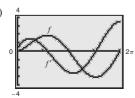


(b) 0.5236, $\pi/2$, 2.6180

(c) $f' > 0$ on $(0, 0.5236)$, $(0.5236, \pi/2)$

   $f' < 0$ on $(\pi/2, 2.6180)$, $(2.6180, \pi)$

**75.** (a)



(b) 1.8366, 4.8158

(c) $f' > 0$ on $(0, 1.8366)$, $(4.8158, 2\pi)$

   $f' < 0$ on $(1.8366, 4.8158)$

**77.** Relative maximum: $(4.49, -4.60)$

**79.** Relative maximum: $(1.27, 0.07)$

   Relative minimum: $(3.38, -1.18)$

**81.** Relative maximum: $(3.96, 1)$

**83.** False. $y' = \frac{1}{2}\sin x(1 - \cos x)^{-1/2}$

**85.** False. $y' = x \sin 2x + \sin^2 x$     **87.** Answers will vary.

## SECTION 8.5   *(page 616)*

> ### Skills Review   *(page 616)*
>
> **1.** $-\dfrac{\sqrt{2}}{2}$   **2.** $-\dfrac{1}{2}$   **3.** $-\dfrac{\sqrt{3}}{2}$   **4.** $\dfrac{\sqrt{3}}{2}$
>
> **5.** $-\dfrac{\sqrt{3}}{3}$   **6.** $-\dfrac{\sqrt{3}}{3}$   **7.** $-1$   **8.** $0$
>
> **9.** $\tan x$   **10.** $\cot x$   **11.** $\sin^2 x$   **12.** $\cos^2 x$
>
> **13.** $1$   **14.** $\cos x$   **15.** $\csc x$   **16.** $\cos x \sin x$
>
> **17.** $\frac{88}{3}$   **18.** $\frac{4}{3}$   **19.** $4$   **20.** $\frac{17}{4}$

**1.** $-2\cos x + 3\sin x + C$     **3.** $t + \csc t + C$

**5.** $-\cot\theta - \sin\theta + C$     **7.** $-\frac{1}{2}\cos 2x + C$

**9.** $\sin x^2 + C$     **11.** $2\tan\dfrac{x}{2} + C$

**13.** $-\frac{1}{3}\ln|\cos 3x| + C$     **15.** $\frac{1}{4}\tan^4 x + C$

**17.** $\dfrac{1}{\pi}\ln|\sin \pi x| + C$     **19.** $\dfrac{1}{2}\ln|\csc 2x - \cot 2x| + C$

**21.** $\ln|\tan x| + C$     **23.** $\ln|\sec x - 1| + C$

**25.** $-\ln|1 + \cos x| + C$     **27.** $\frac{1}{2}\tan^2 x + C$

**29.** $-\cos e^x + C$     **31.** $e^{\sin x} + C$

**33.** $-\cos^2 x + x + C$ or $\sin^2 x + x + C$

**35.** $x\sin x + \cos x + C$     **37.** $x\tan x + \ln|\cos x| + C$

**39.** $\dfrac{3\sqrt{3}}{8} \approx 0.6495$     **41.** $2(\sqrt{3} - 1) \approx 1.4641$

**43.** $\frac{1}{2}$     **45.** $\ln(\cos 0) - \ln(\cos 1) \approx 0.6156$

**47.** $4$     **49.** $\dfrac{\pi^2}{2} + 2 \approx 6.9348$     **51.** $2$

**53.** 545.53 trillion Btu

**55.** (a) 88.63 million barrels

   (b) 88.63 million barrels

   (c) 109 million barrels

**57.** 17.69 in.     **59.** (a) $C \approx \$27.50$     (b) $C \approx \$18.08$

**61.** 0.5093 L     **63.** 0.9777     **65.** 3.8202     **67.** True

## REVIEW EXERCISES FOR CHAPTER 8
*(page 623)*

**1.** $\dfrac{15\pi}{4}, -\dfrac{\pi}{4}$     **3.** $\dfrac{7\pi}{2}, -\dfrac{\pi}{2}$

**5.** $495°, -225°$     **7.** $315°, -45°$

**9.** $\dfrac{7\pi}{6}$   **11.** $-\dfrac{\pi}{3}$   **13.** $-\dfrac{8\pi}{3}$   **15.** $\dfrac{11\pi}{18}$

**17.** $240°$     **19.** $-120°$     **21.** $b = 4\sqrt{3}, \theta = 60°$

**23.** $a = \dfrac{5\sqrt{3}}{2}, c = 5, \theta = 60°$     **25.** About 15.38 ft

**27.** $\dfrac{\pi}{3}$     **29.** $\dfrac{\pi}{6}$     **31.** $60°$     **33.** $60°$     **35.** $\dfrac{\sqrt{2}}{2}$

**37.** $-\sqrt{3}$     **39.** $-\dfrac{\sqrt{3}}{2}$     **41.** $\sqrt{3}$     **43.** $-1$

**45.** $-\frac{1}{2}$     **47.** 0.6494     **49.** 3.2361     **51.** $-0.3420$
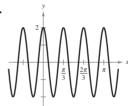
**53.** $-0.2588$     **55.** $r \approx 146.19$     **57.** $x \approx 68.69$

**59.** $\dfrac{2\pi}{3}, \dfrac{4\pi}{3}$     **61.** $\dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$     **63.** $\dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$
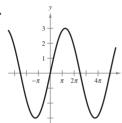
**65.** About 81.18 ft

**67.**
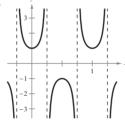


**69.**
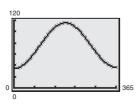
**71.**



**73.**



**75.**



**77.** $5\pi \cos 5\pi x$     **79.** $-x \sec^2 x - \tan x$

**81.** $-\dfrac{x \sin x + 2 \cos x}{x^3}$

**83.** $2 \sin x \cos x + 1 = \sin 2x + 1$

**85.** $-4 \cot x \csc^4 x$     **87.** $e^x(\cot x - \csc^2 x)$

**89.** $y = -2x + \dfrac{\pi}{2}$     **91.** $y = \dfrac{1}{2}$     **93.** $y = 0$

**95.** Relative maximum: $\left(\dfrac{\pi}{6}, \dfrac{\pi + 6\sqrt{3}}{12}\right)$

   Relative minimum: $\left(\dfrac{5\pi}{6}, \dfrac{5\pi - 6\sqrt{3}}{12}\right)$

**97.** Relative maxima: $\left(\dfrac{\pi}{2}, 2\right), \left(\dfrac{3\pi}{2}, 0\right)$

   Relative minima: $\left(\dfrac{7\pi}{6}, -\dfrac{1}{4}\right), \left(\dfrac{11\pi}{6}, -\dfrac{1}{4}\right)$

**99.** (a) 115.50 thousand, day 183

   (b) 34.01 thousand, day 1

**101.** $-3 \cos x - 2 \sin x + C$     **103.** $\frac{1}{4} \sin^4 x + C$

**105.** $\pi + 2$     **107.** $\dfrac{2\sqrt{3}}{3}$     **109.** 0     **111.** 2

**113.** $\frac{2}{3}$     **115.** $\frac{5}{3}$     **117.** About 19.95 in.

## CHAPTER TEST   *(page 626)*

| Function | θ (deg) | θ (rad) | Function Value |
|---|---|---|---|
| **1.** sin | 67.5° | $\dfrac{3\pi}{8}$ | 0.9239 |
| **2.** cos | 36° | $\dfrac{\pi}{5}$ | 0.8090 |
| **3.** tan | 15° | $\dfrac{\pi}{12}$ | 0.2679 |

| Function | θ (deg) | θ (rad) | Function Value |
|---|---|---|---|
| **4.** cot | $-30°$ | $-\dfrac{\pi}{6}$ | $-\sqrt{3}$ |
| **5.** sec | $-40°$ | $-\dfrac{2\pi}{9}$ | 1.3054 |
| **6.** csc | $-225°$ | $-\dfrac{5\pi}{4}$ | $\sqrt{2}$ |

**7.** About 27.37 in.
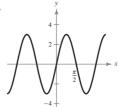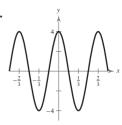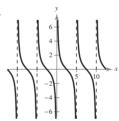
**8.** $\dfrac{\pi}{4}, \dfrac{3\pi}{4}$     **9.** $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$     **10.** $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$

**11.**



**12.**



**13.**



**14.** (a) $y' = \sin 2x - \sin x$

   (b) Relative maxima: $\left(\dfrac{\pi}{3}, \dfrac{1}{4}\right), \left(\dfrac{5\pi}{3}, \dfrac{1}{4}\right)$

   Relative minima: $(0, 0), (\pi, -2)$

**15.** (a) $y' = \sec\left(x - \dfrac{\pi}{4}\right) \tan\left(x - \dfrac{\pi}{4}\right)$

   (b) Relative maximum: $\left(\dfrac{5\pi}{4}, -1\right)$

   Relative minimum: $\left(\dfrac{\pi}{4}, 1\right)$

**16.** (a) $y' = \dfrac{\cos(x + \pi)}{[3 - \sin(x + \pi)]^2}$

   (b) Relative maximum: $\left(\dfrac{3\pi}{2}, \dfrac{1}{2}\right)$

   Relative minimum: $\left(\dfrac{\pi}{2}, \dfrac{1}{4}\right)$

**17.** $-\dfrac{1}{5} \cos 5x + C$     **18.** $\dfrac{2 - \sqrt{2}}{2\pi}$

**19.** $-\frac{1}{2} \ln|\csc x^2 + \cot x^2| + C$     **20.** $\frac{9}{2}$

**21.** $\sin \sqrt{x} + C$     **22.** $e - 1 \approx 1.7183$

**23.** (a) \$243.6 thousand     (b) \$29.06 thousand
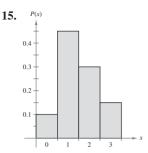
# CHAPTER 9
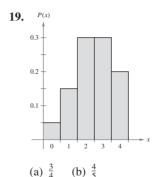
## SECTION 9.1   *(page 634)*

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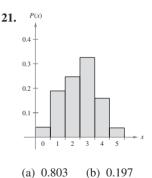
**Skills Review**   *(page 634)*

**1.** 1    **2.** 1    **3.** 2    **4.** 2    **5.** $\frac{1}{2}$

**6.** 1    **7.** 37.50%    **8.** 81.82%

**9.** 54.17%    **10.** 43.75%

---

**1.** (a) $S = \{HHH, HHT, HTH, HTT, THH,$
$\quad\quad THT, TTH, TTT\}$

(b) $A = \{HHH, HHT, HTH, THH\}$

(c) $B = \{HTT, THT, TTH, TTT\}$

**3.** (a) $S = \{III, IIO, IIU, IOI, IOO, IOU, IUI, IUO, IUU,$
$\quad\quad OII, OIO, OIU, OOI, OOO, OOU, OUI, OUO,$
$\quad\quad OUU, UII, UIO, UIU, UOI, UOO, UOU, UUI,$
$\quad\quad UUO, UUU\}$

(b) $A = \{III, IIO, IIU, IOI, IUI, OII, UII\}$

(c) $B = \{III, IIO, IIU, IOI, IOU, IUI, IUO, IUU, OII,$
$\quad\quad OIU, OUI, OUU, UII, UIO, UIU, UOI, UOU,$
$\quad\quad UUI, UUO, UUU\}$

**5.**

| Random variable | 0 | 1 | 2 |
|---|---|---|---|
| Frequency | 1 | 2 | 1 |

**7.**

| Random variable | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Frequency | 1 | 3 | 3 | 1 |

**9.** 0.24    **11.** 0.0145    **13.** $P(3) = 0.25$

**15.**
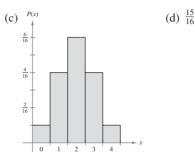


The table represents a probability distribution.

**17.** The table does not represent a probability distribution because the sum of the probabilities does not equal 1 and $P(4) < 0$.

**19.**



(a) $\frac{3}{4}$    (b) $\frac{4}{5}$

**21.**



(a) 0.803    (b) 0.197

**23.** (a) $S = \{gggg, gggb, ggbg, gbgg, bggg, ggbb,$
$\quad\quad gbgb, gbbg, bgbg, bbgg, bggb, gbbb,$
$\quad\quad bgbb, bbgb, bbbg, bbbb\}$

(b)

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(x)$ | $\frac{1}{16}$ | $\frac{4}{16}$ | $\frac{6}{16}$ | $\frac{4}{16}$ | $\frac{1}{16}$ |
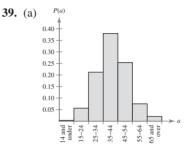
(c)

(d) $\frac{15}{16}$

**25.** $E(x) = 3$
$\quad\; V(x) = 0.875$
$\quad\; \sigma \approx 0.9354$

**27.** $E(x) = 0.8$
$\quad\; V(x) = 8.16$
$\quad\; \sigma \approx 2.8566$

**29.** (a) Mean: 2.5
$\quad\quad$ Variance: 1.25
(b) Mean: 5
$\quad\quad$ Variance: 2.5

**31.** (a) $E(x) = 18.5$
$\quad\quad \sigma = 8.0777$
(b) \$91,575

**33.** \$201    **35.** $-\$0.0526$    **37.** City 1

**39.** (a)

(b) 0.648    (c) 0.729    (d) 0.059

**41.** (a)

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(x)$ | $\frac{14}{50}$ | $\frac{26}{50}$ | $\frac{7}{50}$ | $\frac{2}{50}$ | $\frac{1}{50}$ |

(b)



(c) $\frac{35}{50}$    (d) $E(x) = 1$, $V(x) = 0.76$, $\sigma \approx 0.87$

Answers will vary.

**43.** Answers will vary.

## SECTION 9.2    *(page 643)*
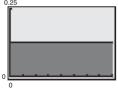
**Skills Review**    *(page 643)*

**1.** Yes    **2.** No    **3.** No    **4.** Yes    **5.** 1

**6.** $\frac{1}{2}$    **7.** 1    **8.** $\frac{1}{4}$    **9.** 1    **10.** 1

**1.**
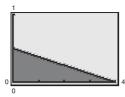


$f(x)$ is a probability density function.

$$\int_0^8 \frac{1}{8}\, dx = \left[\frac{1}{8}x\right]_0^8 = 1$$

**3.**
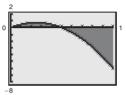


$f(x)$ is a probability density function.

$$\int_0^4 \frac{4 - x}{8}\, dx = \left[\frac{1}{2}x - \frac{1}{16}x^2\right]_0^4 = 1$$

**5.**
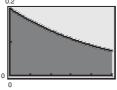


$f(x)$ is not a probability density function because

$$\int_0^1 6x(1 - 2x)dx = \left[3x^2 - 4x^3\right]_0^1 = -1 \neq 1$$

and $f(x) < 0$ over

the interval $\left(\frac{1}{2}, 1\right]$.
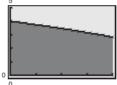
**7.**



$f(x)$ is not a probability density function because

$$\int_0^5 \frac{1}{5}e^{-x/5}\, dx = \left[-e^{-x/5}\right]_0^5 \approx 0.632 \neq 1.$$

**9.**
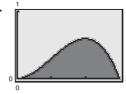


$f(x)$ is not a probability density function because

$$\int_0^2 2\sqrt{4 - x}\, dx = \left[-\frac{4}{3}(4 - x)^{3/2}\right]_0^2 \approx 6.90 \neq 1.$$

**11.**
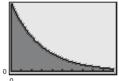


$f(x)$ is a probability density function.

$$\int_0^3 \frac{4}{27}x^2(3 - x)\, dx = \frac{4}{27}\left[x^3 - \frac{x^4}{4}\right]_0^3 = 1$$

**13.**
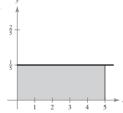


$f(x)$ is a probability density function.
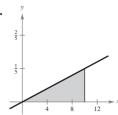
$$\int_0^\infty \frac{1}{3}e^{-x/3}\, dx = \lim_{b\to\infty}\left[-e^{-x/3}\right]_0^b = 1$$

**15.** $\frac{2}{15}$    **17.** $\frac{3}{32}$    **19.** $\frac{1}{2}$

**21.**



(a) $\frac{3}{5}$   (b) $\frac{2}{5}$   (c) $\frac{2}{5}$   (d) $\frac{4}{5}$

**23.**
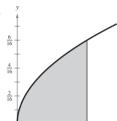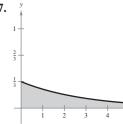


(a) $\frac{9}{25}$   (b) $\frac{1}{5}$   (c) $\frac{9}{25}$   (d) $\frac{24}{25}$

**25.**



(a) $\dfrac{\sqrt{2}}{4} \approx 0.354$

(b) $1 - \dfrac{\sqrt{2}}{4} \approx 0.646$

(c) $\frac{1}{8}\left(3\sqrt{3} - 1\right) \approx 0.525$

(d) $\dfrac{3\sqrt{3}}{8} \approx 0.650$

**27.**



(a) $1 - e^{-2/3} \approx 0.4866$

(b) $e^{-2/3} \approx 0.5134$

(c) $e^{-1/3} - e^{-4/3} \approx 0.4529$

(d) $0$

**29.** (a) $\frac{1}{6}$  (b) $\frac{2}{5}$    **31.** (a) $0.87$   (b) $0.34$

**33.** (a) $1 - e^{-2/3} \approx 0.487$   (b) $e^{-2/3} - e^{-4/3} \approx 0.250$

    (c) $e^{-2/3} \approx 0.513$

**35.** (a) $1 - e^{-6/5} \approx 0.699$   (b) $e^{-2/5} - e^{-6/5} = 0.369$

    (c) $e^{-8/5} \approx 0.202$

**37.** (a) $1 - 2e^{-1} \approx 0.264$   (b) $2e^{-1} - 3e^{-2} \approx 0.330$

    (c) $3e^{-2} \approx 0.406$

**39.** (a) $0.75$. There is a 75% probability that the community will receive up to 10 inches of rain.

    (b) $0.25$. There is a 25% probability that the community will receive 10 to 15 inches of rain.
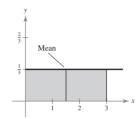
(c) $0.25$. There is a 25% probability that the community will receive up to 5 inches of rain.

(d) About $0.095$. There is a probability of approximately 9.5% that the community will receive 12 to 15 inches of rain.

## SECTION 9.3   *(page 652)*

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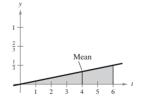
**Skills Review**   *(page 652)*

**1.** $5$    **2.** $8$    **3.** $3\ln 2$    **4.** $9\ln 2$

**5.** $\frac{4}{3}$    **6.** $\frac{4}{3}$    **7.** $\frac{3}{4}$    **8.** $\frac{2}{9}$
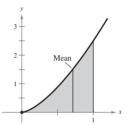
**9.** (a) $\frac{1}{4}$   (b) $\frac{1}{2}$    **10.** (a) $\frac{1}{2}$   (b) $\frac{11}{16}$

---

**1.** (a) $\dfrac{3}{2}$   (b) $\dfrac{3}{4}$   (c) $\dfrac{\sqrt{3}}{2}$
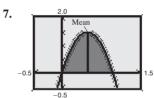


**3.** (a) $4$   (b) $2$   (c) $\sqrt{2}$



**5.** (a) $\dfrac{5}{7}$   (b) $\dfrac{20}{441}$   (c) $\dfrac{2\sqrt{5}}{21}$



**7.**



Mean: $\frac{1}{2}$

**9.**



Mean ≈ 0.848
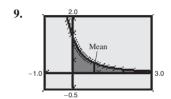
**11.** 9 ln 2 ≈ 6.238

**13.** Uniform density function

Mean: 5

Variance: $\frac{25}{3}$

Standard deviation: $\frac{5\sqrt{3}}{3} \approx 2.887$

**15.** Exponential density function

Mean: 8

Variance: 64

Standard deviation: 8

**17.** Normal density function

Mean: 100

Variance: 121

Standard deviation: 11

**19.** Mean: 0

Standard deviation: 1

$P(0 \le x \le 0.85) \approx 0.3023$

**21.** Mean: 6

Standard deviation: 6

$P(x \ge 2.23) \approx 0.6896$

**23.** Mean: 8

Standard deviation: 2

$P(3 \le x \le 13) \approx 0.9876$

**25.** (a) About 0.309    (b) About 0.159

(c) About 0.841    (d) About 0.669

**27.** (a) Mean: 10:05 A.M.

Standard deviation: $\frac{5\sqrt{3}}{3} \approx 2.9$ minutes

(b) $\frac{3}{10}$

**29.** (a) $f(t) = \frac{1}{2}e^{-t/2}$

(b) $1 - e^{-1/2} \approx 0.3935$

**31.** (a) $f(t) = \frac{1}{5}e^{-t/5}$    (b) $1 - e^{-2} \approx 0.865$

**33.** (a) 1.5 standard deviations    (b) About 93.32%

**35.** (a) $\mu = 3, \sigma = \frac{3\sqrt{5}}{5} \approx 1.342$

(b) 3    (c) About 0.626

**37.** $\mu = \frac{4}{7}$

$V(x) = \frac{8}{147}$

**39.** (a) 10    (b) About 0.1912

**41.** Mean: $\frac{11}{2}$ = median

**43.** Mean: $\frac{1}{6}$

Median: $\dfrac{2 - \sqrt{2}}{4} \approx 0.1464$

**45.** Mean: 5

Median: 5 ln 2 ≈ 3.4657
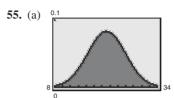
**47.** $\dfrac{1}{-0.28}$ ln 0.5 ≈ $2.48

**49.** (a) Expected value: 6

Standard deviation: $3\sqrt{2} \approx 4.243$

(b) About 0.6151

**51.** $\mu \approx 12.25$

**53.** (a) 35.96%

(b) No, about 16% of the employees will be paid more than $16.00/hr.

**55.** (a)
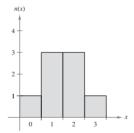


(b) 0.272    (c) 0.154
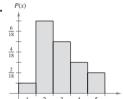
## CHAPTER 9 REVIEW EXERCISES    *(page 658)*

**1.** $S = \{$January, February, March, April, May, June, July, August, September, October, November, December$\}$

**3.** If the essays are numbered 1, 2, 3, and 4, $S = \{123, 124, 134, 234\}$.

**5.** $S = \{0, 1, 2, 3\}$

**7.**

| $x$ | 0 | 1 | 2 | 3 |
|------|---|---|---|---|
| $n(x)$ | 1 | 3 | 3 | 1 |

**9.**



(a) $\frac{5}{6}$   (b) $\frac{5}{9}$
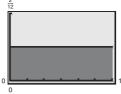
**11.** (a) $\frac{5}{36}$   (b) $\frac{5}{6}$   (c) $\frac{1}{6}$   (d) $\frac{1}{36}$

**13.** 19.5   **15.** (a) 20.5   (b) $80,975

**17.** $V(x) = 42{,}686.9475$   **19.** $V(x) \approx 1.1611$

$\sigma \approx 206.6082$            $\sigma \approx 1.0775$

**21.**
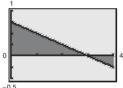


$f(x)$ is a probability density function.
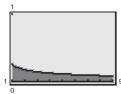
$\int_0^{12} \frac{1}{12}\, dx = \left[\frac{x}{12}\right]_0^{12} = 1.$

**23.**



Although $\int_0^4 \frac{1}{4}(3 - x)\, dx = \left[\frac{3}{4}x - \frac{x^2}{8}\right]_0^4 = 1, f(x)$ is not a probability density function because $f(x) < 0$ over the interval $(3, 4]$.

**25.**



$f(x)$ is a probability density function.

$\int_1^9 \frac{1}{4\sqrt{x}}\, dx = \frac{1}{4}\left[2\sqrt{x}\right]_1^9 = 1$

**27.** $\frac{9}{25}$   **29.** $\frac{2}{3}$   **31.** (a) $\frac{1}{2}$   (b) $\frac{1}{4}$

**33.** 3.5   **35.** 6

**37.** Variance: $\frac{9}{20}$
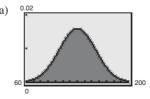
Standard deviation: $\frac{3\sqrt{5}}{10}$

**39.** Variance: 4

Standard deviation: 2

**41.** $\frac{1}{2}$   **43.** $4 \ln 2 \approx 2.7726$

**45.** (a) $1 - e^{-10/15} = 1 - e^{-2/3} \approx 0.4866$

(b) $e^{-10/15} - e^{-20/15} = e^{-2/3} - e^{-4/3} \approx 0.2498$

**47.** About 0.00383

**49.** (a)



(b) 0.150   (c) 0.655

**51.** 0.3829
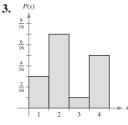
## CHAPTER TEST   *(page 661)*

**1.** (a) $S = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT,$
$HTTH, HTTT, THHH, THHT, THTH, THTT,$
$TTHH, TTHT, TTTH, TTTT\}$; Random variable $x$ assigns numbers to each possible outcome, depending on the number of heads that turn up.
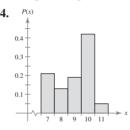
| Random variable, $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Frequency of $x$, $n(x)$ | 1 | 4 | 6 | 4 | 1 |

(b) $P(x \geq 2) = \frac{11}{16}$

**2.** $P(\text{red non-face card}) = \frac{5}{13}$

**3.**



(a) $\frac{5}{8}$   (b) $\frac{3}{8}$

**4.**



(a) 0.95   (b) 0.66

**5.** $E(x) = 1.8$
$V(x) = 1.16$
$\sigma \approx 1.0770$
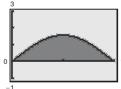
**6.** $E(x) = -0.056$
$V(x) \approx 1.8649$
$\sigma \approx 1.3656$
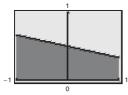
**7.**



$f(x)$ is a probability density function.

$$\int_0^1 \frac{\pi}{2} \sin \pi x \, dx = \left[ -\frac{1}{2} \cos \pi x \right]_0^1 = 1$$
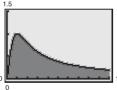
**8.**



$f(x)$ is a probability density function.

$$\int_{-1}^1 \frac{3 - x}{6} \, dx = \left[ -\frac{x}{12}(x - 6) \right]_{-1}^1 = 1$$

**9.**



$f(x)$ is not a probability density function because

$$\int_0^\infty \frac{2x}{x^2 + 1} \, dx = \left[ \ln(x^2 + 1) \right]_0^\infty = \infty \neq 1.$$

**10.** (a) $\frac{15}{64}$   (b) $\frac{27}{64}$   **11.** (a) $\frac{7}{16}$   (b) $\frac{225}{256}$

**12.** (a) 0.6321   (b) 0.3679

**13.** $\mu = 7$
$V(x) = \frac{49}{3}$
$\sigma \approx 4.041$

**14.** $\mu = \frac{5}{8}$
$V(x) \approx 0.0594$
$\sigma \approx 0.2437$

**15.** $\mu = 1$
$V(x) = 1$
$\sigma = 1$

**16.** $P(100 \leq x \leq 120) \approx 0.683$

# CHAPTER 10

## SECTION 10.1   *(page 669)*

### Skills Review   *(page 669)*

**1.** 0   **2.** 0   **3.** 2   **4.** $\infty$   **5.** 0   **6.** 0

**7.** $\dfrac{n - 2}{n}$,   $n \neq -2$   **8.** $\dfrac{n - 3}{n - 4}$,   $n \neq -4$

**9.** $\dfrac{3n^2 + 1}{n^3}$   **10.** $\dfrac{2n + 1}{(n - 1)(n + 2)}$,   $n \geq 2$

**1.** 1, 3, 5, 7, 9    **3.** 3, 9, 27, 81, 243

**5.** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$    **7.** $3, \frac{9}{2}, \frac{27}{6}, \frac{81}{24}, \frac{243}{120}$

**9.** $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}$    **11.** Converges to 0

**13.** Converges to 1    **15.** Converges to $\frac{1}{2}$    **17.** Diverges

**19.** Converges to 0    **21.** Diverges    **23.** Converges to 3

**25.** Converges to 0    **27.** Converges to 0

**29.** Diverges    **31.** Diverges    **33.** Diverges
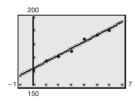
**35.** $3n - 2$    **37.** $5n - 6$    **39.** $\dfrac{n + 1}{n + 2}$

**41.** $\dfrac{(-1)^{n-1}}{2^{n-2}}$    **43.** $\dfrac{n + 1}{n}$    **45.** $2(-1)^n$    **47.** $\dfrac{(-1)^n x^n}{n}$

**49.** 2, 5, 8, 11, 14, 17, . . . ; Pattern: each term is 1 less than 3 times $n$.

**51.** $1, \frac{5}{3}, \frac{7}{3}, 3, \frac{11}{3}, \frac{13}{3}, . . .$ ; Pattern: each term is $\frac{1}{3}$ more than $\frac{2}{3}$ times $n$.

**53.** $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \frac{3}{16}, -\frac{3}{32}, . . .$ ; Pattern: each term is 3 times the quantity $\left(-\frac{1}{2}\right)$ raised to the $(n - 1)$ power.

**55.** 2, 6, 18, 54, 162, 486, . . . ; Pattern: each term is 3 raised to the $n$ power times $\frac{2}{3}$.

**57.** Geometric, $20\left(\frac{1}{2}\right)^{n-1}$

**59.** Arithmetic, $\frac{2}{3}n + 2$    **61.** $\dfrac{3n + 1}{4n}$

**63.** $9045.00, $9090.23, $9135.68, $9181.35, $9227.26, $9273.40, $9319.76, $9366.36, $9413.20, $9460.26

**65.** (a)

| Year | 1 | 2 | 3 |
|---|---|---|---|
| Balance | $2200 | $4620 | $7282 |

| Year | 4 | 5 | 6 |
|---|---|---|---|
| Balance | $10,210.20 | $13,431.22 | $16,974.34 |
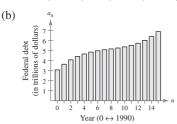
(b) $126,005.00   (c) $973,703.62

**67.** $S_6 = 240$, $S_7 = 440$, $S_8 = 810$, $S_9 = 1490$, $S_{10} = 2740$

**69.** (a) $a_n = 4.95n + 161.8$



(b) \$221.25/wk

**71.** (a) 3.08, 3.643, 4.084, 4.421, 4.672, 4.855, 4.988, 5.089, 5.176, 5.267, 5.38, 5.533, 5.744, 6.031, 6.412, 6.905

(b)



**73.** (a) $1.3(0.85)^n$ billion dollars

(b)
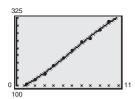
| Year | 1 | 2 |
|---|---|---|
| Budget amount | \$1.105 billion | \$0.939 billion |

| Year | 3 | 4 |
|---|---|---|
| Budget amount | \$0.798 billion | \$0.679 billion |

(c) Converges to 0

**75.** \$2995

**77.** (a)

$a_n = -0.108906n^3 + 1.99009n^2 + 13.2085n + 86.988$



(b) 351.6 billion

**79.**    $a_1 = 2$

$a_{10} \approx 2.5937$

$a_{100} \approx 2.7048$

$a_{1000} \approx 2.7169$

$a_{10,000} \approx 2.7181$

## SECTION 10.2    *(page 681)*

**Skills Review**    *(page 681)*

**1.** $\frac{77}{60}$    **2.** $\frac{73}{24}$    **3.** $\frac{31}{16}$    **4.** $\frac{40}{9}$    **5.** $\frac{21}{8}$

**6.** $\frac{31}{32}$    **7.** $\frac{3}{4}$    **8.** 0    **9.** 1    **10.** $\frac{1}{2}$

**1.** $S_1 = 1$                        **3.** $S_1 = 3$

$S_2 = \frac{5}{4} = 1.25$            $S_2 = -\frac{3}{2} = -1.5$

$S_3 = \frac{49}{36} \approx 1.361$      $S_3 = \frac{21}{4} = 5.25$

$S_4 = \frac{205}{144} \approx 1.424$    $S_4 = -\frac{39}{8} = -4.875$

$S_5 = \frac{5269}{3600} \approx 1.464$  $S_5 = \frac{165}{16} = 10.3125$

**5.** Geometric series: $r = \frac{3}{2} > 1$

**7.** Geometric series: $r = 1.055 > 1$

**9.** $n$th-Term Test: $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1 \neq 0$

**11.** $n$th-Term Test: $\lim\limits_{n\to\infty} \dfrac{n^2}{n^2+1} = 1 \neq 0$

**13.** $r = \frac{3}{4} < 1$    **15.** $r = 0.9 < 1$    **17.** 2    **19.** $\frac{2}{3}$

**21.** $\frac{16}{3}$    **23.** $\frac{10}{9}$    **25.** $\frac{3}{2}$    **27.** $\frac{1}{2}$

**29.** $\dfrac{17}{6}$    **31.** $\lim\limits_{n\to\infty} \dfrac{n+10}{10n+1} = \dfrac{1}{10} \neq 0$; diverges

**33.** $\lim\limits_{n\to\infty} \dfrac{n!+1}{n!} = 1 \neq 0$; diverges

**35.** $\lim\limits_{n\to\infty} \dfrac{3n-1}{2n+1} = \dfrac{3}{2} \neq 0$; diverges

**37.** Geometric series: $r = 1.075 > 1$; diverges

**39.** Geometric series: $r = \frac{1}{4} < 1$; converges    **41.** $\frac{4}{9}$

**43.** $\frac{9}{11}$    **45.** (a) $80{,}000(1 - 0.9^n)$  (b) 80,000

**47.** About 152.42 ft    **49.** \$7808.24

**51.** $\displaystyle\sum_{n=0}^{\infty} 100(0.75)^n = \$400$ million

**53.** $V = 225{,}000(0.7)^n$; \$37,815.75    **55.** \$3,801,020.63

**57.** $\displaystyle\sum_{n=1}^{\infty} \left(\dfrac{1}{2}\right)^n = -1 + \sum_{n=0}^{\infty} \left(\dfrac{1}{2}\right)^n = -1 + \dfrac{1}{1-\frac{1}{2}} = 1$

**59.** $P(2) = \dfrac{1}{8}$; $\displaystyle\sum_{n=0}^{\infty} \dfrac{1}{2}\left(\dfrac{1}{2}\right)^n = \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

**61.** About \$18,018.3 million

**63.** About 36.67 revolutions

**65.** 6    **67.** About 0.5431    **69.** $\dfrac{e^2}{e-1} \approx 4.3003$

**71.** False. $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, but $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

## SECTION 10.3    *(page 690)*

> ### Skills Review    *(page 690)*
>
> **1.** $\dfrac{1}{n+1}$    **2.** $n+1$    **3.** $\dfrac{3n}{n+1}$    **4.** $\dfrac{n+1}{n^2}$
>
> **5.** 1    **6.** 5    **7.** 1    **8.** $\frac{1}{3}$
>
> **9.** Geometric series    **10.** Not a geometric series

**1.** $p$-series    **3.** Not a $p$-series    **5.** Not a $p$-series

**7.** $p$-series    **9.** Converges    **11.** Diverges

**13.** Converges    **15.** Diverges    **17.** Converges

**19.** Converges    **21.** Converges    **23.** Converges

**25.** Diverges    **27.** Converges    **29.** Diverges

**31.** Converges    **33.** About 1.1777; maximum error $\leq \frac{1}{32}$.

**35.** About 1.9953; maximum error $\leq \dfrac{\sqrt{10}}{5} \approx 0.6325$.

**37.** $\displaystyle\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{1/[(n+1)^{3/2}]}{1/(n^{3/2})}$

$\qquad = \displaystyle\lim_{n\to\infty}\left(\dfrac{n}{n+1}\right)^{3/2} = 1$

**39.** $\displaystyle\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{1/(n+1)^3}{1/n^3}$

$\qquad = \displaystyle\lim_{n\to\infty}\left(\dfrac{n}{n+1}\right)^{3} = 1$

**41.** a; diverges: $p = \frac{3}{4} < 1$

**42.** d; diverges: $p = 1$, harmonic series

**43.** e; converges: $p = \frac{5}{2} > 1$    **44.** b; diverges: $p = \frac{2}{5} < 1$

**45.** f; converges: $p = \frac{3}{2} > 1$    **46.** c; converges: $p = 2 > 1$

**47.** Diverges; $n$th-Term Test

**49.** Converges; $p$-Series Test; about 7.82

**51.** Converges; Geometric Series Test; $\dfrac{-\pi}{\pi-4} \approx 3.6598$

**53.** Converges; Geometric Series Test; $\frac{3}{5}$

**55.** Converges; $p$-Series Test; about 0.4429

**57.** Diverges; Geometric Series Test

**59.** Diverges; Ratio Test

**61.** Diverges; $n$th-Term Test

**63.** Converges; Geometric Series Test; $\frac{10}{3}$

**65.** $\displaystyle\sum_{n=1}^{100}\dfrac{1}{n^2} \approx 1.635,\ \dfrac{\pi^2}{6} \approx 1.644934$

## MID-CHAPTER QUIZ    *(page 692)*

**1.** $-\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \frac{1}{256}, -\frac{1}{1024}$    **2.** $\frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}$

**3.** $-5, 5, -5, 5, -5$    **4.** $-1, 0, \frac{1}{6}, \frac{1}{12}, \frac{1}{40}$

**5.** Converges to 0    **6.** Converges to $\frac{1}{2}$

**7.** Converges to 0    **8.** Diverges    **9.** $\dfrac{n-1}{n^2}$

**10.** $(-1)^n \cdot 3^{1/n}$    **11.** $2^{(-1)^n}$

**12.** $S_1 = 0$    **13.** $S_1 = 1$

$\quad S_2 = \frac{1}{2} = 0.5$    $\quad S_2 = 0$

$\quad S_3 = \frac{5}{6} = 0.8\overline{3}$    $\quad S_3 = \frac{3}{4} = 0.75$

$\quad S_4 = \frac{23}{24} = 0.958\overline{3}$    $\quad S_4 = \frac{1}{4} = 0.25$

$\quad S_5 = \frac{119}{120} = 0.991\overline{6}$    $\quad S_5 = \frac{9}{16} = 0.5625$

**14.** Diverges    **15.** Converges    **16.** Diverges

**17.** Converges    **18.** Converges    **19.** Diverges

**20.** 12    **21.** 1    **22.** $\frac{50}{9}$

**23.** $1011.25, 1022.63, 1034.13, \ldots$,

$\quad \$1000\left(1 + \dfrac{0.045}{4}\right)^n = \$1000(1.01125)^n;\ \$1093.62$

**24.** About \$228.40 billion

## SECTION 10.4    *(page 701)*

> ### Skills Review    *(page 701)*
>
> **1.** $f(g(x)) = (x-1)^2$    **2.** $f(g(x)) = 6x + 3$
>
> $\quad g(f(x)) = x^2 - 1$    $\quad g(f(x)) = 6x + 1$
>
> **3.** $f(g(x)) = \sqrt{x^2 + 4}$
>
> $\quad g(f(x)) = x + 4,\ x \geq -4$
>
> **4.** $f(g(x)) = e^{x^2}$    **5.** $f'(x) = 5e^x$
>
> $\quad g(f(x)) = e^{2x}$    $\quad f''(x) = 5e^x$
>
> $\quad\quad\quad\quad\quad\quad\quad f'''(x) = 5e^x$
>
> $\quad\quad\quad\quad\quad\quad\quad f^{(4)}(x) = 5e^x$
>
> **6.** $f'(x) = \dfrac{1}{x}$    **7.** $f'(x) = 6e^{2x}$
>
> $\quad f''(x) = -\dfrac{1}{x^2}$    $\quad f''(x) = 12e^{2x}$
>
> $\quad f'''(x) = \dfrac{2}{x^3}$    $\quad f'''(x) = 24e^{2x}$
>
> $\quad f^{(4)}(x) = -\dfrac{6}{x^4}$    $\quad f^{(4)}(x) = 48e^{2x}$
>
> **8.** $f'(x) = \dfrac{1}{x}$    **9.** $\dfrac{n+1}{3}$    **10.** $\dfrac{n+3}{n+1}$
>
> $\quad f''(x) = -\dfrac{1}{x^2}$
>
> $\quad f'''(x) = \dfrac{2}{x^3}$
>
> $\quad f^{(4)}(x) = -\dfrac{6}{x^4}$

**1.** $1, \dfrac{x}{4}, \left(\dfrac{x}{4}\right)^2, \left(\dfrac{x}{4}\right)^3, \left(\dfrac{x}{4}\right)^4$

**3.** $-1, (x+1), -\dfrac{(x+1)^2}{2}, \dfrac{(x+1)^3}{6}, -\dfrac{(x+1)^4}{24}$

**5.** 2     **7.** 1     **9.** $\infty$     **11.** 0     **13.** 4

**15.** 5     **17.** 1     **19.** 3     **21.** $\frac{1}{2}$     **23.** $\infty$

**25.** $e \displaystyle\sum_{n=0}^{\infty} \dfrac{(x-1)^n}{n!}, \quad R = \infty$     **27.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{(3x)^n}{n!}, \quad R = \infty$

**29.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n, \quad R = 1$

**31.**

$1 + \dfrac{1}{2}(x-1) + \displaystyle\sum_{n=2}^{\infty} \dfrac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-1)^n}{2^n \cdot n!},$

$R = 1$

**33.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n (n+1)x^n, \quad R = 1$

**35.** $1 + \displaystyle\sum_{n=1}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} x^n, \quad R = 1$

**37.** $R = 2$ (all parts)     **39.** $R = 1$ (all parts)

**41.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{x^{3n}}{n!}$     **43.** $3 \displaystyle\sum_{n=0}^{\infty} \dfrac{x^{3n+2}}{n!}$     **45.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{4n}$

**47.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+2}}{n+1}$     **49.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n(x-1)^{n+1}}{n+1}$

**51.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n(x-1)^{n-1}$

**53.** 1.6487     **55.** $-0.6931$     **57.** $-2.3018$

## SECTION 10.5   *(page 710)*

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**Skills Review**   *(page 710)*

**1.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{3^n x^n}{n!}$   **2.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n 3^n x^n}{n!}$

**3.** $4 \displaystyle\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

**4.** $\ln 5 + \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}(x-1)^n}{n}$

**5.** $1 + \dfrac{x}{4} - \dfrac{3x^2}{4^2 2!} + \dfrac{3 \cdot 7 x^3}{4^3 3!} - \dfrac{3 \cdot 7 \cdot 11 x^4}{4^4 4!} + \cdots$

**6.** $1 + \dfrac{x}{2} - \dfrac{x^2}{2^2 2!} + \dfrac{1 \cdot 3 x^3}{2^3 3!} - \dfrac{1 \cdot 3 \cdot 5 x^4}{2^4 4!} + \cdots$

**7.** $\frac{47}{60}$     **8.** $\frac{311}{576}$     **9.** $\frac{5}{12}$     **10.** $\frac{77}{192}$

---

**1.** (a) $S_1(x) = 1 + x$     (b) $S_2(x) = 1 + x + \dfrac{x^2}{2}$

(c) $S_3(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$

(d) $S_4(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$

**3.** (a) $S_1(x) = 1 + 2x$     (b) $S_2(x) = 1 + 2x + 2x^2$

(c) $S_3(x) = 1 + 2x + 2x^2 + \dfrac{4x^3}{3}$

(d) $S_4(x) = 1 + 2x + 2x^2 + \dfrac{4x^3}{3} + \dfrac{2x^4}{3}$

**5.** (a) $S_1(x) = x$     (b) $S_2(x) = x - \dfrac{x^2}{2}$

(c) $S_3(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$

(d) $S_4(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}$

**7.** (a) $S_1(x) = 1 + \dfrac{x}{2}$     (b) $S_2(x) = 1 + \dfrac{x}{2} - \dfrac{x^2}{8}$

(c) $S_3(x) = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \dfrac{x^3}{16}$

(d) $S_4(x) = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \dfrac{x^3}{16} - \dfrac{5x^4}{128}$

**9.** (a) $S_1(x) = 1 - 2x$     (b) $S_2(x) = 1 - 2x + 3x^2$

(c) $S_3(x) = 1 - 2x + 3x^2 - 4x^3$

(d) $S_4(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4$

**11.** (a) $S_1(x) = x$     (b) $S_2(x) = x - x^2$

(c) $S_3(x) = x - x^2 + x^3$

(d) $S_4(x) = x - x^2 + x^3 - x^4$

**13.**

| $x$ | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
|---|---|---|---|---|---|
| $f(x)$ | 1.0000 | 1.1331 | 1.2840 | 1.4550 | 1.6487 |
| $S_1(x)$ | 1.0000 | 1.1250 | 1.2500 | 1.3750 | 1.5000 |
| $S_2(x)$ | 1.0000 | 1.1328 | 1.2813 | 1.4453 | 1.6250 |
| $S_3(x)$ | 1.0000 | 1.1331 | 1.2839 | 1.4541 | 1.6458 |
| $S_4(x)$ | 1.0000 | 1.1331 | 1.2840 | 1.4549 | 1.6484 |

**15.** (a) $S_2(x) = 1 - x^2$     (b) $S_4(x) = 1 - x^2 + x^4$

(c) $S_6(x) = 1 - x^2 + x^4 - x^6$

(d) $S_8(x) = 1 - x^2 + x^4 - x^6 + x^8$

**17.** $S_4(x) = 1 - \dfrac{x}{3} + \dfrac{2x^2}{9} - \dfrac{14x^3}{81} + \dfrac{35x^4}{243}$     **19.** d     **20.** c

**21.** a    **22.** b    **23.** 0.607    **25.** 0.4055
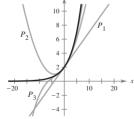
**27.** 0.74286    **29.** 0.481    **31.** 7    **33.** $\dfrac{1}{6!} \approx 0.00139$

**35.** (a) Answers will vary.    (b) 1    (c) $10

**37.**



## SECTION 10.6    *(page 717)*

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### Skills Review    *(page 717)*

**1.**    $f(2.4) = -0.04$    **2.**    $f(-0.6) = 0.064$
$f'(2.4) = 2.8$    $f'(-0.6) = 3.48$

**3.** $f(0.35) = 0.01$    **4.**    $f(1.4) = 0.30$
$f'(0.35) = 4.03$    $f'(1.4) = 12.88$

**5.** $4.9 \le x \le 5.1$    **6.** $0.798 \le x \le 0.802$

**7.** $5.97 \le x \le 6.03$    **8.** $-3.505 \le x \le -3.495$

**9.** $\left(\dfrac{3 + \sqrt{13}}{2}, 2 + \sqrt{13}\right)\left(\dfrac{3 - \sqrt{13}}{2}, 2 - \sqrt{13}\right)$

**10.** $\left(\dfrac{1 - \sqrt{5}}{2}, \dfrac{3 - \sqrt{5}}{2}\right), \left(\dfrac{1 + \sqrt{5}}{2}, \dfrac{3 + \sqrt{5}}{2}\right)$

---

**1.** $x_2 \approx 1.7324$, $x_3 \approx 1.7321$

**3.** Newton's Method: 0.682, Calculator: 0.682

**5.** Newton's Method: 1.25, Calculator: 1.25

**7.** Newton's Method: 0.567, Calculator: 0.567
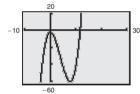
**9.** Newton's Method: $\pm 0.753$, Calculator: $\pm 0.753$
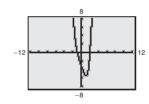
**11.** Newton's Method: $-4.596$, $-1.042$, 5.638
Calculator: $-4.596$, $-1.042$, 5.638

**13.** 2.893    **15.** 0.567
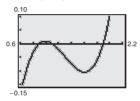
**17.** 11.8033    **19.** $-0.8937$, 2.0720
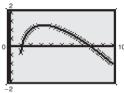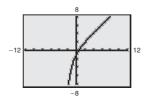


**21.** 0.9, 1.1, 1.9    **23.** 1.1459, 7.8541
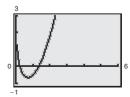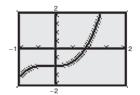


**25.** 0.5671    **27.** 0.2359, 1.3385



**29.** 0.8655



**31.** Newton's Method fails because $f'(x_1) = 0$.

**33.** Newton's Method fails because $1 = x_1 = x_3 = \ldots$;
$0 = x_2 = x_4 = \ldots$. So, the limit does not exist.

**35.** $x_{n+1} = \dfrac{x_n^2 + a}{2x_n}$    **37.** 2.646    **39.** 1.565

**41.** $f(x) = \dfrac{1}{x} - a$

$f'(x) = -\dfrac{1}{x^2}$

Newton's Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n - \dfrac{\dfrac{1}{x_n} - a}{-\dfrac{1}{x_n^2}}$

$= x_n(2 - ax_n)$

**43.** (1.939, 0.240)    **45.** $x \approx 1.563$ miles down the coast

**47.** About $384,356    **49.** $x \approx 40.45$ (about 4045 products)

**51.** False. Let $f(x) = \dfrac{x^2 - 1}{x - 1}$.

## REVIEW EXERCISES    *(page 724)*

**1.** $\dfrac{3}{2}, \dfrac{9}{4}, \dfrac{27}{8}, \dfrac{81}{16}, \dfrac{243}{32}$    **3.** $4, 8, \dfrac{32}{3}, \dfrac{32}{3}, \dfrac{128}{15}$

**5.** Converges to 0    **7.** Diverges    **9.** Converges to 5

**11.** Converges to 0    **13.** $\dfrac{1}{n!}$,   $n = 1, 2, 3, \ldots$

**15.** $(-1)^n \dfrac{2^n}{3^{n+1}}$,   $n = 0, 1, 2, \ldots$

**17.** (a) $15{,}000 + 10{,}000(n - 1)$   (b) $\$175{,}000$

**19.** $\$1.07, \$1.14, \$1.23, \$1.31, \$1.40, \$1.50, \$1.61, \$1.72,$
$\$1.84, \$1.97$

**21.** $S_0 = 1$

$S_1 = \dfrac{5}{2} = 2.5$

$S_2 = \dfrac{19}{4} = 4.75$

$S_3 = \dfrac{65}{8} = 8.125$

$S_4 = \dfrac{211}{16} = 13.1875$

**23.** $S_1 = \dfrac{1}{2} = 0.5$

$S_2 = \dfrac{11}{24} \approx 0.4583$

$S_3 = \dfrac{331}{720} \approx 0.4597$

$S_4 = \dfrac{18{,}535}{40{,}320} \approx 0.4597$

$S_5 = \dfrac{1{,}668{,}151}{3{,}628{,}800} \approx 0.4597$

**25.** Diverges    **27.** Converges

**29.** $\lim\limits_{n\to\infty} \dfrac{2n}{n + 5} = 2 \neq 0$    **31.** $\lim\limits_{n\to\infty} \dfrac{n^2}{n^2 + 1} = 1 \neq 0$

**33.** $\dfrac{5}{4}$    **35.** 6    **37.** Diverges

**39.** Diverges    **41.** Converges to $\dfrac{13}{4}$

**43.** $\$4{,}371{,}379.65$    **45.** $\$2000$    **47.** $\$2181.82$

**49.** About 2.782%    **51.** Converges    **53.** Converges

**55.** a    **56.** c    **57.** d    **58.** b

**59.** About 1.0172; error $\leq \dfrac{1}{(5)4^5} \approx 1.9531 \times 10^{-4}$

**61.** About 2.09074; error $\leq \dfrac{1}{(1/4)(6)^{1/4}} \approx 2.5558$

**63.** Converges    **65.** Diverges    **67.** Converges

**69.** $R = 10$    **71.** $R = 1$    **73.** $R = 0$

**75.** $\displaystyle\sum_{n=0}^{\infty} \left(-\dfrac{1}{2}\right)^n \dfrac{x^n}{n!}$

**77.** $1 + \displaystyle\sum_{n=1}^{\infty} (-1)^n \left[\dfrac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n \, n!}\right](x - 1)^n$

**79.** $1 + \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \left[\dfrac{1 \cdot 3 \cdot 7 \cdots (4n - 5)}{4^n \, n!}\right] x^n$

**81.** $\ln 2 + \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{(x/2)^n}{n}$

**83.** $1 + 2x^2 + x^4 + \cdots$    **85.** $x^2 \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!} = \sum_{n=0}^{\infty} \dfrac{x^{n+2}}{n!}$

**87.** $x^2 \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$

**89.** $\dfrac{1}{9} - \dfrac{2}{27}x + \dfrac{1}{27}x^2 - \dfrac{4}{243}x^3 + \dfrac{5}{729}x^4 - \dfrac{2}{729}x^5 + \dfrac{7}{6561}x^6$

**91.** $\ln 3 + \dfrac{1}{3}(x - 1) - \dfrac{1}{18}(x - 1)^2 + \dfrac{1}{81}(x - 1)^3$
$\quad - \dfrac{1}{324}(x - 1)^4 + \dfrac{1}{1215}(x - 1)^5 - \dfrac{1}{4374}(x - 1)^6$

**93.** 4.7705    **95.** 0.9163    **97.** $\dfrac{1}{32}$    **99.** 0.301

**101.** 0.1233    **103.** 0.5, $\$12.50$

**105.** $-1.532, -0.347, 1.879$    **107.** 0.258

**109.** 1.341    **111.** 0.773

## CHAPTER TEST    (page 727)

**1.** (a) $\dfrac{3}{5}, \dfrac{9}{25}, \dfrac{27}{125}, \dfrac{81}{625}, \dfrac{243}{3125}$    (b) Converges    (c) 0

**2.** (a) $\dfrac{1}{7}, \dfrac{1}{4}, \dfrac{9}{31}, \dfrac{4}{13}, \dfrac{25}{79}$    (b) Converges    (c) $\dfrac{1}{3}$

**3.** (a) $\dfrac{1}{6}, -\dfrac{1}{6}, \dfrac{1}{6}, -\dfrac{1}{6}, \dfrac{1}{6}$    (b) Diverges

(c) The limit does not exist.

**4.** (a) $4, 8, 6, \dfrac{8}{3}, \dfrac{5}{6}$    (b) Converges    (c) 0

**5.** $\dfrac{(-1)^{n+1} n}{n^2 + 1}$    **6.** Converges    **7.** Diverges

**8.** Converges    **9.** Diverges    **10.** $\dfrac{7}{3}$

**11.** (a) $S_0 = -1$    (b) $R = 3$

$S_1 = -1 + \dfrac{x}{3}$

$S_2 = -1 + \dfrac{x}{3} - \dfrac{x^2}{9}$

$S_3 = -1 + \dfrac{x}{3} - \dfrac{x^2}{9} + \dfrac{x^3}{27}$

$S_4 = -1 + \dfrac{x}{3} - \dfrac{x^2}{9} + \dfrac{x^3}{27} - \dfrac{x^4}{81}$

**12.** (a) $S_0 = 1$    (b) $R = \infty$

$S_1 = 1 + \dfrac{x}{2}$

$S_2 = 1 + \dfrac{x}{2} + \dfrac{x^2}{6}$

$S_3 = 1 + \dfrac{x}{2} + \dfrac{x^2}{6} + \dfrac{x^3}{24}$

$S_4 = 1 + \dfrac{x}{2} + \dfrac{x^2}{6} + \dfrac{x^3}{24} + \dfrac{x^4}{120}$

**13.** (a)    (b) $R = 1$

$S_0 = \dfrac{1}{16}$

$S_1 = \dfrac{1}{16} - \dfrac{(x - 3)}{25}$

$S_2 = \dfrac{1}{16} - \dfrac{(x - 3)}{25} + \dfrac{(x - 3)^2}{36}$

$S_3 = \dfrac{1}{16} - \dfrac{(x - 3)}{25} + \dfrac{(x - 3)^2}{36} - \dfrac{(x - 3)^3}{49}$

$S_4 = \dfrac{1}{16} - \dfrac{(x - 3)}{25} + \dfrac{(x - 3)^2}{36} - \dfrac{(x - 3)^3}{49} + \dfrac{(x - 3)^4}{64}$

**14.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-4x)^n}{n!}$    **15.** $1 + \dfrac{2x}{3} - \dfrac{x^2}{9} + \dfrac{4x^3}{81} - \dfrac{7x^4}{243} + \cdots$
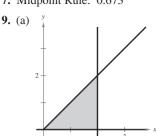
**16.** Taylor polynomial: 0.2232, Calculator: 0.2231

**17.** Taylor polynomial: 1.338, Calculator: 1.342

**18.** (a) $a_n = 0.5413n^2 - 3.221n + 6.15$

(b) 2003 ($n \approx 2.975$); $a_3 = 1.36\%$;
Comparisons will vary.

**19.** Newton's Method: 1.213, Calculator: 1.213

# APPENDIX A    *(page A8)*

**1.** Left Riemann sum: 0.518

Right Riemann sum: 0.768

**3.** Left Riemann sum: 0.746

Right Riemann sum: 0.646

**5.** Left Riemann sum: 0.859

Right Riemann sum: 0.659

**7.** Midpoint Rule: 0.673

**9.** (a)
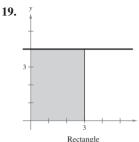
(b) Answers will vary.
(c) Answers will vary.
(d) Answers will vary.
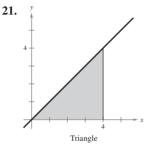
(e)

| $n$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| Left sum, $S_L$ | 1.6 | 1.8 | 1.96 | 1.98 |
| Right sum, $S_R$ | 2.4 | 2.2 | 2.04 | 2.02 |

(f) Answers will vary.

**11.** $\displaystyle\int_0^5 3 \, dx$

**13.** $\displaystyle\int_{-4}^4 (4 - |x|) \, dx = \int_{-4}^0 (4 + x) \, dx + \int_0^4 (4 - x) \, dx$

**15.** $\displaystyle\int_{-2}^2 (4 - x^2) \, dx$    **17.** $\displaystyle\int_0^2 \sqrt{x + 1} \, dx$

**19.**



Rectangle

$A = 12$

**21.**



Triangle

$A = 8$

**23.**



Trapezoid

$A = 14$

**25.**



Triangle

$A = 1$

**27.**



Semicircle

$A = \dfrac{9\pi}{2}$
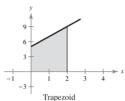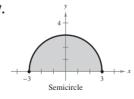
**29.** Answers will vary.    **31.** $>$

# APPENDIX C

## SECTION C.1    *(page A24)*

**Skills Review**    *(page A24)*

**1.** $y' = 6x + 2$    **2.** $y' = -6x^2 - 8$

$y'' = 6$        $y'' = -12x$

**3.** $y' = -6e^{2x}$    **4.** $y' = -6xe^{x^2}$

$y'' = -12e^{2x}$      $y'' = -6e^{x^2}(2x^2 + 1)$

**5.** $\dfrac{1 - x}{y}$    **6.** $\dfrac{2}{3y^2 + 4}$    **7.** $-\dfrac{y}{2x}$

**8.** $-\dfrac{y}{x}$    **9.** $k = 2 \ln 3 - \ln \dfrac{17}{2} \approx 0.0572$

**10.** $k = \ln 10 - \dfrac{\ln 41}{2} \approx 0.4458$

**1.** $y' = 3x^2$

**3.** $y' = -2e^{-2x}$ and $y' + 2y = -2e^{-2x} + 2(e^{-2x}) = 0$

**5.** $y' = 6x^2$ and $y' - \dfrac{3}{x}y = 6x^2 - \dfrac{3}{x}(2x^3) = 0$

**7.** $y'' = 2$ and $x^2 y'' - 2y = x^2(2) - 2(x^2) = 0$

**9.** $y' = 4e^{2x}$, $y'' = 8e^{2x}$, and

$y'' - y' - 2y = 8e^{2x} - 4e^{2x} - 2(2e^{2x}) = 0$

**11.** $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$    **13.** $\dfrac{dy}{dx} = 4Ce^{4x} = 4y$

**15.** $\dfrac{dy}{dt} = -\dfrac{1}{3}Ce^{-t/3}$ and

$$3\dfrac{dy}{dt} + y - 7 = 3\left(-\dfrac{1}{3}Ce^{-t/3}\right) + (Ce^{-t/3} + 7) - 7 = 0$$

**17.** $xy' - 3x - 2y = x(2Cx - 3) - 3x - 2(Cx^2 - 3x) = 0$

**19.** $xy' + y = x\left(2x + 2 - \dfrac{C}{x^2}\right) + \left(x^2 + 2x + \dfrac{C}{x}\right)$

$$= x(3x + 4)$$

**21.** $2y'' + 3y' - 2y = 2\left(\tfrac{1}{4}C_1e^{x/2} + 4C_2e^{-2x}\right)$

$$+ 3\left(\tfrac{1}{2}C_1e^{x/2} - 2C_2e^{-2x}\right)$$

$$- 2(C_1e^{x/2} + C_2e^{-2x}) = 0$$

**23.** $y' - \dfrac{ay}{x} = \left(\dfrac{4bx^3}{4 - a} + aCx^{a-1}\right) - \dfrac{a}{x}\left(\dfrac{bx^4}{4 - a} + Cx^a\right)$

$$= bx^3$$

**25.** $y' + 2xy = -\dfrac{4Cxe^{x^2}}{(1 - Ce^{x^2})^2} + 2x\left(\dfrac{2}{1 + Ce^{x^2}}\right) = xy^2$

**27.** $y' = \ln x + 1 + C$ and

$$x(y' - 1) - (y - 4) = x(\ln x + 1 + C - 1)$$

$$- (x\ln x + Cx + 4 - 4) = 0$$

**29.** $2x + 2yy' = Cy'$

$$y' = \dfrac{2x}{C - 2y} = \dfrac{2xy}{Cy - 2y^2}$$

$$= \dfrac{2xy}{(x^2 + y^2) - 2y^2} = \dfrac{2xy}{x^2 - y^2}$$

**31.** $x + y = \dfrac{C}{x}$

$$y'' = \dfrac{2C}{x^3}$$
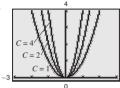
$$x^2y'' - 2(x + y) = \dfrac{2C}{x} - \dfrac{2C}{x} = 0$$
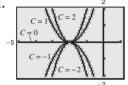
**33.** Solution    **35.** Not a solution    **37.** Not a solution
**39.** Solution    **41.** $y = 3e^{-2x}$    **43.** $y = 5 + 0.5 \ln x$
**45.** $y = 3e^{4x} + 2e^{-3x}$    **47.** $y = \tfrac{4}{3}(3 - x)e^{2x/3}$
**49.**



**51.**
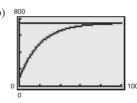


**53.** $y = x^3 + C$    **55.** $y = x + 3\ln|x| + C$
**57.** $y = \dfrac{1}{2}\ln\left(\dfrac{x - 1}{x + 1}\right) + C$
**59.** $y = \tfrac{2}{5}(x - 3)^{3/2}(x + 2) + C$

**61.** $y^2 = \tfrac{1}{4}x^3$    **63.** $y = 3e^x$
**65.** (a) $N = 750 - 650e^{-0.0484t}$

(b)

(c) $N \approx 214$

**67.**

| Year, $t$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| Units, $x$ | 3867 | 7235 | 10,169 | 12,725 | 14,951 |

**69.** Because

$$\dfrac{ds}{dh} = -\dfrac{13}{\ln 3}\left(\dfrac{1/2}{h/2}\right) = -\dfrac{13}{\ln 3}\dfrac{1}{h}, \text{ and } -\dfrac{13}{\ln 3}$$

is a constant, we can conclude that the equation is a solution of $ds/dh = k/h$ where $k = -13/(\ln 3)$.

**71.** $k = 0.07$

**73.** False. From Example 1, $y = e^x$ is a solution of $y'' - y = 0$, but $y = e^x + 1$ is not.

## SECTION C.2   *(page A32)*

**Skills Review**   *(page A32)*

**1.** $\tfrac{2}{5}x^{5/2} + C$    **2.** $\tfrac{1}{4}t^4 - \tfrac{3}{4}t^{4/3} + C$

**3.** $2\ln|x - 5| + C$    **4.** $\tfrac{1}{4}\ln|2y^2 + 1| + C$

**5.** $\tfrac{1}{2}e^{2y} + C$    **6.** $-\tfrac{1}{2}e^{1-x^2} + C$    **7.** $C = -10$

**8.** $C = 5$    **9.** $k = \dfrac{\ln 5}{2} \approx 0.8047$

**10.** $k = -2\ln 3 - \ln 2 \approx -2.8904$

**1.** Yes    **3.** Yes

$$(y + 3)\,dy = x\,dx \qquad dy = \left(\dfrac{1}{x} + 1\right)dx$$

**5.** No. The variables cannot be separated.
**7.** $y = x^2 + C$    **9.** $y = \sqrt[3]{x + C}$
**11.** $C = 2x^2 - (y + 1)^2$    **13.** $y = Ce^{x^2/2}$
**15.** $y^2 = \tfrac{1}{2}e^t + C$
**17.** $y = 1 - \left(C - \dfrac{x}{2}\right)^2$
**19.** $y = C(2 + x)^2$    **21.** $y = Cx$
**23.** $3y^2 + 2y^3 = 3x^2 + C$    **25.** $y = -e^{-x} - x + C$
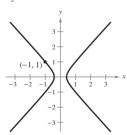
**27.** $y^2 = 2e^x + 14$     **29.** $y = -4 - e^{-x^2/2}$

**31.** $P = 5e^{6t}$

**33.** $5y^2 = 6x^2 - 1$ or $6x^2 - 5y^2 = 1$



**35.** $v = 34.56(1 - e^{-0.1t})$     **37.** $T \approx 383.298°F$

**39.** (a) $T \approx 7.277°F$     (b) $t \approx 5.158$ hr

**41.** $N = 30 + Ce^{-kt}$     **43.** $y = Cx^{-k}$

## SECTION C.3   (page A37)

> ### Skills Review   (page A37)
>
> **1.** $e^x + 1$     **2.** $e^{3x} + 1$     **3.** $\dfrac{1}{x^3}$     **4.** $x^2 e^x$
>
> **5.** $2e^{2x} + C$     **6.** $\frac{1}{6}e^{3x^2} + C$     **7.** $\frac{1}{2}\ln|2x + 5| + C$
> **8.** $\frac{1}{2}\ln|x^2 + 2x + 3| + C$     **9.** $\frac{1}{12}(4x - 3)^3 + C$
> **10.** $\frac{1}{6}(x^2 - 1)^3 + C$

**1.** $y' + \dfrac{-3}{2x^2}y = \dfrac{x}{2}$     **3.** $y' + \dfrac{1}{x}y = e^x$

**5.** $y' + \dfrac{1}{1-x}y = \dfrac{1}{x-1}$     **7.** $y = 2 + Ce^{-3x}$

**9.** $y = e^{-x}(x + C)$     **11.** $y = x^2 + 2x + \dfrac{C}{x}$

**13.** $y = \dfrac{1}{5} + Ce^{-(5/2)x^2}$     **15.** $y = \dfrac{x^3 - 3x + C}{3(x - 1)}$

**17.** $y = e^{1/x^2}\left(-\dfrac{1}{2x^2} + C\right)$     **19.** $y = Ce^{-x} + 4$

**21.** $y = Ce^{x^2} - 1$

**23.** c     **24.** d     **25.** a     **26.** b

**27.** $y = 3e^x$     **29.** $xy = 4$

**31.** $y = 1 + 5e^{-x^3}$     **33.** $y = x^2(5 - \ln|x|)$

**35.** $S = t + 95(1 - e^{-t/5})$

**37.** $p = 400 - 3x$     **39.** $p = 15(4 + e^{-t})$

**41.** (a) $A = \dfrac{P}{r^2}(rt - 1 + e^{-rt})$     (b) $A \approx \$18,924,053.07$

**43.** $v = -\dfrac{gm}{k} + Ce^{-kt/m}$     **45.** Answers will vary.

## SECTION C.4   (page A44)

> ### Skills Review   (page A44)
>
> **1.** $y = \frac{3}{2}x^2 + C$     **2.** $y^2 = 3x + C$
>
> **3.** $y = Ce^{x^2}$     **4.** $y^4 = \frac{1}{2}(x - 4)^2 + C$
>
> **5.** $y = 2 + Ce^{-2x}$     **6.** $y = xe^{-2x} + Ce^{-2x}$
>
> **7.** $y = 1 + Ce^{-x^2/2}$     **8.** $y = \frac{1}{4}x^2 + Cx^{-2}$
>
> **9.** $\dfrac{dy}{dx} = Cx^2$     **10.** $\dfrac{dx}{dt} = C(x - t)$

**1.** $y = e^{(x \ln 2)/3} \approx e^{0.2310x}$     **3.** $y = 4e^{-(x \ln 4)/4}$
$\approx 4e^{-0.3466x}$

**5.** $y = \frac{1}{2}e^{(\ln 2)x} \approx \frac{1}{2}e^{0.6931x}$     **7.** $\$4451.08$

**9.** $S = L(1 - e^{-kt})$     **11.** $y = \dfrac{20}{1 + 19e^{-0.5889x}}$
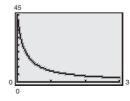
**13.** $y = \dfrac{5000}{1 + 19e^{-0.10156x}}$     **15.** $N = \dfrac{500}{1 + 4e^{-0.2452t}}$
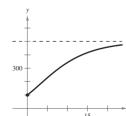
**17.** $\dfrac{dP}{dn} = kP(L - P)$, $P = \dfrac{CL}{e^{-Lkn} + C}$

**19.** $y = \dfrac{360}{8 + 41t}$     **21.** $y = 500e^{-1.6094e^{-0.1451t}}$



**23.** 34 beavers     **25.** 92%

**27.** (a) $y = Ce^{kt}$     (b) $\approx 6.2$ hr

**29.** 38.843 lb/gal     **31.** $\approx 3.15$ hr

**33.** $P = Ce^{kt} - \dfrac{N}{k}$     **35.** $A = \dfrac{P}{r}(e^{rt} - 1)$

**37.** $\$7,305,295.15$     **39.** (a) $C = C_0 e^{-Rt/V}$     (b) 0

**41.** (a) $C(t) = \dfrac{Q}{R}(1 - e^{-Rt/V})$     (b) $\dfrac{Q}{R}$

# Answers to Checkpoints

## CHAPTER 0

### SECTION 0.1

**Checkpoint 1**  $x < 5$  or  $(-\infty, 5)$

**Checkpoint 2**  $x < -2$  or  $x > 5$; $(-\infty, -2) \cup (5, \infty)$

**Checkpoint 3**  $200 \le x \le 400$; so the daily production levels during the month varied between a low of 200 units and a high of 400 units.

### SECTION 0.2

**Checkpoint 1**  $8$; $8$; $-8$

**Checkpoint 2**  $2 \le x \le 10$

**Checkpoint 3**  $\$4027.50 \le C \le \$11,635$

### SECTION 0.3

**Checkpoint 1**  $\frac{4}{9}$

**Checkpoint 2**  $8$

**Checkpoint 3**  (a) $3x^6$  (b) $8x^{7/2}$  (c) $4x^{4/3}$

**Checkpoint 4**  (a) $x(x^2 - 2)$  (b) $2x^{1/2}(1 + 4x)$

**Checkpoint 5**  $\dfrac{(3x - 1)^{3/2}(13x - 2)}{(x + 2)^{1/2}}$

**Checkpoint 6**  $\dfrac{x^2(5 + x^3)}{3}$

**Checkpoint 7**  (a) $[2, \infty)$  (b) $(2, \infty)$  (c) $(-\infty, \infty)$

### SECTION 0.4

**Checkpoint 1**  (a) $\dfrac{-2 \pm \sqrt{2}}{2}$  (b) $4$  (c) No real zeros

**Checkpoint 2**  (a) $x = -3$ and $x = 5$  (b) $x = -1$

(c) $x = \frac{3}{2}$ and $x = 2$

**Checkpoint 3**  $(-\infty, -2] \cup [1, \infty)$

**Checkpoint 4**  $-1, \frac{1}{2}, 2$

### SECTION 0.5

**Checkpoint 1**  (a) $\dfrac{x^2 + 2}{x}$  (b) $\dfrac{3x + 1}{(x + 1)(2x + 1)}$

**Checkpoint 2**  (a) $\dfrac{3x + 4}{(x + 2)(x - 2)}$  (b) $-\dfrac{x + 1}{3x(x + 2)}$

**Checkpoint 3**

(a) $\dfrac{(A + B + C)x^2 + (A + 3B)x + (-2A + 2B - C)}{(x + 1)(x - 1)(x + 2)}$

(b) $\dfrac{(A + C)x^2 + (-A + B + 2C)x + (-2A - 2B + C)}{(x + 1)^2(x - 2)}$

**Checkpoint 4**  (a) $\dfrac{3x + 8}{4(x + 2)^{3/2}}$  (b) $\dfrac{1}{\sqrt{x^2 + 4}}$
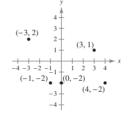
**Checkpoint 5**  $\dfrac{\sqrt{x^2 + 4}}{x^2}$
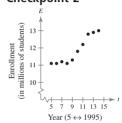
**Checkpoint 6**  (a) $\dfrac{5\sqrt{2}}{4}$  (b) $\dfrac{x + 2}{4\sqrt{x + 2}}$  (c) $\dfrac{\sqrt{6} + \sqrt{3}}{3}$

(d) $\dfrac{\sqrt{x + 2} - \sqrt{x}}{2}$

## CHAPTER 1

### SECTION 1.1

**Checkpoint 1**          **Checkpoint 2**



**Checkpoint 3**  $5$

**Checkpoint 4**  $d_1 = \sqrt{20}, d_2 = \sqrt{45}, d_3 = \sqrt{65}$

$d_1^2 + d_2^2 = 20 + 45 = 65 = d_3^2$

**Checkpoint 5**  25 yd

**Checkpoint 6**  $(-2, 5)$
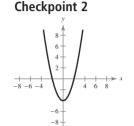
**Checkpoint 7**  $\$13.25$ billion

**Checkpoint 8**  $(-1, -4), (1, -2), (1, 2), (-1, 0)$

### SECTION 1.2

**Checkpoint 1**          **Checkpoint 2**

**Checkpoint 3** (a) x-intercepts: $(3, 0), (-1, 0)$
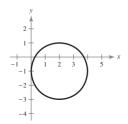
y-intercept: $(0, -3)$

(b) x-intercept: $(-4, 0)$

y-intercepts: $(0, 2), (0, -2)$

**Checkpoint 4** $(x + 2)^2 + (y - 1)^2 = 25$

**Checkpoint 5** $(x - 2)^2 + (y + 1)^2 = 4$
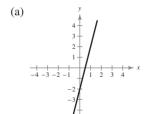


**Checkpoint 6** 12,500 units
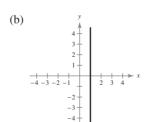
**Checkpoint 7** 4 million units at \$122/unit

**Checkpoint 8** The projection obtained from the model is \$9456.26 million, which is close to the *Value Line* projection.

## SECTION 1.3

**Checkpoint 1**

(a)



(b)



**Checkpoint 2** Yes, $\frac{27}{312} \approx 0.08654 > \frac{1}{12} = 0.08\overline{3}$.

**Checkpoint 3** The y-intercept $(0, 875)$ tells you that the original value of the copier is \$875. The slope of $m = -175$ tells you that the value decreases by \$175/yr.

**Checkpoint 4** (a) 2  (b) $-\frac{1}{2}$

**Checkpoint 5** $y = 2x + 4$

**Checkpoint 6** $S = 0.79t + 2.06$; \$6.80

**Checkpoint 7** (a) $y = \frac{1}{2}x$  (b) $y = -2x + 5$

**Checkpoint 8** $V = -1375t + 12,000$

## SECTION 1.4

**Checkpoint 1** (a) Yes, $y = x - 1$.

(b) No, $y = \pm\sqrt{4 - x^2}$.

(c) No, $y = \pm\sqrt{2 - x}$.

(d) Yes, $y = x^2$.

**Checkpoint 2** (a) Domain: $[-1, \infty)$; Range: $[0, \infty)$

(b) Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

**Checkpoint 3** $f(0) = 1, f(1) = -3, f(4) = -3$

No, $f$ is not one-to-one.

**Checkpoint 4** (a) $x^2 + 2x\,\Delta x + (\Delta x)^2 - 2x - 2\,\Delta x + 3$

(b) $2x + \Delta x - 2, \Delta x \neq 0$

**Checkpoint 5** (a) $2x^2 + 5$  (b) $4x^2 + 4x + 3$

**Checkpoint 6** (a) $f^{-1}(x) = 5x$  (b) $f^{-1}(x) = \frac{1}{3}(x - 2)$

**Checkpoint 7** $f^{-1}(x) = \sqrt{x - 2}$

**Checkpoint 8** $f(x) = x^2 + 4$

$y = x^2 + 4$

$x = y^2 + 4$

$x - 4 = y^2$

$\pm\sqrt{x - 4} = y$

## SECTION 1.5

**Checkpoint 1** 6

**Checkpoint 2** (a) 4  (b) Does not exist  (c) 4

**Checkpoint 3** 5

**Checkpoint 4** 12

**Checkpoint 5** 7

**Checkpoint 6** $\frac{1}{4}$

**Checkpoint 7** (a) $-1$  (b) 1

**Checkpoint 8** 1

**Checkpoint 9** $\lim_{x \to 1^-} f(x) = 12$ and $\lim_{x \to 1^+} f(x) = 14$

$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

**Checkpoint 10** Does not exist

## SECTION 1.6

**Checkpoint 1** (a) $f$ is continuous on the entire real line.

(b) $f$ is continuous on the entire real line.

**Checkpoint 2** (a) $f$ is continuous on $(-\infty, 1)$ and $(1, \infty)$.
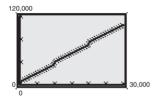
(b) $f$ is continuous on $(-\infty, 2)$ and $(2, \infty)$.

(c) $f$ is continuous on the entire real line.

**Checkpoint 3** $f$ is continuous on $[2, \infty)$.

**Checkpoint 4** $f$ is continuous on $[-1, 5]$.

**Checkpoint 5**



**Checkpoint 6**   $A = 10,000(1 + 0.02)^{[\![4t]\!]}$

# CHAPTER 2

## SECTION 2.1

**Checkpoint 1**   3

**Checkpoint 2**   For the months on the graph to the left of July, the tangent lines have positive slopes. For the months to the right of July, the tangent lines have negative slopes. The average daily temperature is increasing prior to July and decreasing after July.

**Checkpoint 3**   4

**Checkpoint 4**   2

**Checkpoint 5**   $m = 8x$

At $(0, 1)$, $m = 0$.

At $(1, 5)$, $m = 8$.

**Checkpoint 6**   $2x - 5$

**Checkpoint 7**   $-\dfrac{4}{t^2}$

## SECTION 2.2

**Checkpoint 1**   (a) 0   (b) 0   (c) 0   (d) 0

**Checkpoint 2**   (a) $4x^3$   (b) $-\dfrac{3}{x^4}$   (c) $2w$   (d) $-\dfrac{1}{t^2}$

**Checkpoint 3**   $f'(x) = 3x^2$

$m = f'(-1) = 3;$

$m = f^{-1}(0) = 0;$

$m = f^{-1}(1) = 3$

**Checkpoint 4**   (a) $8x$   (b) $\dfrac{8}{\sqrt{x}}$

**Checkpoint 5**   (a) $\frac{1}{4}$   (b) $-\frac{2}{5}$

**Checkpoint 6**   (a) $-\dfrac{9}{2x^3}$   (b) $-\dfrac{9}{8x^3}$

**Checkpoint 7**   (a) $\dfrac{\sqrt{5}}{2\sqrt{x}}$   (b) $\dfrac{1}{3x^{2/3}}$

**Checkpoint 8**   $-1$

**Checkpoint 9**   $y = -x + 2$

**Checkpoint 10**   $R'(13) \approx \$1.18/\text{yr}$

## SECTION 2.3

**Checkpoint 1**   (a) $0.5\overline{6}$ mg/ml/min

(b) 0 mg/ml/min

(c) $-1.5$ mg/ml/min

**Checkpoint 2**   (a) $-16$ ft/sec   (b) $-48$ ft/sec

(c) $-80$ ft/sec

**Checkpoint 3**   When $t = 1.75$, $h'(1.75) = -56$ ft/sec.

When $t = 2$, $h'(2) = -64$ ft/sec.

**Checkpoint 4**   $h = -16t^2 + 16t + 12$

$v = h' = -32t + 16$

**Checkpoint 5**   When $x = 100$, $\dfrac{dP}{dx} = \$16/\text{unit}$.

Actual gain $= \$16.06$

**Checkpoint 6**   $p = 11 - \dfrac{x}{2000}$

**Checkpoint 7**   Revenue: $R = 2000x - 4x^2$

Marginal revenue: $\dfrac{dR}{dx} = 2000 - 8x$

**Checkpoint 8**   $\dfrac{dP}{dx} = \$1.44/\text{unit}$

Actual increase in profit $\approx \$1.44$
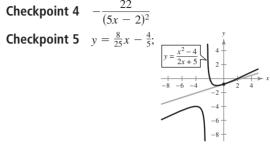
## SECTION 2.4

**Checkpoint 1**   $-27x^2 + 12x + 24$

**Checkpoint 2**   $\dfrac{2x^2 - 1}{x^2}$

**Checkpoint 3**   (a) $18x^2 + 30x$   (b) $12x + 15$

**Checkpoint 4**   $-\dfrac{22}{(5x - 2)^2}$

**Checkpoint 5**   $y = \frac{8}{25}x - \frac{4}{5};$



**Checkpoint 6**   $\dfrac{-3x^2 + 4x + 8}{x^2(x + 4)^2}$

**Checkpoint 7**   (a) $\frac{2}{5}x + \frac{4}{5}$   (b) $3x^3$

**Checkpoint 8**  $\dfrac{2x^2 - 4x}{(x - 1)^2}$

**Checkpoint 9**

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\dfrac{dP}{dt}$ | 0 | $-50$ | $-16$ | $-6$ | $-2.77$ | $-1.48$ | $-0.88$ | $-0.56$ |

As $t$ increases, the rate at which the blood pressure drops decreases.

## SECTION 2.5

**Checkpoint 1**  (a) $u = g(x) = x + 1$
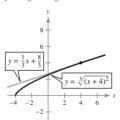
$$y = f(u) = \frac{1}{\sqrt{u}}$$

(b) $u = g(x) = x^2 + 2x + 5$

$$y = f(u) = u^3$$

**Checkpoint 2**  $6x^2(x^3 + 1)$

**Checkpoint 3**  $4(2x + 3)(x^2 + 3x)^3$

**Checkpoint 4**  $y = \frac{1}{3}x + \frac{8}{3}$



**Checkpoint 5**  (a) $-\dfrac{8}{(2x + 1)^2}$  (b) $-\dfrac{6}{(x - 1)^4}$

**Checkpoint 6**  $\dfrac{x(3x^2 + 2)}{\sqrt{x^2 + 1}}$

**Checkpoint 7**  $-\dfrac{12(x + 1)}{(x - 5)^3}$

**Checkpoint 8**  About $3.27/yr

## SECTION 2.6

**Checkpoint 1**  $f'(x) = 18x^2 - 4x$, $f''(x) = 36x - 4$, $f'''(x) = 36$, $f^{(4)}(x) = 0$

**Checkpoint 2**  18
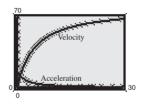
**Checkpoint 3**  $\dfrac{120}{x^6}$

**Checkpoint 4**  $s(t) = -16t^2 + 64t + 80$

$v(t) = s'(t) = -32t + 64$

$a(t) = v'(t) = s''(t) = -32$

**Checkpoint 5**  $-9.8$ m/sec$^2$

**Checkpoint 6**



Acceleration approaches zero.

## SECTION 2.7

**Checkpoint 1**  $-\dfrac{2}{x^3}$

**Checkpoint 2**  (a) $12x^2$  (b) $6y\dfrac{dy}{dx}$  (c) $1 + 5\dfrac{dy}{dx}$

(d) $y^3 + 3xy^2\dfrac{dy}{dx}$

**Checkpoint 3**  $\frac{3}{4}$

**Checkpoint 4**  $\dfrac{dy}{dx} = -\dfrac{x - 2}{y - 1}$

**Checkpoint 5**  $\frac{5}{9}$

**Checkpoint 6**  $\dfrac{dx}{dp} = -\dfrac{2}{p^2(0.002x + 1)}$

## SECTION 2.8

**Checkpoint 1**  9

**Checkpoint 2**  $12\pi \approx 37.7$ ft$^2$/sec

**Checkpoint 3**  $72\pi \approx 226.2$ in.$^2$/min

**Checkpoint 4**  $1500/day

**Checkpoint 5**  $28,400/wk

# CHAPTER 3

## SECTION 3.1

**Checkpoint 1**  $f'(x) = 4x^3$

$f'(x) < 0$ if $x < 0$; therefore, $f$ is decreasing on $(-\infty, 0)$.

$f'(x) > 0$ if $x > 0$; therefore, $f$ is increasing on $(0, \infty)$.

**Checkpoint 2**  $\dfrac{dW}{dt} = 0.116t + 0.19 > 0$ when $5 \le t \le 14$,

which implies that the consumption of bottled water was increasing from 1995 through 2004.

**Checkpoint 3**   Increasing on $(-\infty, -2)$ and $(2, \infty)$

Decreasing on $(-2, 2)$

**Checkpoint 4**   Increasing on $(0, \infty)$

Decreasing on $(-\infty, 0)$

**Checkpoint 5**   Because $f'(x) = -3x^2 = 0$ when $x = 0$ and because $f$ is decreasing on $(-\infty, 0) \cup (0, \infty)$, $f$ is decreasing on $(-\infty, \infty)$.

**Checkpoint 6**   $(0, 3000)$

## SECTION 3.2

**Checkpoint 1**   Relative maximum at $(-1, 5)$
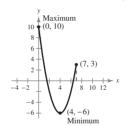
Relative minimum at $(1, -3)$

**Checkpoint 2**   Relative minimum at $(3, -27)$

**Checkpoint 3**   Relative maximum at $(1, 1)$

Relative minimum at $(0, 0)$

**Checkpoint 4**   Absolute maximum at $(0, 10)$

Absolute minimum at $(4, -6)$



**Checkpoint 5**

| $x$ (units) | 24,000 | 24,200 | 24,300 | 24,400 |
|---|---|---|---|---|
| $P$ (profit) | \$24,760 | \$24,766 | \$24,767.50 | \$24,768 |

| $x$ (units) | 24,500 | 24,600 | 24,800 | 25,000 |
|---|---|---|---|---|
| $P$ (profit) | \$24,767.50 | \$24,766 | \$24,760 | \$24,750 |

## SECTION 3.3

**Checkpoint 1**   (a) $f'' = -4$; because $f''(x) < 0$ for all $x$, $f$ is concave downward for all $x$.

(b) $f''(x) = \dfrac{1}{2x^{3/2}}$; because $f''(x) > 0$ for all $x > 0$, $f$ is concave upward for all $x > 0$.

**Checkpoint 2**   Because $f''(x) > 0$ for $x < -\dfrac{2\sqrt{3}}{3}$ and

$x > \dfrac{2\sqrt{3}}{3}$, $f$ is concave upward on

$\left(-\infty, -\dfrac{2\sqrt{3}}{3}\right)$ and $\left(\dfrac{2\sqrt{3}}{3}, \infty\right)$.

Because $f''(x) < 0$ for $-\dfrac{2\sqrt{3}}{3} < x < \dfrac{2\sqrt{3}}{3}$,

$f$ is concave downward on $\left(-\dfrac{2\sqrt{3}}{3}, \dfrac{2\sqrt{3}}{3}\right)$.

**Checkpoint 3**   $f$ is concave upward on $(-\infty, 0)$ and $(1, \infty)$.
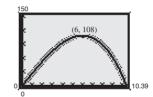
$f$ is concave downward on $(0, 1)$.

Points of inflection: $(0, 1)$, $(1, 0)$

**Checkpoint 4**   Relative minimum: $(3, -26)$

**Checkpoint 5**   Point of diminishing returns: $x = \$150$ thousand

## SECTION 3.4

**Checkpoint 1**



Maximum volume $= 108$ in.$^3$

**Checkpoint 2**   $x = 6$, $y = 12$

**Checkpoint 3**   $\left(\sqrt{\frac{1}{2}}, \frac{7}{2}\right)$ and $\left(-\sqrt{\frac{1}{2}}, \frac{7}{2}\right)$

**Checkpoint 4**   8 in. by 12 in.

## SECTION 3.5

**Checkpoint 1**   125 units yield a maximum revenue of \$1,562,500.

**Checkpoint 2**   400 units

**Checkpoint 3**   \$6.25/unit

**Checkpoint 4**   \$4.00

**Checkpoint 5**   Demand is elastic when $0 < x < 144$.

Demand is inelastic when $144 < x < 324$.

Demand is of unit elasticity when $x = 144$.

## SECTION 3.6

**Checkpoint 1**   (a) $\lim\limits_{x \to 2^-} \dfrac{1}{x - 2} = -\infty$;  $\lim\limits_{x \to 2^+} \dfrac{1}{x - 2} = \infty$

(b) $\lim\limits_{x \to -3^-} \dfrac{-1}{x + 3} = \infty$;  $\lim\limits_{x \to -3^+} \dfrac{-1}{x + 3} = -\infty$
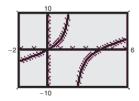
**Checkpoint 2**   $x = 0, x = 4$

**Checkpoint 3**   $x = 3$

**Checkpoint 4**   $\lim\limits_{x\to 2^-} \dfrac{x^2 - 4x}{x - 2} = \infty; \ \lim\limits_{x\to 2^+} \dfrac{x^2 - 4x}{x - 2} = -\infty$



**Checkpoint 5**   2

**Checkpoint 6**   (a)  $y = 0$

(b)  $y = \frac{1}{2}$

(c)  No horizontal asymptote

**Checkpoint 7**   $C = 0.75x + 25,000$

$$\overline{C} = 0.75 + \frac{25,000}{x}$$

$$\lim\limits_{x\to\infty} \overline{C} = \$0.75/\text{unit}$$

**Checkpoint 8**   No, the cost function is not defined at $p = 100$, which implies that it is not possible to remove 100% of the pollutants.

## SECTION 3.7

### Checkpoint 1

| | $f(x)$ | $f'(x)$ | $f''(x)$ | Shape of graph |
|---|---|---|---|---|
| $x$ in $(-\infty, -1)$ | | $-$ | $+$ | Decreasing, concave upward |
| $x = -1$ | $-32$ | $0$ | $+$ | Relative minimum |
| $x$ in $(-1, 1)$ | | $+$ | $+$ | Increasing, concave upward |
| $x = 1$ | $-16$ | $+$ | $0$ | Point of inflection |
| $x$ in $(1, 3)$ | | $+$ | $-$ | Increasing, concave downward |
| $x = 3$ | $0$ | $0$ | $-$ | Relative maximum |
| $x$ in $(3, \infty)$ | | $-$ | $-$ | Decreasing, concave downward |

### Checkpoint 2

| | $f(x)$ | $f'(x)$ | $f''(x)$ | Shape of graph |
|---|---|---|---|---|
| $x$ in $(-\infty, 0)$ | | $-$ | $+$ | Decreasing, concave upward |
| $x = 0$ | $5$ | $0$ | $0$ | Point of inflection |
| $x$ in $(0, 2)$ | | $-$ | $-$ | Decreasing, concave downward |
| $x = 2$ | $-11$ | $-$ | $0$ | Point of inflection |
| $x$ in $(2, 3)$ | | $-$ | $+$ | Decreasing, concave upward |
| $x = 3$ | $-22$ | $0$ | $+$ | Relative minimum |
| $x$ in $(3, \infty)$ | | $+$ | $+$ | Increasing, concave upward |

### Checkpoint 3

| | $f(x)$ | $f'(x)$ | $f''(x)$ | Shape of graph |
|---|---|---|---|---|
| $x$ in $(-\infty, 0)$ | | $+$ | $-$ | Increasing, concave downward |
| $x = 0$ | $0$ | $0$ | $-$ | Relative maximum |
| $x$ in $(0, 1)$ | | $-$ | $-$ | Decreasing, concave downward |
| $x = 1$ | Undef. | Undef. | Undef. | Vertical asymptote |
| $x$ in $(1, 2)$ | | $-$ | $+$ | Decreasing, concave upward |
| $x = 2$ | $4$ | $0$ | $+$ | Relative minimum |
| $x$ in $(2, \infty)$ | | $+$ | $+$ | Increasing, concave upward |

### Checkpoint 4

| | $f(x)$ | $f'(x)$ | $f''(x)$ | Shape of graph |
|---|---|---|---|---|
| $x$ in $(-\infty, -1)$ | | $+$ | $+$ | Increasing, concave upward |
| $x = -1$ | Undef. | Undef. | Undef. | Vertical asymptote |
| $x$ in $(-1, 0)$ | | $+$ | $-$ | Increasing, concave downward |
| $x = 0$ | $-1$ | $0$ | $-$ | Relative maximum |
| $x$ in $(0, 1)$ | | $-$ | $-$ | Decreasing, concave downward |
| $x = 1$ | Undef. | Undef. | Undef. | Vertical asymptote |
| $x$ in $(1, \infty)$ | | $-$ | $+$ | Decreasing, concave upward |

**Checkpoint 5**

|  | $f(x)$ | $f'(x)$ | $f''(x)$ | Shape of graph |
|---|---|---|---|---|
| $x$ in $(0, 1)$ |  | $-$ | $+$ | Decreasing, concave upward |
| $x = 1$ | $-4$ | $0$ | $+$ | Relative minimum |
| $x$ in $(1, \infty)$ |  | $+$ | $+$ | Increasing, concave upward |

## SECTION 3.8

**Checkpoint 1**   $dy = 0.32$; $\Delta y = 0.32240801$

**Checkpoint 2**   $dR = \$22$; $\Delta R = \$21$

**Checkpoint 3**   $dP = \$10.96$; $\Delta P = \$10.98$

**Checkpoint 4**   (a) $dy = 12x^2\, dx$   (b) $dy = \frac{2}{3}\, dx$

   (c) $dy = (6x - 2)\, dx$   (d) $dy = -\dfrac{2}{x^3}\, dx$

**Checkpoint 5**   $S = 1.96\pi$ in.$^2 \approx 6.1575$ in.$^2$

   $dS = \pm 0.056\pi$ in.$^2 \approx \pm 0.1759$ in.$^2$
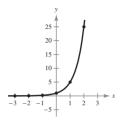
# CHAPTER 4

## SECTION 4.1

**Checkpoint 1**   (a) 243   (b) 3   (c) 64

   (d) 8   (e) $\frac{1}{2}$   (f) $\sqrt{10}$

**Checkpoint 2**   (a) $5.453 \times 10^{-13}$   (b) $1.621 \times 10^{-13}$

   (c) $2.629 \times 10^{-14}$
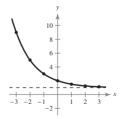
**Checkpoint 3**

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $\frac{1}{125}$ | $\frac{1}{25}$ | $\frac{1}{5}$ | 1 | 5 | 25 | 125 |



**Checkpoint 4**

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 9 | 5 | 3 | 2 | $\frac{3}{2}$ | $\frac{5}{4}$ | $\frac{9}{8}$ |
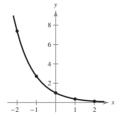


Horizontal asymptote: $y = 1$

## SECTION 4.2

**Checkpoint 1**

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | $e^2 \approx 7.389$ | $e \approx 2.718$ | 1 | $\frac{1}{e} \approx 0.368$ | $\frac{1}{e^2} \approx 0.135$ |



**Checkpoint 2**   After 0 h, $y = 1.25$ g.

   After 1 h, $y \approx 1.338$ g.

   After 10 h, $y \approx 1.498$ g.

   $\lim\limits_{t \to \infty} \dfrac{1.50}{1 + 0.2e^{-0.5t}} = 1.50$ g

**Checkpoint 3**   (a) \$4870.38   (b) \$4902.71

   (c) \$4918.66   (d) \$4919.21

   All else being equal, the more often interest is compounded, the greater the balance.

**Checkpoint 4**   (a) 7.12%   (b) 7.25%

**Checkpoint 5**   \$16,712.90

## SECTION 4.3

**Checkpoint 1**   At $(0, 1)$, $y = x + 1$.

   At $(1, e)$, $y = ex$.
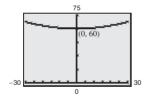
**Checkpoint 2**   (a) $3e^{3x}$   (b) $-\dfrac{6x^2}{e^{2x^3}}$   (c) $8xe^{x^2}$   (d) $-\dfrac{2}{e^{2x}}$

**Checkpoint 3**  (a) $xe^x(x + 2)$   (b) $\frac{1}{2}(e^x - e^{-x})$
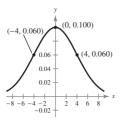
(c) $\dfrac{e^x(x - 2)}{x^3}$   (d) $e^x(x^2 + 2x - 1)$

**Checkpoint 4**



**Checkpoint 5**  $18.39/unit (80,000 units)

**Checkpoint 6**
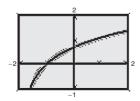


Points of inflection:  $(-4, 0.060), (4, 0.060)$

## SECTION 4.4

**Checkpoint 1**

| $x$ | $-1.5$ | $-1$ | $-0.5$ | $0$ | $0.5$ | $1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $-0.693$ | $0$ | $0.405$ | $0.693$ | $0.916$ | $1.099$ |



**Checkpoint 2**  (a) 3   (b) $x + 1$

**Checkpoint 3**  (a) $\ln 2 - \ln 5$   (b) $\frac{1}{3}\ln(x + 2)$

(c) $\ln x - \ln 5 - \ln y$

(d) $\ln x + 2 \ln(x + 1)$

**Checkpoint 4**  (a) $\ln x^4 y^3$   (b) $\ln \dfrac{x + 1}{(x + 3)^2}$

**Checkpoint 5**  (a) $\ln 6$   (b) $5 \ln 5$

**Checkpoint 6**  (a) $e^4$   (b) $e^3$

**Checkpoint 7**  7.9 yr

## SECTION 4.5

**Checkpoint 1**  $\dfrac{1}{x}$

**Checkpoint 2**  (a) $\dfrac{2x}{x^2 - 4}$   (b) $x(1 + 2 \ln x)$

(c) $\dfrac{2 \ln x - 1}{x^3}$

**Checkpoint 3**  $\dfrac{1}{3(x + 1)}$

**Checkpoint 4**  $\dfrac{2}{x} + \dfrac{x}{x^2 + 1}$

**Checkpoint 5**  Relative minimum:
$(2, 2 - 2 \ln 2) \approx (2, 0.6137)$

**Checkpoint 6**  $\dfrac{dp}{dt} = -1.3\%/mo$
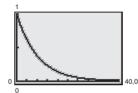
The average score would decrease at a greater rate than the model in Example 6.

**Checkpoint 7**  (a) 4   (b) $-2$   (c) $-5$   (d) 3

**Checkpoint 8**  (a) 2.322   (b) 2.631   (c) 3.161

(d) $-0.5$

**Checkpoint 9**



As time increases, the derivative approaches 0. The rate of change of the amount of carbon isotopes is proportional to the amount present.

## SECTION 4.6

**Checkpoint 1**  About 2113.7 yr

**Checkpoint 2**  $y = 25e^{0.6931t}$

**Checkpoint 3**  $r = \frac{1}{8} \ln 2 \approx 0.0866$ or 8.66%

**Checkpoint 4**  About 12.42 mo

# CHAPTER 5

## SECTION 5.1

**Checkpoint 1**  (a) $\displaystyle\int 3 \, dx = 3x + C$

(b) $\displaystyle\int 2x \, dx = x^2 + C$

(c) $\displaystyle\int 9t^2 \, dt = 3t^3 + C$

**Checkpoint 2**  (a) $5x + C$   (b) $-r + C$   (c) $2t + C$

**Checkpoint 3**  $\frac{5}{2}x^2 + C$

**Checkpoint 4**  (a) $-\dfrac{1}{x} + C$   (b) $\dfrac{3}{4}x^{4/3} + C$

**Checkpoint 5**  (a) $\frac{1}{2}x^2 + 4x + C$   (b) $x^4 - \frac{5}{2}x^2 + 2x + C$

**Checkpoint 6**  $\frac{2}{3}x^{3/2} + 4x^{1/2} + C$

**Checkpoint 7**  General solution:  $F(x) = 2x^2 + 2x + C$

Particular solution:  $F(x) = 2x^2 + 2x + 4$

**Checkpoint 8**  $s(t) = -16t^2 + 32t + 48$. The ball hits the ground 3 seconds after it is thrown, with a velocity of $-64$ feet per second.

**Checkpoint 9**  $C = -0.01x^2 + 28x + 12.01$

$C(200) = \$5212.01$

## SECTION 5.2

**Checkpoint 1**  (a) $\dfrac{(x^3 + 6x)^3}{3} + C$   (b) $\dfrac{2}{3}(x^2 - 2)^{3/2} + C$

**Checkpoint 2**  $\frac{1}{36}(3x^4 + 1)^3 + C$

**Checkpoint 3**  $2x^9 + \frac{12}{5}x^5 + 2x + C$

**Checkpoint 4**  $\frac{5}{3}(x^2 + 1)^{3/2} + C$

**Checkpoint 5**  $-\frac{1}{3}(1 - 2x)^{3/2} + C$

**Checkpoint 6**  $\frac{1}{3}(x^2 + 4)^{3/2} + C$

**Checkpoint 7**  About \$32,068

## SECTION 5.3
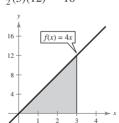
**Checkpoint 1**  (a) $3e^x + C$   (b) $e^{5x} + C$

(c) $e^x - \dfrac{x^2}{2} + C$

**Checkpoint 2**  $\frac{1}{2}e^{2x+3} + C$

**Checkpoint 3**  $2e^{x^2} + C$

**Checkpoint 4**  (a) $2\ln|x| + C$   (b) $\ln|x^3| + C$

(c) $\ln|2x + 1| + C$

**Checkpoint 5**  $\frac{1}{4}\ln|4x + 1| + C$

**Checkpoint 6**  $\frac{3}{2}\ln(x^2 + 4) + C$

**Checkpoint 7**  (a) $4x - 3\ln|x| - \dfrac{2}{x} + C$

(b) $2\ln(1 + e^x) + C\,dx$

(c) $\dfrac{x^2}{2} + x + 3\ln|x + 1| + C$

## SECTION 5.4
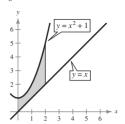
**Checkpoint 1**  $\frac{1}{2}(3)(12) = 18$
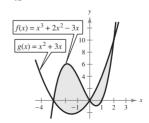


**Checkpoint 2**  $\frac{22}{3}$ units$^2$

**Checkpoint 3**  68

**Checkpoint 4**  (a) $\frac{1}{4}(e^4 - 1) \approx 13.3995$

(b) $-\ln 5 + \ln 2 \approx -0.9163$

**Checkpoint 5**  $\frac{13}{2}$

**Checkpoint 6**  (a) About \$14.18   (b) \$141.79

**Checkpoint 7**  \$13.70

**Checkpoint 8**  (a) $\frac{2}{5}$   (b) 0

**Checkpoint 9**  About \$12,295.62

## SECTION 5.5

**Checkpoint 1**  $\frac{8}{3}$ units$^2$



**Checkpoint 2**  $\frac{32}{3}$ units$^2$

**Checkpoint 3**  $\frac{9}{2}$ units$^2$

**Checkpoint 4**  $\frac{253}{12}$ units$^2$



**Checkpoint 5**  Consumer surplus:  40

Producer surplus:  20

**Checkpoint 6**  The company can save \$39.36 million.

## SECTION 5.6

**Checkpoint 1**  $\frac{37}{8}$ units$^2$

**Checkpoint 2**  0.436 unit$^2$

**Checkpoint 3**  5.642 units$^2$

**Checkpoint 4**  About 1.463

# CHAPTER 6

## SECTION 6.1

**Checkpoint 1**　$\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

**Checkpoint 2**　$\frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C$

**Checkpoint 3**　$\frac{d}{dx}[x\ln x - x + C] = x\left(\frac{1}{x}\right) + \ln x - 1$

$$= \ln x$$

**Checkpoint 4**　$e^x(x^3 - 3x^2 + 6x - 6) + C$

**Checkpoint 5**　$e - 2$

**Checkpoint 6**　$538,145

**Checkpoint 7**　$721,632.08

## SECTION 6.2

**Checkpoint 1**　$\dfrac{5}{x+3} - \dfrac{4}{x+4}$

**Checkpoint 2**　$\ln|x(x+2)^2| + \dfrac{1}{x+2} + C$

**Checkpoint 3**　$\dfrac{1}{2}x^2 - 2x - \dfrac{1}{x} + 4\ln|x+1| + C$

**Checkpoint 4**　$ky(1-y) = \dfrac{kbe-kt}{(1+be^{-kt})^2}$

$$y = (1+be^{-kt})^{-1}$$

$$\frac{dy}{dt} = \frac{kbe^{-kt}}{(1+be^{-kt})^2}$$

Therefore, $\dfrac{dy}{dt} = ky(1-y)$

**Checkpoint 5**　$y = 4$

**Checkpoint 6**　$y = \dfrac{4000}{1 + 39e^{-0.31045t}}$

## SECTION 6.3

**Checkpoint 1**　$\frac{2}{3}(x-4)\sqrt{2+x} + C$　(Formula 19)

**Checkpoint 2**　$\sqrt{x^2+16} - 4\ln\left|\dfrac{4+\sqrt{x^2+16}}{x}\right| + C$
(Formula 23)

**Checkpoint 3**　$\dfrac{1}{4}\ln\left|\dfrac{x-2}{x+2}\right| + C$　(Formula 29)

**Checkpoint 4**　$\frac{1}{3}[1 - \ln(1+e) + \ln 2] \approx 0.12663$
(Formula 37)

**Checkpoint 5**　$x(\ln x)^2 + 2x - 2x\ln x + C$　(Formula 42)

**Checkpoint 6**　About 18.2%

## SECTION 6.4

**Checkpoint 1**　3.2608

**Checkpoint 2**　3.1956

**Checkpoint 3**　1.154
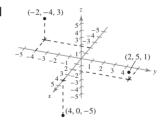
## SECTION 6.5

**Checkpoint 1**　(a) Converges; $\frac{1}{2}$　(b) Diverges

**Checkpoint 2**　1

**Checkpoint 3**　$\frac{1}{2}$

**Checkpoint 4**　2

**Checkpoint 5**　Diverges

**Checkpoint 6**　Diverges

**Checkpoint 7**　0.0038 or $\approx 0.4\%$

**Checkpoint 8**　No, you do not have enough money to start the scholarship fund because you need $125,000.　($125,000 > $120,000)

# CHAPTER 7

## SECTION 7.1
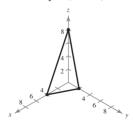
**Checkpoint 1**



**Checkpoint 2**　$2\sqrt{6}$

**Checkpoint 3**　$\left(-\frac{5}{2}, 2, -2\right)$

**Checkpoint 4**　$(x-4)^2 + (y-3)^2 + (z-2)^2 = 25$

**Checkpoint 5**　$(x-1)^2 + (y-3)^2 + (z-2)^2 = 38$

**Checkpoint 6**　Center: $(-3, 4, -1)$; radius: 6

**Checkpoint 7**　$(x+1)^2 + (y-2)^2 = 16$

## SECTION 7.2

**Checkpoint 1**   $x$-intercept: $(4, 0, 0)$;

   $y$-intercept: $(0, 2, 0)$;

   $z$-intercept: $(0, 0, 8)$



**Checkpoint 2**   Hyperboloid of one sheet
   $xy$-trace: circle, $x^2 + y^2 = 1$; $yz$-trace:
   hyperbola, $y^2 - z^2 = 1$; $xz$-trace: hyperbola,
   $x^2 - z^2 = 1$; $z = 3$ trace: circle,
   $x^2 + y^2 = 10$

**Checkpoint 3**   (a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = z$; elliptic paraboloid

   (b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} - z^2 = 0$; elliptic cone

## SECTION 7.3

**Checkpoint 1**   (a) 0   (b) $\frac{9}{4}$

**Checkpoint 2**   Domain: $x^2 + y^2 \le 9$

   Range: $0 \le z \le 3$

**Checkpoint 3**   Steep; nearly level

**Checkpoint 4**   Alaska is mainly used for forest land. Alaska
   does not contain any manufacturing centers,
   but it does contain a mineral deposit of
   petroleum.

**Checkpoint 5**   $f(1500, 1000) \approx 127{,}542$ units

   $f(1000, 1500) \approx 117{,}608$ units

   $x$, person-hours, has a greater effect on
   production.

**Checkpoint 6**   (a) $M = \$733.76/\text{mo}$

   (b) Total paid $= (30 \times 12) \times 733.76$

   $= \$264{,}153.60$

## SECTION 7.4

**Checkpoint 1**   $\dfrac{\partial z}{\partial x} = 4x - 8xy^3$

   $\dfrac{\partial z}{\partial y} = -12x^2y^2 + 4y^3$

**Checkpoint 2**   $f_x(x, y) = 2xy^3$; $f_x(1, 2) = 16$

   $f_y(x, y) = 3x^2y^2$; $f_y(1, 2) = 12$

**Checkpoint 3**   In the $x$-direction: $f_x(1, -1, 49) = 8$

   In the $y$-direction: $f_y(1, -1, 49) = -18$

**Checkpoint 4**   Substitute product relationship

**Checkpoint 5**   $\dfrac{\partial w}{\partial x} = xy + 2xy \ln(xz)$

   $\dfrac{\partial w}{\partial y} = x^2 \ln xz$

   $\dfrac{\partial w}{\partial z} = \dfrac{x^2 y}{z}$

**Checkpoint 6**   $f_{xx} = 8y^2$

   $f_{yy} = 8x^2 + 8$

   $f_{xy} = 16xy$

   $f_{yx} = 16xy$

**Checkpoint 7**   $f_{xx} = 0$     $f_{xy} = e^y$     $f_{xz} = 2$

   $f_{yx} = e^y$     $f_{yy} = xe^y + 2$     $f_{yz} = 0$

   $f_{zx} = 2$     $f_{zy} = 0$     $f_{zz} = 0$

## SECTION 7.5

**Checkpoint 1**   $f(-8, 2) = -64$:  relative minimum

**Checkpoint 2**   $f(0, 0) = 1$:  relative maximum

**Checkpoint 3**   $f(0, 0) = 0$:  saddle point

**Checkpoint 4**   $P(3.11, 3.81) = \$744.81$ maximum profit

**Checkpoint 5**   $V\left(\frac{4}{3}, \frac{2}{3}, \frac{8}{3}\right) = \frac{64}{27}$ units$^3$

## SECTION 7.6

**Checkpoint 1**   $V\left(\frac{4}{3}, \frac{2}{3}, \frac{8}{3}\right) = \frac{64}{27}$ units$^3$

**Checkpoint 2**   $f(187.5, 50) \approx 13{,}474$  units

**Checkpoint 3**   About 26,740 units

**Checkpoint 4**   $P(3.35, 4.26) = \$758.08$ maximum profit

**Checkpoint 5**   $f(2, 0, 2) = 8$

## SECTION 7.7

**Checkpoint 1**   For $f(x)$, $S \approx 9.1$.

   For $g(x)$, $S \approx 0.47515$.

   The quadratic model is a better fit.

**Checkpoint 2**   $f(x) = \frac{6}{5}x + \frac{23}{10}$

**Checkpoint 3**   $y = 20{,}041.5t + 103{,}455.5$

   In 2010, $y \approx 303{,}870.5$ subscribers

**Checkpoint 4**   $y = 6.595t^2 + 143.50t + 1971.0$

   In 2010, $y = \$7479$.

## SECTION 7.8
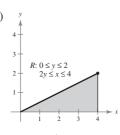
**Checkpoint 1** (a) $\frac{1}{4}x^4 + 2x^3 - 2x - \frac{1}{4}$

(b) $\ln|y^2 + y| - \ln|2y|$

**Checkpoint 2** $\frac{25}{2}$

**Checkpoint 3** $\int_2^4 \int_1^5 dx\, dy = 8$

**Checkpoint 4** $\frac{4}{3}$

**Checkpoint 5** (a)



(b) $\int_0^4 \int_0^{x/2} dy\, dx$

(c) $\int_0^2 \int_{2y}^4 dx\, dy = 4 = \int_0^4 \int_0^{x/2} dy\, dx$

**Checkpoint 6** $\int_{-1}^3 \int_{x^2}^{2x+3} dy\, dx = \frac{32}{3}$

## SECTION 7.9

**Checkpoint 1** $\frac{16}{3}$

**Checkpoint 2** $e - 1$

**Checkpoint 3** $\frac{176}{15}$

**Checkpoint 4** Integration by parts

**Checkpoint 5** 3

# CHAPTER 8

## SECTION 8.1

**Checkpoint 1** (a) 150°  (b) 30°  (c) 135°  (d) 30°

**Checkpoint 2** (a) $\frac{5\pi}{4}$  (b) $-\frac{\pi}{4}$  (c) $\frac{4\pi}{3}$  (d) $\frac{5\pi}{6}$

**Checkpoint 3** (a) 300°  (b) 210°  (c) 270°  (d) $-135°$

**Checkpoint 4** $\frac{1}{2}$ ft²

## SECTION 8.2

**Checkpoint 1** $\sin \frac{\pi}{6} = \frac{1}{2}$

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\tan \frac{\pi}{6} = \frac{\sqrt{3}}{3}$

**Checkpoint 2** (a) $\frac{1}{2}$  (b) $-\frac{\sqrt{2}}{2}$  (c) $-\sqrt{3}$

**Checkpoint 3** (a) $-\frac{1}{2}$  (b) $\sqrt{2}$  (c) $\frac{\sqrt{6} - \sqrt{2}}{4}$

(d) 1  (e) 1  (f) 1

**Checkpoint 4** About 52.5 ft
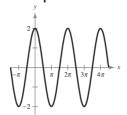
**Checkpoint 5** About 245.76°

**Checkpoint 6** (a) $\frac{\pi}{4}, \frac{7\pi}{4}$  (b) $\frac{2\pi}{3}, \frac{5\pi}{3}$  (c) $\frac{7\pi}{6}, \frac{11\pi}{6}$
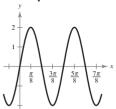
**Checkpoint 7** $\theta = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$

## SECTION 8.3

**Checkpoint 1**



**Checkpoint 2**
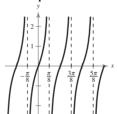


**Checkpoint 3**



**Checkpoint 4**
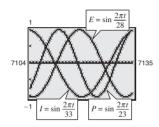
| $x$ | $-0.10$ | $-0.05$ | $-0.01$ |
|---|---|---|---|
| $\frac{1 - \cos x}{x}$ | $-0.0500$ | $-0.0250$ | $-0.0050$ |

| $x$ | 0.01 | 0.05 | 0.10 |
|---|---|---|---|
| $\frac{1 - \cos x}{x}$ | 0.0050 | 0.0250 | 0.0500 |

$\lim\limits_{x \to 0} \frac{1 - \cos x}{x} = 0$

**Checkpoint 5** Answers will vary.

**Checkpoint 6**



## SECTION 8.4

**Checkpoint 1**   (a) $-4 \sin 4x$   (b) $2x \cos(x^2 - 1)$

(c) $\dfrac{1}{2} \sec^2 \dfrac{x}{2}$

**Checkpoint 2**   (a) $\dfrac{1}{2\sqrt{x}} \cos \sqrt{x}$   (b) $-6x^2 \sin x^3$

**Checkpoint 3**   (a) $3 \sin^2 x \cos x$   (b) $-8 \sin 2x \cos^3 2x$

**Checkpoint 4**   (a) $4 \sec(4x) \tan(4x)$   (b) $-2x \csc^2 x^2$

**Checkpoint 5**   (a) $-\dfrac{\sin 2x}{\sqrt{\cos 2x}}$   (b) $\dfrac{\sec^2 3x}{\sqrt[3]{\tan^2 3x}}$

**Checkpoint 6**   (a) $-x^2 \sin x + 2x \cos x$

(b) $2t \cos 2t + \sin 2t$

**Checkpoint 7**   Relative maximum: $\left( \dfrac{7\pi}{6}, \dfrac{7\pi + 6\sqrt{3}}{12} \right)$

Relative minimum: $\left( \dfrac{11\pi}{6}, \dfrac{11\pi - 6\sqrt{3}}{12} \right)$

**Checkpoint 8**   Relative maximum: $\left( \dfrac{\pi}{6}, \dfrac{3\sqrt{3}}{4} \right)$

Relative minimum: $\left( \dfrac{5\pi}{6}, -\dfrac{3\sqrt{3}}{4} \right)$

**Checkpoint 9**   About 1721 lb/day

**Checkpoint 10**   About $-3.9°/h$

## SECTION 8.5

**Checkpoint 1**   $-5 \cos x + C$

**Checkpoint 2**   $\sin x^4 + C$

**Checkpoint 3**   $\frac{1}{5} \tan 5x + C$

**Checkpoint 4**   $-\csc 2x + C$

**Checkpoint 5**   $-\frac{1}{8} \cos^4 2x + C$

**Checkpoint 6**   $\ln|\sin x| + C$

**Checkpoint 7**   $1$

**Checkpoint 8**   $1$

**Checkpoint 9**   $\frac{1}{2} \ln|\sec 2x + \tan 2x| + C$

**Checkpoint 10**   $82.68°$

## *CHAPTER 9*
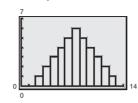
## SECTION 9.1

**Checkpoint 1**
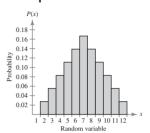
(a) $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2),$
    $(2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4),$
    $(3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),$
    $(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2),$
    $(6, 3), (6, 4), (6, 5), (6, 6)\}$

(b) $E = \{(1, 6), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 3),$
    $(4, 4), (4, 5), (4, 6), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),$
    $(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

**Checkpoint 2**
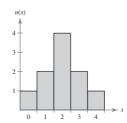


**Checkpoint 3**



**Checkpoint 4**   $1$

**Checkpoint 5**   $2.37$ units/day

**Checkpoint 6**   $V(x) = 1.2$, $\sigma \approx 1.095$



## SECTION 9.2

**Checkpoint 1**   $\displaystyle\int_0^2 \dfrac{1}{2}x \, dx = 1$

**Checkpoint 2**   $\displaystyle\int_0^\infty 2e^{-2x} \, dx = 1$

**Checkpoint 3**   $\frac{3}{16}$

**Checkpoint 4**   $0.4$

**Checkpoint 5**   $0.368$

**Checkpoint 6**    (a) About 0.259   (b) About 0.259

Because the graph of the probability density function is symmetric about the line $x = 3$, you know that the area under the graph over the interval $[0, 2]$ is equal to the area under the graph over the interval $[4, 6]$.

From Example 6, you know that the probability over $[2, 4]$ is about 0.481. If you subtract 0.481 from 1 you obtain 0.519. Then divide 0.519 by 2 to obtain approximately 0.259, which is the probability from parts (a) and (b).

## SECTION 9.3

**Checkpoint 1**    2

**Checkpoint 2**    $V(x) = \dfrac{4}{5}, \sigma \approx \sqrt{\dfrac{4}{5}} \approx 0.8944$

**Checkpoint 3**    84.6%

**Checkpoint 4**    Mean: 0.5; median: 0.35

**Checkpoint 5**    $\mu = 1; \sigma \approx 0.577$

**Checkpoint 6**    0.61 or 61%

**Checkpoint 7**    0.159

# *CHAPTER 10*

## SECTION 10.1

**Checkpoint 1**    (a) $a_1 = 2, a_2 = 5, a_3 = 8, a_4 = 11$

(b) $a_1 = \frac{1}{2}, a_2 = \frac{2}{5}, a_3 = \frac{3}{10}, a_4 = \frac{4}{17}$

**Checkpoint 2**    (a) 0   (b) 2

**Checkpoint 3**    0

**Checkpoint 4**

| $n$ | 1 | 2 | 3 | 4 | $\cdots$ | $n$ |
|---|---|---|---|---|---|---|
| $f^{(n-1)}(x)$ | $e^{2x}$ | $2e^{2x}$ | $4e^{2x}$ | $8e^{2x}$ | $\cdots$ | $2^{n-1}e^{ex}$ |
| $f^{(n-1)}(0)$ | 1 | $2^1$ | $2^2$ | $2^3$ | $\cdots$ | $2^{n-1}$ |

Diverges

**Checkpoint 5**    $\dfrac{(-1)^{n+1}n^2}{(n+1)!}$

**Checkpoint 6**    $A_n = 1000(1.015)^n$

## SECTION 10.2

**Checkpoint 1**    $\displaystyle\sum_{i=1}^{4} 4\left(-\dfrac{1}{2}\right)^n$

**Checkpoint 2**    $\frac{1}{3}$

**Checkpoint 3**    (a) 4   (b) $\frac{1}{2}$

**Checkpoint 4**    (a) Diverges   (b) Diverges

**Checkpoint 5**    $S_5 \approx 5.556, S_{50} = 5.\overline{5}, S_{500} = 5.\overline{5}$

**Checkpoint 6**    \$1021.17

**Checkpoint 7**    (a) Converges to $\frac{5}{30}$

(b) Diverges

(c) Converges to $\frac{20}{3}$

**Checkpoint 8**    About 40,000 units

**Checkpoint 9**    140 ft

## SECTION 10.3

**Checkpoint 1**    (a) $p$-series with $p = \pi$

(b) $p$-series with $p = \frac{3}{2}$

(c) Geometric series

**Checkpoint 2**    (a) Converges

(b) Converges

(c) Diverges

**Checkpoint 3**    Converges to approximately 20.086

**Checkpoint 4**    Diverges

**Checkpoint 5**    Diverges

## SECTION 10.4

**Checkpoint 1**    (a) 2   (b) $-3$   (c) 0

**Checkpoint 2**    Radius of convergence is infinite.

**Checkpoint 3**    $R = 3$

**Checkpoint 4**    $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-x)^n}{n!};$   radius of convergence is infinite.

**Checkpoint 5**    $\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}(x-1)^n}{n}$

**Checkpoint 6**    (a) $\displaystyle\sum_{n=0}^{\infty} \dfrac{(2x)^n}{n!}$   (b) $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n(2x)^n}{n!}$
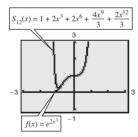
**Checkpoint 7**

$(1 + x)^{1/2} = 1 + \dfrac{x}{2} - \dfrac{x^2}{2^2 2!} + \dfrac{3x^3}{2^3 3!} - \dfrac{3 \cdot 5x^4}{2^4 4!} + \cdots$

**Checkpoint 8**    (a) $1 - \displaystyle\sum_{n=0}^{\infty} \dfrac{(-x)^n}{n!}$

(b) $e^2 \displaystyle\sum_{n=0}^{\infty} \dfrac{(-x)^n}{n!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}(x-1)^n}{n}$

## SECTION 10.5

**Checkpoint 1**  $S_{12}(x) = 1 + 2x^3 + 2x^6 + \dfrac{4x^9}{3} + \dfrac{2x^{12}}{3}$



**Checkpoint 2**  $e^{-0.5} \approx 0.607$ with a maximum error of 0.0003.

**Checkpoint 3**  1.970

**Checkpoint 4**  0.0005; 8 times

**Checkpoint 5**  1
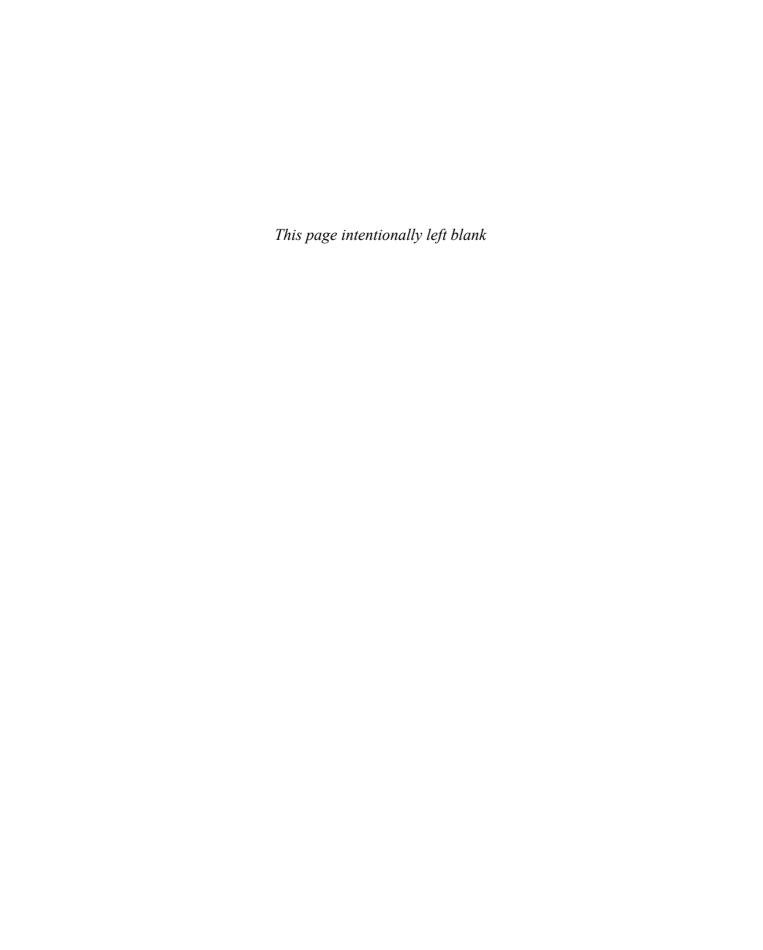
## SECTION 10.6

**Checkpoint 1**  1.735811

**Checkpoint 2**  $-0.453398$

**Checkpoint 3**  1.319074

**Checkpoint 4**  0.567143

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# Index

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# Applications (continued from front endsheets)

## Basic Differentiation Rules

**1.** $\dfrac{d}{dx}[cu] = cu'$

**2.** $\dfrac{d}{dx}[u \pm v] = u' \pm v'$

**3.** $\dfrac{d}{dx}[uv] = uv' + vu'$

**4.** $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{vu' - uv'}{v^2}$

**5.** $\dfrac{d}{dx}[c] = 0$

**6.** $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$

**7.** $\dfrac{d}{dx}[x] = 1$

**8.** $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$

**9.** $\dfrac{d}{dx}[e^u] = e^u u'$

**10.** $\dfrac{d}{dx}[\log_a u] = \dfrac{u'}{(\ln a)u}$

**11.** $\dfrac{d}{dx}[a^u] = (\ln a)a^u u'$

**12.** $\dfrac{d}{dx}[\sin u] = (\cos u)u'$

**13.** $\dfrac{d}{dx}[\cos u] = -(\sin u)u'$

**14.** $\dfrac{d}{dx}[\tan u] = (\sec^2 u)u'$

**15.** $\dfrac{d}{dx}[\cot u] = -(\csc^2 u)u'$

**16.** $\dfrac{d}{dx}[\sec u] = (\sec u \tan u)u'$

**17.** $\dfrac{d}{dx}[\csc u] = -(\csc u \cot u)u'$

## Basic Integration Formulas

**1.** $\displaystyle\int kf(u)\, du = k\int f(u)\, du$

**2.** $\displaystyle\int [f(u) \pm g(u)]\, du = \int f(u)\, du \pm \int g(u)\, du$

**3.** $\displaystyle\int du = u + C$

**4.** $\displaystyle\int a^u\, du = \left(\dfrac{1}{\ln a}\right)a^u + C$

**5.** $\displaystyle\int e^u\, du = e^u + C$

**6.** $\displaystyle\int \ln u\, du = u(-1 + \ln u) + C$

**7.** $\displaystyle\int \sin u\, du = -\cos u + C$

**8.** $\displaystyle\int \cos u\, du = \sin u + C$

**9.** $\displaystyle\int \tan u\, du = -\ln|\cos u| + C$

**10.** $\displaystyle\int \cot u\, du = \ln|\sin u| + C$

**11.** $\displaystyle\int \sec u\, du = \ln|\sec u + \tan u| + C$

**12.** $\displaystyle\int \csc u\, du = -\ln|\csc u + \cot u| + C$

**13.** $\displaystyle\int \sec^2 u\, du = \tan u + C$

**14.** $\displaystyle\int \csc^2 u\, du = -\cot u + C$

## Trigonometric Identities

*Pythagorean Identities*

$\sin^2 \theta + \cos^2 \theta = 1$

$\tan^2 \theta + 1 = \sec^2 \theta$

$\cot^2 \theta + 1 = \csc^2 \theta$

*Sum or Difference of Two Angles*

$\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi$

$\cos(\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi$

$\tan(\theta \pm \phi) = \dfrac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi}$

*Double Angle*

$\sin 2\theta = 2 \sin \theta \cos \theta$

$\cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$

*Reduction Formulas*

$\sin(-\theta) = -\sin \theta$

$\cos(-\theta) = \cos \theta$

$\tan(-\theta) = -\tan \theta$

$\sin \theta = -\sin(\theta - \pi)$

$\cos \theta = -\cos(\theta - \pi)$

$\tan \theta = \tan(\theta - \pi)$

*Half Angle*

$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$

$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$

# ALGEBRA

## Quadratic Formula:

If $p(x) = ax^2 + bx + c$, $a \neq 0$ and $b^2 - 4ac \geq 0$, then the real zeros of $p$ are $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$.

*Example*

If $p(x) = x^2 + 3x - 1$, then $p(x) = 0$ if

$$x = \frac{-3 \pm \sqrt{13}}{2}.$$

## Special Factors:

$x^2 - a^2 = (x - a)(x + a)$

$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

$x^4 - a^4 = (x - a)(x + a)(x^2 + a^2)$

$x^4 + a^4 = \left(x^2 + \sqrt{2}\,ax + a^2\right)\left(x^2 - \sqrt{2}\,ax + a^2\right)$

$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-1})$, for $n$ odd

$x^n + a^n = (x + a)(x^{n-1} - ax^{n-2} + \cdots + a^{n-1})$, for $n$ odd

$x^{2n} - a^{2n} = (x^n - a^n)(x^n + a^n)$

*Examples*

$x^2 - 9 = (x - 3)(x + 3)$

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

$x^3 + 4 = \left(x + \sqrt[3]{4}\right)\left(x^2 - \sqrt[3]{4}x + \sqrt[3]{16}\right)$

$x^4 - 4 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)(x^2 + 2)$

$x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$

$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$

$x^7 + 1 = (x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$

$x^6 - 1 = (x^3 - 1)(x^3 + 1)$

## Exponents and Radicals:

$a^0 = 1, \ a \neq 0$

$a^{-x} = \dfrac{1}{a^x}$

$a^x a^y = a^{x+y}$

$\dfrac{a^x}{a^y} = a^{x-y}$

$(a^x)^y = a^{xy}$

$(ab)^x = a^x b^x$

$\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

$\sqrt{a} = a^{1/2}$

$\sqrt[n]{a} = a^{1/n}$

$\sqrt[n]{a^m} = a^{m/n} = \left(\sqrt[n]{a}\right)^m$

$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

$\sqrt[n]{\left(\dfrac{a}{b}\right)} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

## Algebraic Errors to Avoid:

$\dfrac{a}{x + b} \neq \dfrac{a}{x} + \dfrac{a}{b}$

(To see this error, let $a = b = x = 1$.)

$\sqrt{x^2 + a^2} \neq x + a$

(To see this error, let $x = 3$ and $a = 4$.)

$a - b(x - 1) \neq a - bx - b$

[Remember to distribute negative signs. The equation should be $a - b(x - 1) = a - bx + b$.]

$\dfrac{\left(\dfrac{x}{a}\right)}{b} \neq \dfrac{bx}{a}$

[To divide fractions, invert and multiply. The equation should be

$$\dfrac{\left(\dfrac{x}{a}\right)}{b} = \dfrac{\left(\dfrac{x}{a}\right)}{\left(\dfrac{b}{1}\right)} = \left(\dfrac{x}{a}\right)\left(\dfrac{1}{b}\right) = \dfrac{x}{ab}.]$$

$\sqrt{-x^2 + a^2} \neq -\sqrt{x^2 - a^2}$

(The negative sign cannot be factored out of the square root.)

$\dfrac{a + bx}{a} \neq 1 + bx$

(This is one of many examples of incorrect dividing out. The equation should be

$$\dfrac{a + bx}{a} = \dfrac{a}{a} + \dfrac{bx}{a} = 1 + \dfrac{bx}{a}.)$$

$\dfrac{1}{x^{1/2} - x^{1/3}} \neq x^{-1/2} - x^{-1/3}$

(This error is a more complex version of the first error.)

$(x^2)^3 \neq x^5$

[This equation should be $(x^2)^3 = x^2 x^2 x^2 = x^6$.]